Abstract

Magic cubes are figures which are composed of the values 1...N³ arranged on a cubic lattice in which the sum of the elements of all rows, columns, pillars and main body diagonals are equal. The entries in the magic cube are replaced by corresponding multiples of a unit charge. Several new magic cube properties which depend only on the order of the cube are reported for the multipole expansion of the electrical potential of these objects when treated as distributions of point charges. The lowest non-zero multipole terms are in the elements of the octupole tensor.

1 Introduction

In a recent paper Rogers and Loly [1] showed that the inertia tensor of a discrete rigid mass distribution, whose masses reflect the placement of numbers in a magic cube has maximal symmetry, akin to that of a continuous spherical distribution. Here we consider the corresponding electrical problem, where the charges in a discrete distribution correspond to the numbers of a magic cube, finding highly symmetrical results.

Magic squares are figures which are arranged in a square array, having the property that all rows, columns and main diagonals add to the same value. This value is referred to as the magic constant of the magic square. Normal magic squares are defined as having a set of values ranging from 1 to N², provided N > 2. This type of magic square has a simple and easily derived formula to predict the magic sum, based only on the order of the square, N [2]:

$$C_2 = \frac{N}{2}(N^2 + 1)$$

(1)

The other property of magic squares which depends only on N is the moment of inertia of the square when the entries are treated as multiples of a unit mass.
The moment of inertia is given by a slightly more complicated expression than the line sum [3].

The concept of magic squares can be pushed further by extending the figure into the third dimension, which produces a figure known as a magic cube. In analogy with magic squares, magic cubes are defined as having entries consisting of 1..N³, with the sum of all rows, columns, pillars and the four main body diagonals equal to the magic constant. The expression for the magic constant of a magic cube is [2]:

\[ C_3 = \frac{N}{2}(N^3 + 1) \] (2)

As in the case of a magic square, the magic constant depends only on the order of the cube \( N \), as expected. The smallest magic cubes are the four fundamental magic cubes of order three. Through reflections and rotations of the cube elements each of these fundamental cubes can be expressed in 48 different ways, giving rise to a total of 192 magic cubes of order three [4]. One of these cubes is displayed below:

| 7 11 24 | 15 25 2 | 20 6 16 |
| 23 9 10 | 1 14 27 | 18 19 5 |
| 12 22 8 | 26 3 13 | 4 17 21 |

Like their 2-dimensional counterparts, the are inertia tensor of a magic cube also depends only upon \( N \) [1]. In an analogous manner, the derivation of the expressions describing the moments of the multipole expansion are only dependent on the semi-magic property of a magic cube. That is, in order for these results to be valid it is only required that the rows, columns and pillars of the cube add to the magic constant.

2 The multipole expansion

In classical electromagnetism it is possible to expand the potential \( V \) of a charge distribution in a series of terms that decrease as powers of \( 1/r \) [5]. The series of terms produced is known as the multipole expansion of the charge distribution.

\[
V(r) = \left( \frac{1}{4\pi \varepsilon_0} \right) \left( \frac{Q}{r} + \frac{\Sigma_i \hat{r}_i p_i}{r^2} + \frac{\Sigma_{i,j} \hat{r}_i \hat{r}_j Q_{ij}}{2r^3} + \frac{\Sigma_{i,j,k} \hat{r}_i \hat{r}_j \hat{r}_k Q_{ijk}}{2r^4} + \ldots \right) (4)
\]

Where \( \hat{r}_i \) are the unit components of the displacement vector. Typically, it is assumed that the lowest non-vanishing term in the expansion is the dominant term and as such most nearly represents the true potential of the charge distribution at large \( r \). Each additional term adds a correction, eventually assembling an increasingly accurate description of the charge distribution in question.
The first term in the multipole expansion is called the monopole moment. The monopole moment is simply the sum of the charges in a given charge distribution; a single point charge located at the origin has the monopole moment as the only non-vanishing term in the expansion. In this case, the monopole moment alone is enough to specify the exact potential at large distances from the distribution. The monopole term in the multipole expansion goes as $1/r$.

The monopole moment is specified by a scalar quantity. The second-order term in the expansion is called the dipole moment of the distribution. In some cases although the overall monopole moment may vanish the dipole moment will not, such as the case of two equal but opposite charges located a distance $d$ apart. The dipole term decreases as $1/r^2$ and is represented by a vector. Specifically, the dipole moment is given by the expression

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i$$

in which $q_i$ represents the value of the $i^{th}$ charge and $r_i$ represents the displacement from the origin to that charge.

The third order multipole term is called the quadrupole moment of the distribution. The potential generated by this term decreases as $1/r^3$, and is represented by a second-rank tensor which is similar in form to a $3 \times 3$ array. The elements of the quadrupole tensor are defined as:

$$Q_{ij} = \sum_{\alpha} q_{\alpha} (3x_{\alpha,i}x_{\alpha,j} - \delta_{i,j}r_{\alpha}^2)$$

where $q_{\alpha}$ represents the value of the charge located in the $\alpha^{th}$ cell and the $x_{\alpha,n}$ values represent the components of the displacement from a specified origin to the $\alpha^{th}$ cell measured in terms of the nearest neighbor displacement.

Finally, the fourth order term in the multipole expansion is the octupole moment of the distribution. This term provides a correction to the potential which goes as $1/r^4$ and is a third-rank tensor, composed of 27 elements. These elements are defined by:

$$O_{ijk} = \sum_{\alpha} q_{\alpha} (5r_{\alpha,i}r_{\alpha,j}r_{\alpha,k} - r^2 (r_{\alpha,i}\delta_{j,k} + r_{\alpha,j}\delta_{i,k} + r_{\alpha,k}\delta_{i,j}))$$

### 3 The dipole moment of a magic cube

In calculating the dipole moment of a magic cube, it is convenient to place the co-ordinate origin at the center of the cube. This choice of origin is physically significant, as the cube can be interpreted as a series of $N$ layers, each layer comprised of $N$ rows and $N$ columns. From this, it is obvious that since each row (or column) of the layer has identical charge as specified by equation (2) and these rows have equal and opposite displacements from the origin, with the origin located at the geometric center of the cube the dipole moment vanishes.
The contributions of each line sum perfectly cancel one another due to their symmetric arrangement. Since this argument hinges on the fact that only the rows, columns and pillars of the cube sum to the same value, this property is true for the larger class of semi-magic cubes.

4 The quadrupole moment of a magic cube

With the origin of the co-ordinate system at the center of the cube and co-ordinate axes perpendicular to the faces of the cube, one can define the quadrupole products (the off-diagonal elements of the quadrupole tensor) as given by equation (5) when $i \neq j$. An additional property of this tensor is that it is a 'symmetric' tensor, as is the inertia tensor [6]. This implies that

$$Q_{ij} = Q_{ji}$$

(8)

and as such the tensor contains only 6 independent elements [6]. Since a magic cube itself is symmetrically composed of $N$ layers of charges equally separated by a unit distance from one another and each layer consists of $N$ rows and columns, it is clear that the quadrupole products will all be equal. It is relevant that this property stems from the semi-magic attributes of the magic cube. Using this fact, one need only calculate a single quadrupole product to discover all the off-diagonal elements of the tensor. Using the magic cube previously given (3), one of these calculations gives:

$$Q_{xy} = 0$$

(9)

In general this is expected since we have equal amounts of charge oppositely displaced from the origin in all directions.

With the origin of the co-ordinate system held fixed at the center of the cube it is further possible to define the quadrupole moments of the tensor (these entries of the quadrupole tensor lie along the main diagonal of the tensor). Because a point charge at the origin is defined as having vanishing multipole moments save for the monopole moment, the tensor reproduces this effect for a spherical distribution. A spherical distribution of charge centered on the origin will have a non-vanishing monopole moment and vanishing higher order terms in the multipole expansion. The quadrupole tensor is 'traceless', that is the sum of the diagonal elements will always equal zero. The trace is an invariant property of a matrix, ensuring that the trace of this tensor always vanishes regardless of the co-ordinate system the tensor is described with respect to (this provides a useful check to the accuracy of the calculation of the quadrupole moments). For the third order cube cited previously the values of $Q_{xx}$, $Q_{yy}$ and $Q_{zz}$ can be calculated:

$$Q_{xx} = Q_{yy} = Q_{zz} = 0$$

(10)

The quadrupole tensor consists of vanishing elements when calculated at the center of the cube. Somewhat surprisingly, this is identical to the quadrupole tensor of a solid sphere with the origin of co-ordinates at the center of the sphere.
These results apply to cubes of arbitrary order as a result of the line sum (Row, Column and Pillar) symmetry, which is referred to as RCP symmetry. Far from a spherical distribution all moments higher than the monopole moment vanish. In terms of the calculation just performed it is clear that the quadrupole tensor does not differentiate between a solid sphere and a magic cube if the origin is placed at the center of each distribution. This result is significant in part because of the number of distributions which are described by this simple result. The number of magic squares per order grow rapidly. While there exist only 4 fundamental order three cubes, Walter Trump recently estimated the number of fourth order cubes at approximately 7 trillion. [7].

5 The octupole moment of a magic cube

With the origin at the center of the cube, we proceed by calculating the octupole moments of the potential. This is a third order tensor and can be represented as a $3 \times 3 \times 3$ cube. Through direct calculation (using the aforementioned magic cube) we can find the elements of this tensor. When $i, j, k$ are all equal, the diagonal elements of the tensor vanish. However, unlike the previous multipole moments, the octupole products (the entries of the tensor which do not lie on the main-diagonal) do not in general vanish. The six elements for which $i, j$ and $k$ assume different values change depending on the magic cube under consideration. These elements have the form

$$O_{ijk} = \sum_{\alpha} q_{\alpha}(5x_{\alpha}y_{\alpha}z_{\alpha})$$

(11)

When $i \neq j \neq k \neq i$. The reason why these elements do not vanish is that over a pillar the x and y co-ordinates stay constant, though over the length of the pillar the z co-ordinate changes. Therefore we cannot remove the magic sum from the product, and since the values in each pillar are different these elements do not have a general form, unlike the entries of the inertia tensor which contains elements that can be totally described in terms of $N$, the order of the cube [1].

This is similar to the situation encountered when the quadrupole moment is calculated for a magic square. In that case the quadrupole products depend on x and y, and in two dimensions these values change along the length of a row/column. Therefore these values will change with the square used in calculating the moments.

6 The multipole expansion of neutral magic cubes

Define a neutral magic cube as a cube having all line sums vanishing. For example, a neutral magic cube can be made from a normal magic cube composed of entries $1..N^3$, and subtracting the center element from all the entries of the cube. This will ensure that the entries of the cube run from $-(\frac{N+1}{2})$ to...
\( \frac{N+1}{2} \). For even order cubes, the average of the eight central values is taken and subtracted from the other entries in the cube again ensuring that the entries of the cube follow a similar distribution to that of an odd order neutral cube. This causes the monopole moment to vanish. When the origin is fixed at the center of the cube, the dipole and quadrupole contributions to the potential also vanish as discussed in the previous case. Thus a neutral magic cube produces no electric potential to third order. However, the octupole tensor behaves in much the same way as for the normal magic cube. That is, the off-diagonal elements in which i,j and k assume different values do not in general vanish and assume values which depend on the cube used to perform the calculation.

7 Conclusion

In closing, these magic cube examples provide a good arena in practice for illustrating properties of the multipole expansion. In his book, "Introduction to Electrodynamics" [5], David J. Griffiths presents the following problem (Problem 3.45, part c):

"Show that the quadrupole moment is independent of origin if the monopole and dipole moments both vanish. (This works all the way up the hierarchy - the lowest non-zero multipole moment is always independent of origin.)"

The dipole and quadrupole moments of semi-magic cubes have been shown to vanish, and neutral cubes also have a vanishing monopole moment. Because of this, the neutral cube octupole moment should be independent of origin. In a direct calculation for magic cubes, the octupole moment is independent of origin, returning the same value regardless of the placement of the origin. These distributions provide a good example of the property mentioned by Griffiths and serve as excellent demonstrations of the properties of the multipole expansion.

8 Acknowledgement

The authors would like to acknowledge Balram Bhakar, whose insight and advice have clarified many questions.

References


