A Class of Symmetrical Condensed Node TLM Methods Derived Directly from Maxwell's Equations

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Abstract—**A series of general transmission line matrix (TLM)-type methods, which include the symmetrical condensed node method, are derived directly from Maxwell's curl equations without recourse to transmission line models. Written as a system of conservation laws, Maxwell's equations in 3-D plus time are decomposed along the orthogonal characteristic directions of a rectangular grid. The Riemann invariants in this method correspond to the voltage pulses of the TLM method. A new method of handling inhomogeneous media is proposed based on a new transfer event. The dispersive nature of these schemes is also investigated.**

I. Introduction

THE transmission line matrix (TLM) method was pioneered by Johns and Beurle [1] for two-dimensional waveguide scattering problems. The method can be considered as a differential equation-based numerical method, capable of providing an approximate solution to the time dependent form of Maxwell's equations in arbitrary media. The method is traditionally viewed as a physical approximation in that the space domain of an electromagnetic field problem is approximated by an orthogonal system of transmission lines, and the exact solution for the voltage variables of the transmission line problem is obtained. The solution to the original field problem is then approximated by a mapping from the voltage variables to the field variables of interest. A wide variety of electromagnetic field problems have been analyzed using the TLM method [2], [3], including the characterization of microwave circuit components [4], and radar cross section calculations [5].

Over the years, new versions of the method have been developed in order to extend it to three dimensions and improve the modeling of arbitrary inhomogeneous media. These versions usually consisted of changing the topology of the transmission lines used to approximate the physical problem. The transmission line matrix method can be considered as: a discrete form of Huygen's principle [6], [7], an extension of the lumped element techniques originated by Kron [8], or as a physical model of mathematical finite differencing [9]–[11]. The most recent and widely used three-dimensional version is the symmetrical condensed node method introduced by Johns [12].

In this paper, a class of TLM-type algorithms are derived directly from Maxwell's equations. It is shown that the curl equations in conservation law form can be decomposed on a discrete space-time grid as systems of equations governing a series of orthogonally propagating plane waves. The new variables which correspond to the voltage pulses in the TLM method are the so-called Riemann invariants [13]. The fact that these methods can be derived directly from Maxwell's equations without recourse to transmission line theory may make them more appealing and understandable to practitioners of electromagnetic modeling.

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II. Maxwell's Equations in Three Spacial Dimensions

The first step is to write Maxwell's curl equations as a system of hyberbolic conservation laws [14]–[16]. These equations are then approximated in each cell of a discrete numerical space-time grid as three systems of equations which can be diagonalized by a transformation of variables. The curl equations are written in conservation law form as

$$\partial_t \boldsymbol{u} + A_E \partial_x \boldsymbol{u} + A_F \partial_y \boldsymbol{u} + A_G \partial_z \boldsymbol{u} = \boldsymbol{0} \tag{1}$$

where

$$A_E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_G = \begin{bmatrix} 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

and $e = 1/\varepsilon$, $m = 1/\mu$ and the solution vector $\boldsymbol{u}$ is the field vector

$$\boldsymbol{u} = \begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} \tag{3}$$

The symmetrization of A_E, A_F, and A_G requires the eigenvalues of these matrices which can be calculated as

$$\begin{aligned} \lambda &= \{-\sqrt{me}, -\sqrt{me}, 0, 0, \sqrt{me}, \sqrt{me}\} \\ &= \{-c, -c, 0, 0, c, c\} \end{aligned} \tag{4}$$

as well as the right and left eigenvector matrices, R and L, which are calculated as

$$R_E = \begin{bmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & Y & 0 & 0 & 0 & -Y \\ -Y & 0 & 0 & 0 & Y & 0 \end{bmatrix},$$

$$L_E = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & 0 & Z & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & 0 & -Z & 0 \end{bmatrix}, \tag{5}$$

$$R_F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -Y & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ Y & 0 & 0 & 0 & -Y & 0 \end{bmatrix},$$

$$L_F = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & -Z & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & Z & 0 & 0 \end{bmatrix}, \tag{6}$$

$$R_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & -Y & 0 \\ -Y & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{bmatrix},$$

$$L_G = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & -Z & 0 \\ 0 & 1 & 0 & Z & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -Z & 0 & 0 \\ 1 & 0 & 0 & 0 & Z & 0 \end{bmatrix}. \tag{7}$$

The plane wave admittance and impedance in a computational cell are defined as

$$\begin{aligned} Y &= \frac{c}{e} = \frac{m}{c} = \sqrt{\frac{m}{e}} = \sqrt{\frac{\varepsilon}{\mu}} = \frac{1}{Z} \quad \text{and} \\ Z &= \frac{c}{m} = \frac{e}{z} = \sqrt{\frac{e}{m}} = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{Y}. \end{aligned} \tag{8}$$

It can be checked that $LR = RL = I$ as required, and the similarity transformations of A_E, A_F, and A_G are written as

$$\begin{aligned} L_E A_E R_E &= L_F A_F R_F = L_G A_G R_G \\ &= \operatorname{diag}(\lambda) = \Lambda. \end{aligned} \tag{9}$$

III. Numerical Approximations

Now if it is assumed that, in a computational cell, propagation along the x-direction involves *no* variation with respect to the y- and z-directions, propagation along the y-direction involves *no* variation with respect to the x- and z-directions, and that propagation along the z-direction involves *no* variation with respect to the x- and y-directions, then equation (1) can be approximated by the three systems of equations

$$\begin{aligned} &\partial_t \boldsymbol{u} + A_E \partial_x \boldsymbol{u} = \boldsymbol{0}, \qquad \partial_t \boldsymbol{u} + A_F \partial_y \boldsymbol{u} = \boldsymbol{0} \\ &\text{and} \quad \partial_t \boldsymbol{u} + A_G \partial_z \boldsymbol{u} = \boldsymbol{0}. \end{aligned} \tag{10}$$

Each of these equations can be uncoupled by diagonalizing the matrices A_E, A_F, and A_G. This is accomplished by the use of the right and left matrices defined above. New variables $\boldsymbol{v}$, called *Riemann invariants* [13], are defined as

$$\begin{aligned} \boldsymbol{v}_E = L_E \boldsymbol{u} &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & 0 & Z & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & 0 & -Z & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} E_y - ZH_z \\ E_z + ZH_y \\ \sqrt{2}\,E_x \\ \sqrt{2}\,H_x \\ E_y + ZH_z \\ E_z - ZH_y \end{bmatrix} = \begin{bmatrix} v_{E1} \\ v_{E2} \\ v_{E3} \\ v_{E4} \\ v_{E5} \\ v_{E6} \end{bmatrix} \end{aligned} \tag{11}$$

$$\begin{aligned} \boldsymbol{v}_F = L_F \boldsymbol{u} &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & -Z & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & Z & 0 & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \\ Hx \\ H_y \\ H_z \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} E_x + ZH_z \\ E_z - ZH_x \\ \sqrt{2}\,E_y \\ \sqrt{2}\,H_y \\ E_x - ZH_z \\ E_z + ZH_x \end{bmatrix} = \begin{bmatrix} v_{F1} \\ v_{F2} \\ v_{F3} \\ v_{F4} \\ v_{F5} \\ v_{F6} \end{bmatrix} \end{aligned} \tag{12}$$

$$v_G = L_G u = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & -Z & 0 \\ 0 & 1 & 0 & Z & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -Z & 0 & 0 \\ 1 & 0 & 0 & 0 & Z & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} E_x - ZH_y \\ E_y + ZH_x \\ \sqrt{2}E_z \\ \sqrt{2}H_z \\ E_y - ZH_x \\ E_x + ZH_y \end{bmatrix} = \begin{bmatrix} v_{G1} \\ v_{G2} \\ v_{G3} \\ v_{G4} \\ v_{G5} \\ v_{G6} \end{bmatrix} \tag{13}$$

and are used in (2) to give

$$\partial_t \boldsymbol{v}_F + \Lambda_E \partial_x \boldsymbol{v}_F = \boldsymbol{0}, \qquad \partial_t + \Lambda_F \partial_y = \boldsymbol{0}$$
$$\text{and} \quad \partial_t \boldsymbol{v}_G + \Lambda_G \partial_z \boldsymbol{v}_G = \boldsymbol{0}$$

which are diagonalized systems of partial differential equations. The solutions are easily found as

$$v_{Ei}(x,t) = f_{Ei}(x - \lambda_i t), \qquad v_{Fi}(y,t) = f_{Fi}(y - \lambda_i t)$$
$$\text{and} \quad v_{Gi}(z,t) = f_{Gi}(z - \lambda_i t),$$

respectively, where the f_{Ei}, f_{Fi}, and f_{Gi} are arbitrary functions. When $(x - \lambda_i t)$ is constant, v_{Ei} is constant; the lines defined by the equations $\Gamma_i : x - \lambda_i t = \text{Const.}$ are called the characteristic directions. Now if initial conditions are given for the v_{Ei}, v_{Fi}, and v_{Gi}, then the f_{Ei}, f_{Fi}, and f_{Gi} will be known. That is,

$$\boldsymbol{f}_E(x) = \boldsymbol{v}_E(x,0) = \boldsymbol{v}_{E0} = L_E \boldsymbol{u}(x,0) = L_E \boldsymbol{u}_0$$

$$= \frac{1}{2}\begin{bmatrix} E_y(x,0) - ZH_z(x,0) \\ E_z(x,0) + ZH_y(x,0) \\ \sqrt{2}E_x(x,0) \\ \sqrt{2}H_x(x,0) \\ E_y(x,0) + ZH_z(x,0) \\ E_z(x,0) - ZH_y(x,0) \end{bmatrix}, \tag{14}$$

$$\boldsymbol{f}_F(y) = \boldsymbol{v}_F(y,0) = \boldsymbol{v}_{F0} = L_F \boldsymbol{u}(y,0) = L_F \boldsymbol{u}_0$$

$$= \frac{1}{2}\begin{bmatrix} E_x(y,0) + ZH_z(y,0) \\ E_z(y,0) - ZH_x(y,0) \\ \sqrt{2}E_y(y,0) \\ \sqrt{2}H_y(y,0) \\ E_x(y,0) - ZH_z(y,0) \\ E_z(y,0) + ZH_x(y,0) \end{bmatrix}, \tag{15}$$

$$\boldsymbol{f}_G(z) = \boldsymbol{v}_G(z,0) = \boldsymbol{v}_{G0} = L_G \boldsymbol{u}(z,0) = L_G \boldsymbol{u}_0$$

$$= \frac{1}{2}\begin{bmatrix} E_x(z,0) - ZH_y(z,0) \\ E_y(z,0) + ZH_x(z,0) \\ \sqrt{2}E_z(z,0) \\ \sqrt{2}H_z(z,0) \\ E_y(z,0) - ZH_x(z,0) \\ E_x(z,0) + ZH_y(z,0) \end{bmatrix}. \tag{16}$$

The first Riemann invariant (RI), $V_{E1} = 1/2\,(E_y - ZH_z)$, is constant on the line defined by $\Gamma_{-x} : x + ct = \text{Const.}$ ($-c$ eigenvalue). Therefore, V_{E1} can be thought of as *propagating* in the negative x-direction. The RI, $V_{E2} = 1/2\,(E_z + ZH_y)$, propagates in the positive x-direction along the line defined by $\Gamma_{+x} : x - ct = K$ (c eigenvalue). The RI, $V_{E3} = \sqrt{2}\,E_x$, is constant on the vertical line $\Gamma_x : x = K$ (0-eigenvalue). That is, the field component normal to the direction of assumed propagation does not change with time. This allows for discontinuous normal components of the fields at the interface between cells of different material constants. A similar analysis can be performed for the y- and z-directions. If a computational grid is used to approximate the fields and the grid spacing is such that $\Delta x_j = c_j \Delta t$ in cell j, then the RI's reach the cell interface after time $\Delta t/2$. The propagating RI's are renamed in order to explicitly denote the direction of propagation and the polarization of the electric field. For example, v_{E1} and v_{E2} are renamed v_{Ly} and v_{Lz}, representing *left* traveling waves in which the electric field is polarized in the y- and z-directions, respectively. The first letter of the descriptive labels *left, right, down, up, back,* and *forward* are used to identify the six orthogonal directions. The propagating RI's are summarized in Table I.

These propagating RI's can be represented as matrix multiplication of the field components as

$$\boldsymbol{V} = \begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \\ V_{Dx} \\ V_{Dz} \\ V_{Ux} \\ V_{Uz} \\ V_{Bx} \\ V_{By} \\ V_{Fy} \\ V_{Fx} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} E_y - ZH_z \\ E_z + ZH_y \\ E_y + ZH_z \\ E_z - ZH_y \\ E_x + ZH_z \\ E_z - ZH_x \\ E_x - ZH_z \\ E_z + ZH_x \\ E_x - ZH_y \\ E_y + ZH_x \\ E_y - ZH_x \\ E_x + ZH_y \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & 0 & Z & 0 \\ 0 & 1 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & 0 & -Z & 0 \\ 1 & 0 & 0 & 0 & 0 & Z \\ 0 & 0 & 1 & -Z & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -Z \\ 0 & 0 & 1 & Z & 0 & 0 \\ 1 & 0 & 0 & 0 & -Z & 0 \\ 0 & 1 & 0 & Z & 0 & 0 \\ 0 & 1 & 0 & -Z & 0 & 0 \\ 1 & 0 & 0 & 0 & Z & 0 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} = A\boldsymbol{u}. \tag{17}$$

The fact that this represents an overdetermined system of equations is key to the derivation to follow. The vector $\boldsymbol{V}$ can be calculated at *any* point in space and time. If the field vector is known on a computational grid at time $t = t^n$, $\boldsymbol{u}(x_i, y_j, z_k, t^n) = \boldsymbol{u}^n_{ijk}$, then the RI's which leave each grid point can be calculated via (17) (see Fig. 1). Thus,

TABLE I
SUMMARY OF PROPAGATING RIEMANN INVARIANTS

Cellular Field Approximation	Riemann Invariants	Field Representation	Characteristic Line (K = constant)	New Notation	Old Notation: out going	Old Notation: in coming
Field Variation in x-Direction Only	V_{E1}	$1/2\,(E_y - ZH_z)$	$\Gamma_{-x} : x + ct = K$ (left)	V_{Ly}	V_3	V_{11}
	V_{E2}	$1/2\,(E_z + ZH_y)$		V_{Lz}	V_6	V_{10}
	V_{E5}	$1/2\,(E_y + ZH_z)$	$\Gamma_{+x} : x - ct = K$ (right)	V_{Ry}	V_{11}	V_3
	V_{E6}	$1/2\,(E_z - ZH_y)$		V_{Rz}	V_{10}	V_6
Field Variation in y-Direction Only	V_{F1}	$1/2\,(E_x + ZH_z)$	$\Gamma_{-y} : y + ct = K$ (down)	V_{Dx}	V_1	V_{12}
	V_{F2}	$1/2\,(E_z - ZH_x)$		V_{Dz}	V_5	V_7
	V_{F5}	$1/2\,(E_x - ZH_z)$	$\Gamma_{+y} : y - ct = K$ (up)	V_{Ux}	V_{12}	V_1
	V_{F6}	$1/2\,(E_z + ZH_x)$		V_{Uz}	V_7	V_5
Field Variation in z-Direction Only	V_{G1}	$1/2\,(E_x - ZH_y)$	$\Gamma_{-z} : z + ct = K$ (back)	V_{Bx}	V_2	V_9
	V_{G2}	$1/2\,(E_y + ZH_x)$		V_{By}	V_4	V_8
	V_{G5}	$1/2\,(E_y - ZH_x)$	$\Gamma_{+z} : z - ct = K$ (forward)	V_{Fy}	V_8	V_4
	V_{G6}	$1/2\,(E_x + ZH_y)$		V_{Fx}	V_9	V_2

at $t = t^0$, the $\boldsymbol{V}^0$ can be calculated from $\boldsymbol{u}^0$ and they are propagated without attenuation or distortion to the cell interface. Once they reach the cell boundaries ($i+1/2$ points), the RI's must cross the interface. The principle which is used to do this is that at the cell boundary points the tangential field components must have *unique* values which are continuous across the boundary. For example, at time $t = t^{(n+1/2)^-}$, the RI's V_{Ry} and V_{Rz} reach the point $(i+1/2)$ from the point (i), and the RI's V_{Ly} and V_{Lz} reach the point $(i+1/2)$ from the point $(i+1)$. The tangential fields at this point can be determined as

$$\begin{bmatrix} E_y \\ E_z \\ H_y \\ H_z \end{bmatrix}_{i+1/2}^{n+1/2} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & Y_{i+1} & 0 & -Y_i \\ -Y_{i+1} & 0 & Y_i & 0 \end{bmatrix} \cdot \begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^-}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & Y_i & 0 & -Y_{i+1} \\ -Y_i & 0 & Y_{i+1} & 0 \end{bmatrix} \cdot \begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^+} \tag{18}$$

which leads to

$$\begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^+} =$$

$$\begin{bmatrix} \frac{2Y_{i+1}}{Y_{i+1}+Y_i} & 0 & \frac{Y_{i+1}-Y_i}{Y_{i+1}+Y_i} & 0 \\ 0 & \frac{2Y_{i+1}}{Y_{i+1}+Y_i} & 0 & \frac{Y_{i+1}-Y_i}{Y_{i+1}+Y_i} \\ \frac{Y_i-Y_{i+1}}{Y_{i+1}+Y_i} & 0 & \frac{2Y_i}{Y_{i+1}+Y_i} & 0 \\ 0 & \frac{Y_i-Y_{i+1}}{Y_{i+1}+Y_i} & 0 & \frac{2Y_i}{Y_{i+1}+Y_i} \end{bmatrix} \cdot \begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^-} \tag{19}$$

if $Y_{i+1} \neq Y_i$ and is simply the traditional TLM *transfer event*

$$\begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^+} = \begin{bmatrix} V_{Ly} \\ V_{Lz} \\ V_{Ry} \\ V_{Rz} \end{bmatrix}_{i+1/2}^{n+1/2^-} \tag{20}$$

if $Y_{i+1} = Y_i$. Thus, (19) gives the proper form for transferring the pulses through inhomogeneous media. Similar expressions can be obtained for the other pulses traveling in the y- and z-directions. The RI's then propagate from $t = t^{(n+1/2)^+}$ to $t^{(n+1)^-}$ without change. Once they reach the integer grid points, they are used to calculate the new outward propagating RI's. At time $(n+1)^-$ the RI's which are calculated for time $(n+1/2)^+$ reach the integer valued grid points, and again the restriction that the field values defined at each grid point should be uniquely defined by both the incoming as well as the outgoing RI's is imposed. If it is assumed that both $\boldsymbol{V}_{ijk}^{n+1^+}$ and $\boldsymbol{V}_{ijk}^{n+1^-}$ are derived from (17), then symbolically the relations

$$\boldsymbol{V}_{ijk}^{n+1^+} = A\boldsymbol{u}_{ijk}^{n+1} \quad \text{and} \quad \boldsymbol{V}_{ijk}^{n+1^-} = A\boldsymbol{u}_{ijk}^{n+1}, \tag{21}$$

can be written. Notice that it is *not* correct to write $\boldsymbol{V}_{ijk}^{n+1^+}$ equal to $\boldsymbol{V}_{ijk}^{n+1^-}$ since the defining equation is an overdetermined system of linear equations. The procedure chosen is to determine $\boldsymbol{u}_{ijk}^{n+1}$ from $\boldsymbol{V}_{ijk}^{n+1^-}$ and then $\boldsymbol{V}_{ijk}^{n+1^+}$ from $\boldsymbol{u}_{ijk}^{n+1}$. Since $\boldsymbol{u}_{ijk}^{n+1}$ is not uniquely defined by $\boldsymbol{V}_{ijk}^{n+1^-}$, an appropriate generalized inverse matrix must be determined.

The Moore–Penrose generalized inverse $A^{\dagger}\boldsymbol{V}_{ijk}^{n+1^-} = \boldsymbol{u}_{ijk}^{n+1}$ has the properties: 1) $AA^{\dagger}A = A$, 2) $A^{\dagger}AA^{\dagger} = A^{\dagger}$, 3) $\left(AA^{\dagger}\right)^{H} = AA^{\dagger}$, and 4) $\left(A^{\dagger}A\right)^{H} = A^{\dagger}A$, where A^{H} denotes the conjugate transpose of the matrix A [17]. It also turns out that $A^{\dagger}$ is the inverse which minimizes the Euclidean norm of $\boldsymbol{u}_{ijk}$. For the present case, this inverse can be calculated as (22) (below) and it can be shown that (23) (below). It is required to show that the chosen inverse matrix defines the field values at the integer grid points uniquely and continuously: $\boldsymbol{u}_{ijk}^{n+1} = A^{\dagger}\boldsymbol{V}_{ijk}^{n+1^-} = A^{\dagger}\boldsymbol{V}_{ijk}^{n+1^+}$. The second property of the Moore–Penrose generalized inverse is now used to determine the relationship between the RI's just before and just after the time $n+1$ by setting

$$\boldsymbol{u}_{ijk}^{n+1} = A^{\dagger}\boldsymbol{V}_{ijk}^{n+1^-} = A^{\dagger}AA^{\dagger}\boldsymbol{V}_{ijk}^{n+1^-} = A^{\dagger}\left(AA^{\dagger}+B\right)\boldsymbol{V}_{ijk}^{n+1^-} = A^{\dagger}\boldsymbol{V}_{ijk}^{n+1^+} = A^{\dagger}AA^{\dagger}\boldsymbol{V}_{ijk}^{n+1^+} = A^{\dagger}\left(AA^{\dagger}+B\right)\boldsymbol{V}_{ijk}^{n+1^+} \tag{24}$$

where any matrix, B, in the null-space of $A^{\dagger}$, say, $B \in \boldsymbol{N}\left(A^{\dagger}\right)$, is added to $AA^{\dagger}$. Thus, a scattering event defines the relationship between $\boldsymbol{V}_{ijk}^{n+1^-}$ and $\boldsymbol{V}_{ijk}^{n+1^+}$ as

$$\boldsymbol{V}_{ijk}^{n+1^+} = S\boldsymbol{V}_{ijk}^{n+1^-} \quad \text{or} \quad \boldsymbol{V}_{ijk}^{n+1^-} = S\boldsymbol{V}_{ijk}^{n+1^+} \tag{25}$$

where the matrix S is defined as

$$S = AA^{\dagger} + B\,. \tag{26}$$

Accepting both expressions (25) would result in the possible contradiction

$$\boldsymbol{V}_{ijk}^{n+1^+} = S\boldsymbol{V}_{ijk}^{n+1^-} = S\left(S\boldsymbol{V}_{ijk}^{n+1^+}\right) = S^2\boldsymbol{V}_{ijk}^{n+1^+}\,. \tag{27}$$

$$A^{\dagger} = \left(A^T A\right)^{-1}A^T = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -Y & 0 & Y & 0 & Y & -Y & 0 \\ 0 & Y & 0 & -Y & 0 & 0 & 0 & 0 & -Y & 0 & 0 & Y \\ -Y & 0 & Y & 0 & Y & 0 & -Y & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$AA^{\dagger} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad A^{\dagger}A = I\,. \tag{23}$$

$$B_a = \alpha\begin{bmatrix} -2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \tag{28}$$

In general, the left side of (25) is chosen since a scheme to move forward in time is desired, but it will be shown that for a particular choice of the B matrix, both expressions in (25) can be satisfied. A series of symmetric matrices, $\{B_\alpha | B_\alpha \in \boldsymbol{N}(A^\dagger), \alpha \in \mathcal{R}\}$, can be determined as (28) (shown on the previous page) and the symmetric scattering matrix becomes (29) (below). When $\alpha = 1/4$, the scattering matrix becomes (30) (below), which can be compared to the symmetrical condensed node TLM method as follows. In the traditional TLM method, the voltage pulses are not denoted by their propagation direction but rather by their location in each cell. The voltage pulses are numbered from 1 to 12 and are scattered according to

$$V_{tlm} = S_{tlm} V_{tlm} \tag{31}$$

where V_{tlm} is the vector of pulses V_1 to V_{12}. A transformation from V_{tlm} to the RI's defined herein can be defined by the use of Table I. In the present case, we can write

$$V^{out} = SV^{in} \tag{32}$$

where the extra notations *out* and *in* have been used to identify the fact that the transformation to the old notation is not the

$$S_\alpha = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$+ \alpha \begin{bmatrix} -2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \tag{29}$$

$$S_{1/4} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

same for each of these. That is,

$$V^{in} = A_{in/tlm} V_{tlm}, \qquad V_{tlm} = A_{tlm/out} V^{out} \tag{33}$$

$$A_{in/tlm} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$A_{tlm/out} = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \end{bmatrix} \tag{34}$$

and we can express the traditional TLM scattering matrix in terms of the present scattering matrix as (35) (below). Since any of these scattering matrices defined by (26) will satisfy the condition for uniquely defined fields at the integer grid points, some other method of determining the appropriate S_α is required. One appealing technique is to consider the energy flow in the propagating waves. Recall that the Poynting vector, defined as $\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H}$ can be written as

$$\begin{aligned} \boldsymbol{P} &= (E_y H_z - E_z H_y)\hat{a}_x + (E_z H_x - E_x H_z)\hat{a}_y \\ &\quad + (E_x H_y - E_y H_x)\hat{a}_z = P_x \hat{a}_x + P_y \hat{a}_y + P_z \hat{a}_z \end{aligned} \tag{36}$$

where P_x, P_y, and P_z denote the energy densities flowing in the x-, y-, and z-directions, respectively. The plane wave propagation assumption relates the electric field in each RI to the magnetic field via the plane wave impedance, that is,

$$E_\xi = \pm Z H_\zeta \tag{37}$$

where the sign is chosen such that $E_\xi \times (\pm H_\zeta)$ defines the direction of propagation by the right-hand rule. For example, left traveling plane waves (i.e., negative x-direction) give $E_y = -ZH_z$. Using this approximation, the sum of the squares of the RI's traveling in the same direction can be shown to be proportional to the energy density traveling in the same direction, that is,

$$\begin{bmatrix} V_{Ly}^2 + V_{Lz}^2 \\ V_{Ry}^2 + V_{Rz}^2 \\ V_{Dx}^2 + V_{Dz}^2 \\ V_{Ux}^2 + V_{Uz}^2 \\ V_{Bx}^2 + V_{By}^2 \\ V_{Fy}^2 + V_{Fx}^2 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} E_y^2 - 2ZE_yH_z + Z^2H_z^2 + E_z^2 + 2ZE_zH_y + Z^2H_y^2 \\ E_y^2 + 2ZE_yH_z + Z^2H_z^2 + E_z^2 - 2ZE_zH_y + Z^2H_y^2 \\ E_x^2 + 2ZE_xH_z + Z^2H_z^2 + E_z^2 - 2ZE_zH_x + Z^2H_x^2 \\ E_x^2 - 2ZE_xH_z + Z^2H_z^2 + E_z^2 + 2ZE_zH_x + Z^2H_x^2 \\ E_x^2 - 2ZE_xH_y + Z^2H_y^2 + E_y^2 + 2ZE_yH_x + Z^2H_x^2 \\ E_y^2 - 2ZE_yH_x + Z^2H_x^2 + E_x^2 + 2ZE_xH_y + Z^2H_y^2 \end{bmatrix} = 2 \begin{bmatrix} -P_x \\ P_x \\ -P_y \\ P_y \\ -P_z \\ P_z \end{bmatrix} \tag{38}$$

$$S_{tlm} = A_{tlm/out} S A_{in/tlm} = \frac{1}{2} \begin{bmatrix} 0&1&1&0&0&0&0&0&1&0&-1&0 \\ 1&0&0&0&0&1&0&0&0&-1&0&1 \\ 1&0&0&1&0&0&0&1&0&0&0&-1 \\ 0&0&1&0&1&0&-1&0&0&0&1&0 \\ 0&0&0&1&0&1&0&-1&0&1&0&0 \\ 0&1&0&0&1&0&1&0&-1&0&0&0 \\ 0&0&0&-1&0&1&0&1&0&1&0&0 \\ 0&0&1&0&-1&0&1&0&0&0&1&0 \\ 1&0&0&0&0&-1&0&0&0&1&0&1 \\ 0&-1&0&0&1&0&1&0&1&0&0&0 \\ -1&0&0&1&0&0&0&1&0&0&0&1 \\ 0&1&-1&0&0&0&0&0&1&0&1&0 \end{bmatrix}. \tag{35}$$

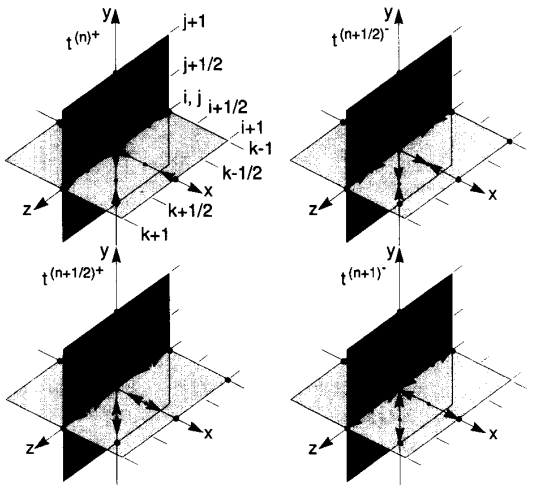

Fig. 1. RI's propagating on computational grid $\{(i,j,k)\}$.

Thus, the total energy density leaving a node can be represented as the Euclidean norm of the RI's. If the scattering matrix is chosen such that the total energy density entering a grid point is equal to the total energy density leaving that grid point (i.e., after the scattering event), then this produces the relations $\|\boldsymbol{V}_{ijk}^{n+1^+}\|_2^2 = \|S_\alpha \boldsymbol{V}_{ijk}^{n+1^-}\|_2^2 = \|\boldsymbol{V}_{ijk}^{n+1^-}\|_2^2$, which can be written as

$$\left(\boldsymbol{V}_{ijk}^{n+1^-}\right)^T \boldsymbol{V}_{ijk}^{n+1^-} = \left(S_\alpha \boldsymbol{V}_{ijk}^{n+1^-}\right)^T \left(S_\alpha \boldsymbol{V}_{ijk}^{n+1^-}\right) = \left(\boldsymbol{V}_{ijk}^{n+1^-}\right)^T S_\alpha^2 \boldsymbol{V}_{ijk}^{n+1^-} \tag{39}$$

since S_α is symmetric. One possible condition on the scattering matrix which satisfies this requirement is that $S_\alpha^2 = I$. If S_α^2 is expanded for any α, it will be found that the diagonal terms are all $8\alpha^2 + 1/2$, and the off-diagonal terms are all either $-4\alpha^2 + 1/4$, $4\alpha^2 - 1/4$, or 0, and therefore this leads to the result that $\{8\alpha^2 + 1/2 = 1, -4\alpha^2 + 1/4 = 0\} \Rightarrow \alpha = \pm(1/4)$ with associated scattering matrices given by $S_{-1/4} = I$ and $S_{1/4}$ given by (26). The first matrix results in *no* scattering at all, while the second results in the same scattering matrix as in the traditional TLM method (with the notational differences discussed previously.) Notice that this choice of scattering matrix allows the satisfaction of both expressions (25) since now choosing $\boldsymbol{V}_{ijk}^{n+1^-} = S_{1/4}\boldsymbol{V}_{ijk}^{n+1^+}$ as well as $\boldsymbol{V}_{ijk}^{n+1^-} = S_{1/4}\boldsymbol{V}_{ijk}^{n+1^+}$ implies

$$\boldsymbol{V}_{ijk}^{n+1^+} = S_{1/4}\boldsymbol{V}_{ijk}^{n+1^-} = S_{1/4}\left(S_{1/4}\boldsymbol{V}_{ijk}^{n+1^+}\right) = \boldsymbol{V}_{ijk}^{n+1^+} \tag{40}$$

which is now not a contradiction. Thus $S_{1/4}$ determines the reverse process as well as the forward process.

IV. Finite Difference Equivalents and Dispersion/Dissipation Analysis

For the case where $Z_j = Z_{j+1} = Z_k = Z_{k+1} = Z$, the transfer event can be written as

$$\begin{aligned}\boldsymbol{V}_{ijk}^{n+1^-} &= \operatorname{diag}(S_x^1, S_x^1, S_x^{-1}, S_x^{-1}, S_y^1, S_y^1, S_y^{-1}, S_y^{-1}, \\ &\quad S_z^1, S_z^1, S_z^{-1}, S_z^{-1}) V_{ijk}^{n^+} \\ &= \begin{bmatrix} (V_{Ly})_{i+1,j,k} \\ (V_{Lz})_{i+1,j,k} \\ (V_{Ry})_{i-1,j,k} \\ (V_{Rz})_{i-1,j,k} \\ (V_{Dx})_{i,j+1,k} \\ (V_{Dz})_{i,j+1,k} \\ (V_{Ux})_{i,j-1,k} \\ (V_{Uz})_{i,j-1,k} \\ (V_{Bx})_{i,j,k+1} \\ (V_{By})_{i,j,k+1} \\ (V_{Fy})_{i,j,k-1} \\ (V_{Fx})_{i,j,k-1} \end{bmatrix}^{n^+} = C\boldsymbol{V}_{ijk}^{n^+}\end{aligned} \tag{41}$$

where C is the diagonal matrix of forward and backward shift operators (i.e., S_x^{-1} represents a shift of -1 units in the x-direction). The equations developed thus far are combined to give

$$\begin{aligned}\boldsymbol{u}_{ijk}^{n+1} &= A^\dagger \boldsymbol{V}_{ijk}^{n+1^+} = A^\dagger \boldsymbol{V}_{ijk}^{n+1^-} = A^\dagger C \boldsymbol{V}_{ijk}^{n^+} \\ &= A^\dagger CS\boldsymbol{V}_{ijk}^{n^-} = A^\dagger C(AA^\dagger + B)\left(\boldsymbol{V}_{ijk}^{n^-}\right) \\ &= (A^\dagger CAA^\dagger + A^\dagger CB)\left(\boldsymbol{V}_{ijk}^{n^-}\right) \\ &= (A^\dagger CAA^\dagger + A^\dagger CB)(A(\boldsymbol{u}_{ijk}^n)) \\ &= (A^\dagger C(AA^\dagger A) + A^\dagger CBA)(\boldsymbol{u}_{ijk}^n) \\ &= (A^\dagger CA + A^\dagger CBA)(\boldsymbol{u}_{ijk}^n).\end{aligned} \tag{42}$$

Now, not only is $B \in \boldsymbol{N}(A^\dagger)$, but $A \in \boldsymbol{N}(B)$, and (42) becomes

$$\boldsymbol{u}_{ijk}^{n+1} = A^\dagger CA(\boldsymbol{u}_{ijk}^n). \tag{43}$$

Notice that in the field formulation, the *effect* of the B matrix is lost. Therefore, a one-time step finite difference scheme can only represent the TLM method if S_0 is used, and a TLM method with α other than zero cannot be represented by a one-time step finite difference scheme. Notice that in [9], Johns states that there is no one-time step scheme for the traditional ($\alpha = 1/4$) method. A two-time level scheme can be written by following the same procedure as in (42), but continuing the procedure for one more time level. Thus,

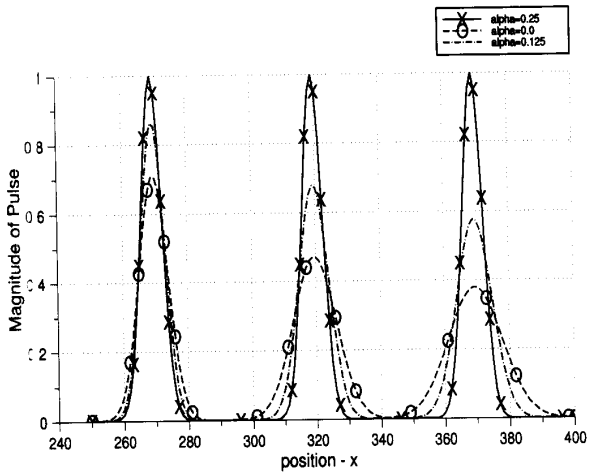

Fig. 2. Dispersion of Gaussian plane wave pulses.

$$
\begin{aligned}
u_{ijk}^{n+1} &= \left(A^{\dagger}CAA^{\dagger} + A^{\dagger}CB\right)\left(\boldsymbol{V}_{ijk}^{n-}\right) \\
&= A^{\dagger}CAA^{\dagger}\left(\boldsymbol{V}_{ijk}^{n-}\right) + A^{\dagger}CB\left(\boldsymbol{V}_{ijk}^{n-}\right) \\
&= A^{\dagger}CAA^{\dagger}Au_{ijk}^{n} + A^{\dagger}CB\left(\boldsymbol{V}_{ijk}^{n-}\right) \\
&= A^{\dagger}CA\left(u_{ijk}^{n}\right) + A^{\dagger}CB\left(\boldsymbol{V}_{ijk}^{n-}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + A^{\dagger}CBC\left(\boldsymbol{V}_{ijk}^{n-1^{+}}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + A^{\dagger}CBCS\left(\boldsymbol{V}_{ijk}^{n-1^{-}}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + A^{\dagger}CBC\left(AA^{\dagger} + B\right)\left(\boldsymbol{V}_{ijk}^{n-1^{-}}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + A^{\dagger}CBCAA^{\dagger} + A^{\dagger}CBCB \\
&\quad \cdot \left(\boldsymbol{V}_{ijk}^{n-1^{-}}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + \left(A^{\dagger}CBCAA^{\dagger} + A^{\dagger}CBCB\right) \\
&\quad \cdot A\left(\boldsymbol{u}_{ijk}^{n-1}\right) \\
&= A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + \left(A^{\dagger}CBCAA^{\dagger}A + A^{\dagger}CBCBA\right) \\
&\quad \cdot \left(\boldsymbol{u}_{ijk}^{n-1}\right) \qquad (44)
\end{aligned}
$$

and again, using $AA^{\dagger}A = A$ and $BA = 0$, we get

$$
\boldsymbol{u}_{ijk}^{n+1} = A^{\dagger}CA\left(\boldsymbol{u}_{ijk}^{n}\right) + A^{\dagger}CBCA\left(\boldsymbol{u}_{ijk}^{n-1}\right) \qquad (45)
$$

in which, now, the effect of the B matrix is evident. This constitutes the equivalent finite difference two-time step scheme for the generalized TLM methods.

The dispersion/dissipation effects of the derived schemes have been investigated by applying the to a propagating Gaussian pulse plane wave (see Fig. 2). The plane wave is assumed to propagate axially through the mesh, and was generated using the TLM total/scattered field formulation discussed in [5]. Notice that, except for the case where $\alpha = 1/4$ (standard TLM), the schemes exhibit dissipation. This also follows from the energy conservation used in deriving the $1/4$ scheme.

V. Conclusions

A Series of TLM-type algorithms have been derived directly from Maxwell's equations using the approximation that disturbances within a computational cell, given by the Riemann invariants, travel as a series of plane waves in the orthogonal directions. This analysis derives not only one but an infinite number of TLM-type methods, based on a parameter α, with one being equivalent to the symmetrical condensed node method ($\alpha = 1/4$). The use of the principle of conservation of power density on the scattering matrices imposes the value of $1/4$ on α. A general formulation of the equivalent finite-difference schemes for these methods has been given, and all but one ($\alpha = 0$) correspond to a two-time level scheme. Each of these methods gives different dispersion and dissipation characteristics which have been investigated numerically.

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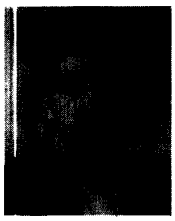

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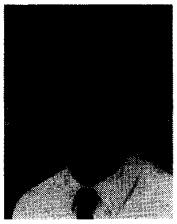

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