# On the Statistics of a Sum of Harmonic Waveforms

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***Abstract*—In this paper, we address certain aspects of the problem of statistically characterizing the electromagnetic field inside an enclosure. The field that we are interested in describing is time-harmonic and a three-dimensional spatial vector; therefore, two random variables are required for each vector component at each location in the enclosure. We could describe either the magnitude (or intensity) and phase, or the real (in-phase) and imaginary (quadrature) parts, of each spatial component. It is the relationship between these two modes of description that is addressed in this paper. We show that this relationship is given by the Blanc–Lapierre transform and when there is a sum of more than one time-harmonic field, by equations first derived by Kluyver. The relationships are derived for any form of distribution taken on by any of the random variable. We also address issues related to the approximation of the probability density function (pdf) of the amplitude of an electromagnetic field given a known pdf of the intensity of this field. The work presented herein fills in some of the gaps which were left in some recent literature wherein the independence of the variables to each other was assumed, that is, the independence of the in-phase to the quadrature variables.**

## I. Introduction

In a number of publications, an attempt has been made to statistically characterize the electromagnetic field inside a leaky enclosure [1]–[10]. Experimentally, the distribution of the intensity of the field can be measured directly and a few statistical models have been suggested. However, in numerical simulations, the probability distribution of the amplitude of the field is required. To the best of our knowledge, the papers addressing this issue start from the assumption that the in-phase and quadrature components are independent; however, this is true only in the particular case of a Gaussian distribution for the amplitude of the field (or, equivalently, a $\chi^2$ distribution of the intensity). In this paper, we provide the proper relation between the distributions of the in-phase and quadrature components and the distributions of the magnitude and intensity of the field. While an exact integral equation is given, a few approximations can be useful when the integrals cannot be calculated analytically. We provide two kinds of approximation: one allows us to calculate the moments of the unknown distribution through those of known distributions, and the second represents the unknown probability density function (pdf) in the form of a generalized Fourier series (i.e., Gram–Charlier expansions).

A traditional approach to statistically modeling the electromagnetic field inside a complex enclosure is to assume that, at any particular point, the field is a superposition of a large number of independent harmonic oscillators. These can be thought of as arising from the excited modes (i.e., eigenfunctions) of the enclosure [1], [2] or from a ray theory approach, as the multiple reflections of rays from the walls of the enclosure. Thus, we assume that at any particular location inside the enclosure each component of electric (or magnetic) field can be represented as

$$E(t) = \exp(\omega_0 t) \sum_{n=1}^{N} a_n e^{j\varphi_n}. \tag{1}$$

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It follows from the Large Number Theorem that, if all the magnitudes have the same distribution and their number approaches infinity, then the distribution of $E(t)$ approaches a Gaussian. We adopt the same model here and show how the pdf of the intensity is related to the pdf of the magnitude of the individual components. A number of practically important examples are considered.

## II. Blanc–Lapierre Transformation and Its Extension to the Case of More Than One Summand

Let us consider the following pair of random processes (in-phase and quadrature components):

$$\begin{aligned} \xi_R(t) &= A(t)\cos[\omega_0 t + \phi(t)] = A(t)\cos\Psi(t) \\ \xi_I(t) &= A(t)\sin[\omega_0 t + \phi(t)] = A(t)\sin\Psi(t) \end{aligned} \tag{2}$$

where $A(t)$ is a slowly varying random function with respect to $\cos(\omega_0 t)$ and $\phi(t)$ is a slowly varying random function with respect to $\omega_0 t$. The random process $A(t)$ is called the envelope (magnitude) of $\xi_R(t)$ and can be described by its pdf $p_A(a)$. The total phase, $\Psi(t)$, can be described by its pdf $p_\Psi(\psi)$. These can also be described by the joint pdf $p_{A,\Psi}(a, \psi)$. The process $\xi_R(t)$ can be described by its pdf $p_{\xi_R}(x)$. It can be shown (see [15] for a detailed derivation) that

$$\begin{aligned} p_{\xi_R}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_{\xi_R}(u)\exp[-jxu]\,du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} J_0(ua)p_A(a)\exp[-jxu]\,da\,du \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{p_A(a)}{a} &= \int_0^{\infty} J_0(ua)\Theta_{\xi_R}(u)u\,du \\ &= \int_0^{\infty} uJ_0(ua)\int_{-\infty}^{\infty} p_{\xi_R}(x)\exp[jxu]\,dx\,du. \end{aligned} \tag{4}$$

Here

$$\theta_{\xi_R}(u) = \int_{-\infty}^{\infty} p_{\xi_R}(x)\exp[jxu]\,dx \tag{5}$$

is the characteristic function of the random process $\xi_R(t)$, defined as the Fourier transform of its pdf [11]. The transform-pair given by (3) and (4) is known as the Blanc–Lapierre transform [15]. It is worth noting that, unlike what was done in [1]–[8], (3) and (4) are derived here without the assumption that the in-phase $\xi_R(t)$ and quadrature $\xi_I(t)$ components are statistically independent. Despite the difference in the equations, both methods produce the same result in the case of Gaussian random process $\xi_R(t)$. However, they differ in all other cases.

It also follows from the derivation of (4) that the joint distribution $p_{\xi_R,\xi_I}(x_1, x_2)$ of in-phase and quadrature components is given by

$$p_{\xi_R,\xi_I}(x_1, x_2) = \frac{1}{2\pi}\,\frac{p_A(a)}{a} = \frac{1}{2\pi}\,\frac{p_A\left(\sqrt{x_1^2+x_2^2}\right)}{\sqrt{x_1^2+x_2^2}}. \tag{6}$$

In the case of a Rayleigh distribution for the magnitude (or, equivalently, $\chi^2$ distribution for the intensity), $p_{\xi_R,\xi_I}(x_1, x_2)$ is just a product of two Gaussian distributions, i.e., the in-phase and quadrature components are independent. However, this is not the case in general. Furthermore, it is shown in [15] that the Gaussian case is the only one where the components are, in fact, independent. It is interesting that the authors of [3]–[6] obtain results, which are apparently in good agreement with experiment, assuming the independence of the components. In our opinion, this may be explained by the fact that, although they may indeed be dependent, the components are uncorrelated[1] —a condition which is somewhat weaker than independence. However,

[1]This follows from the even symmetry of the joint pdf (6).

the difference does appear in the high-order moments and may not be important for the specific case considered by those authors.

The Blanc–Lapierre transform can be extended to the case of a sum of $N$ complex processes. Indeed, let a random complex process $Z$ be a sum of $N$ independent stationary random processes $Z_n$, $n = 1, 2, \ldots, N$

$$Z = X + jY = \sum_{n=1}^{N} Z_n = \sum_{n=1}^{N} X_n + j \sum_{n=1}^{N} Y_n \tag{7}$$

where $X_n = \mathrm{Re}(Z_n), Y_n = \mathrm{Im}(Z_n)$. Let us assume that we know the joint pdf $p_{X_n, Y_N}(x_n, y_n)$ for each individual $Z_n$. This can be recast in terms of a joint pdf of the magnitude $A_n$ and the phase $\Psi_n$ as

$$\begin{aligned} Z_n &= A_n \exp(j\Psi_n), \\ A_n &= \sqrt{X_n^2 + Y_n^2}, \\ \Psi_n &= \operatorname{atan} \frac{Y_n}{X_n} \end{aligned} \tag{8}$$

$$\begin{aligned} P_{A_n, \Psi_n}(a_n, \varphi_n) &= a_n p_{X_n, Y_N}(a_n \cos\varphi_n, a_n \sin\varphi_n) \\ &= \frac{1}{2\pi} p_{A_n}(a_n). \end{aligned} \tag{9}$$

The process $Z$ itself can also be described by the joint pdf of its real and imaginary part $p_Z(x, y)$ or, equivalently, by its characteristic function

$$\Theta_Z(u, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_Z(x, y) \exp[j(ux + \nu y)]\, dx\, dy. \tag{10}$$

Since $Z$ is a sum of $N$ independent random variables its characteristic function is a product of the characteristic functions of each single summand,[2] [11] i.e.,

$$\begin{aligned} \Theta_X(u) &= \prod_{n=1}^{N} \Theta_{X_n}(u) \\ &= \prod_{n=1}^{N} \int_0^{\infty} J_0(ua_n) p_{A_n}(a_n)\, da_n \\ &= \prod_{n=1}^{N} \langle J_0(ua_n) \rangle \end{aligned} \tag{11}$$

where

$$\Theta_{X_n}(u) = \int_0^{\infty} J_0(ua_n) p_{A_n}(a_n)\, da_n \tag{12}$$

is the characteristic function of the $n$th component of the sum (1) and the expectation $\langle \bullet \rangle$ pertains to the random variable $A$. In turn, according to the Blanc–Lapierre transformation (4), $p_A(a)$ can be expressed in terms of $\Theta_X(u)$ as

$$\begin{aligned} p_A(a) &= a \int_0^{\infty} J_0(ua) \Theta_X(u) u\, du \\ &= a \int_0^{\infty} J_0(ua) \prod_{n=1}^{N} \langle J_0(ua_n) \rangle u\, du \\ &= a \int_0^{\infty} J_0(ua) \left[ \prod_{n=1}^{N} \int_0^{\infty} J_0(ua_n) p_{A_n}(a_n)\, da_n \right] u\, du. \end{aligned} \tag{13}$$

This equation relates the distribution of the magnitude of the sum through distributions of the magnitude of each individual component.

[2]This is equivalent to the fact that PDF of $X$ is the convolution of PDF of each $X_n$ [11].

In practice, it is the value of magnitude squared, $I = A^2$, which is available for measurements. In this case, the pdf of the intensity $I$ can be easily found to be

$$\begin{aligned} p_I(i) &= \frac{p_A\left(\sqrt{i}\right)}{2\sqrt{i}} \\ &= \frac{\sqrt{i} \int_0^{\infty} J_0\left(u\sqrt{i}\right) \left[ \prod_{n=1}^{N} \int_0^{\infty} J_0(ua_n) p_{A_n}(a_n)\, da_n \right] u\, du}{2\sqrt{i}} \\ &= \frac{1}{2} \int_0^{\infty} J_0\left(u\sqrt{i}\right) \prod_{n=1}^{N} \langle J_0(ua_n) \rangle u\, du. \end{aligned} \tag{14}$$

The last equation was originally derived in [16].

## III. Relations Between Moments of $p_{A_n}(a_n)$ and $p_I(i)$

Equation (14) can rarely be used to produce analytical results. However, a useful relation between the moments of the sum and those of its components can be analytically obtained. We start by relating the moments $m_{Ik}$ of a distribution to the Laplace transform of its pdf

$$\begin{aligned} \Phi(\lambda) &= \int_0^{\infty} e^{-\lambda i} p_I(i)\, di \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(-\lambda i)^k}{k!} p_I(i)\, di \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_0^{\infty} i^k p_I(i)\, di \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} m_{Ik}. \end{aligned} \tag{15}$$

Another Taylor-type expansion for $\Phi(\lambda)$ can be obtained directly from (14), first noting that

$$\begin{aligned} \Phi(\lambda) &= \int_0^{\infty} e^{-\lambda i} p_I(i)\, di \\ &= \int_0^{\infty} e^{-\lambda i} \tfrac{1}{2} \int_0^{\infty} J_0\left(u\sqrt{i}\right) \prod_{n=1}^{N} \langle J_0(ua_n) \rangle u\, du\, di \\ &= \int_0^{\infty} \prod_{n=1}^{N} \langle J_0(ua_n) \rangle u\, du \tfrac{1}{2} \int_0^{\infty} e^{-\lambda i} J_0\left(u\sqrt{i}\right) di \\ &= \frac{1}{2} \int_0^{\infty} \prod_{n=1}^{N} \langle J_0(ua_n) \rangle u\, du\, \frac{1}{\lambda} \exp\left(-\frac{u^2}{4\lambda}\right). \end{aligned} \tag{16}$$

Furthermore, using the following substitution of variables in (16)

$$x = \frac{u^2}{4\lambda}, \quad u = \sqrt{4\lambda x}, \quad du = \sqrt{\lambda} \frac{dx}{\sqrt{x}} \tag{17}$$

so that (16) becomes

$$\Phi(\lambda) = \frac{1}{2} \int_0^{\infty} \prod_{n=1}^{N} \left\langle J_0\left(\sqrt{4\lambda x} a_n\right) \right\rangle \exp(-x)\, dx \tag{18}$$

and expanding $J_0(\sqrt{4\lambda x} a_n)$ into the Taylor series with respect to variable $\lambda$, one can obtain using [12, eq. 9.1.10],

$$\Phi(\lambda) = \sum_{m=0}^{\infty} (-1)^m c_m \lambda^m m! \tag{19}$$

where

$$c_m = \sum_{k_N=0}^{\infty} \sum_{k_{N-1}=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \cdot \left. \frac{\langle a_{N-1}^{2k_N} \rangle \langle a_{N-1}^{2k_{N-1}} \rangle \cdots \langle a_1^{2k_1} \rangle}{k_N! k_N! k_{N-1}! k_{N-1}! \cdots k_1! k_1!} \right|_{\sum_{n=1}^{N} k_n = m}. \quad (20)$$

Comparing this to (15), we see that

$$\begin{aligned} m_{kI} &= k!k!c_k \\ &= k!k! \sum_{k_N=0}^{\infty} \sum_{k_{N-1}=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \\ &\quad \cdot \left. \frac{\langle a_{N-1}^{2k_N} \rangle \langle a_{N-1}^{2k_{N-1}} \rangle \cdots \langle a_1^{2k_1} \rangle}{k_N! k_N! k_{N-1}! k_{N-1}! \cdots k_1! k_1!} \right|_{\sum_{n=1}^{N} k_n = k} \end{aligned} \quad (21)$$

which allows us to express the moments of the intensity of the sum through the moments of the magnitude of individual components.

## IV. Gram–Charlier Series for $p_{\xi_R}(x)$ and $p_I(i)$

As has been mentioned above, it is the distribution of the intensity $p_I(i)$ which is usually available for measurements. At the same time, it is important for numerical simulations of the coupling field to know and reproduce the statistic of the amplitude $p_{\xi_R}(x)$. The exact analytical form, given by (3)–(6), is rarely achievable (however, a number of results can be found in [15]). Important relations like (21) give us insight into how the distribution behaves but does not really allow its numerical simulation. In this section, we derive some results allowing the reconstruction of the pdf $p_{\xi_R}(x)$ from the pdf of the intensity $p_I(i)$.

As the first step, let us rewrite the Blanc–Lapierre transformation (3) in terms of $p_I(i)$, using the fact that

$$p_I(i) = \frac{p_A\left(\sqrt{i}\right)}{2\sqrt{i}} \Rightarrow p_A\left(\sqrt{i}\right) = 2\sqrt{i}\, p_I(i) \quad (22)$$

and thus

$$\begin{aligned} p_{\xi_R}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} J_0(ua) p_A(a)\, da \exp[-jxu]\, du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} J_0\left(u\sqrt{i}\right) p_A\left(\sqrt{i}\right) \\ &\quad \cdot \exp[-jxu]\, du\, d\sqrt{i} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} J_0\left(u\sqrt{i}\right) 2\sqrt{i}\, p_I(i) \frac{di}{2\sqrt{i}} \\ &\quad \cdot \exp[-jxu]\, du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-jxu]\, du \int_0^{\infty} J_0\left(u\sqrt{i}\right) p_I(i)\, di. \end{aligned} \quad (23)$$

Now let the pdf $p_I(i)$ be represented by its expansion through the Laguerre polynomials $L_k(\beta i)$, i.e.,

$$p_I(i) = \beta \exp[-\beta i] \sum_{k=0}^{\infty} \alpha_k L_k(\beta i) \quad (24)$$

where

$$\alpha_k = \int_0^{\infty} p_I(i) L_k(\beta i)\, di \quad (25)$$

and $1/\beta$ is a mean value of $I$. In this case, (23) becomes

$$\begin{aligned} p_{\xi_R}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-jxu]\, du \int_0^{\infty} J_0\left(u\sqrt{i}\right) \beta \\ &\quad \cdot \exp[-\beta i] \sum_{k=0}^{\infty} \alpha_k L_k(\beta i)\, di \\ &= \frac{\beta}{2\pi} \sum_{k=0}^{\infty} \alpha_k \int_{-\infty}^{\infty} \exp[-jxu]\, du \\ &\quad \cdot \int_0^{\infty} J_0\left(u\sqrt{i}\right) L_k(\beta i) \exp[-\beta i]\, di \\ &= \frac{\beta}{2\pi} \sum_{k=0}^{\infty} \alpha_k \int_{-\infty}^{\infty} F_k(u) \exp[-jxu]\, du \end{aligned} \quad (26)$$

where

$$F_k(u) = \int_0^{\infty} J_0\left(u\sqrt{i}\right) L_k(\beta i) \exp[-\beta i]\, di. \quad (27)$$

Using [14, p. 847, integral 7.421.2] one can obtain

$$F_k(u) = \frac{1}{k!\beta^{k+1}} \left(\frac{u}{2}\right)^{2k} \exp\left(-\frac{u^2}{4\beta}\right). \quad (28)$$

Rewriting (26) by taking into account (28), one can obtain

$$\begin{aligned} p_{\xi_R}(x) &= \frac{\beta}{2\pi} \sum_{k=0}^{\infty} \alpha_k \int_{-\infty}^{\infty} F_k(u) \exp[-jxu]\, du \\ &= \frac{\beta}{2\pi} \sum_{k=0}^{\infty} \alpha_k \int_{-\infty}^{\infty} \frac{1}{k!\beta^{k+1}} \left(\frac{u}{2}\right)^{2k} \\ &\quad \cdot \exp\left(-\frac{u^2}{4\beta}\right) \exp[-jxu]\, du \\ &= \frac{\sqrt{\beta}}{\pi} \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} \int_{-\infty}^{\infty} t^{2k} \\ &\quad \cdot \exp(-t^2) \exp\left[-j\left(2\sqrt{\beta}\, x\right) t\right]\, dt \\ &= \frac{\sqrt{\beta}}{\pi} \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} G_k(x) \end{aligned} \quad (29)$$

where we use the notation

$$\begin{aligned} G_k(x) &= \int_{-\infty}^{\infty} t^{2k} \exp(-t^2) \exp\left[-j\left(2\sqrt{\beta}\, x\right) t\right]\, dt \\ &= \exp[-\beta x^2] \int_{-\infty}^{\infty} t^{2k} \exp\left[-\left(t + j\sqrt{\beta}\, x\right)^2\right]\, dt. \end{aligned} \quad (30)$$

Comparing the standard integral 3.462-4 in [14, p. 338]

$$\int_{-\infty}^{\infty} x^n \exp[-(x-\beta)^2]\, dx = (2j)^{-n} \sqrt{\pi}\, H_n(j\beta) \quad (31)$$

one can obtain

$$\begin{aligned} G_k(x) &= \exp[-\beta x^2] \int_{-\infty}^{\infty} t^{2k} \exp\left[-\left(t + j\sqrt{\beta}\, x\right)^2\right]\, dt \\ &= 2^{-2k}(-1)^k \exp[-\beta x^2] \sqrt{\pi}\, H_{2k}\left(\sqrt{\beta}\, x\right) \end{aligned} \quad (32)$$

and thus

$$\begin{aligned} p_{\xi_R}(x) &= \frac{\sqrt{\beta}}{\pi} \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} G_k(x) \\ &= \frac{\sqrt{\beta}}{\sqrt{\pi}} \exp[-\beta x^2] \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_k}{2^{2k} k!} H_{2k}\left(\sqrt{\beta}\, x\right). \end{aligned} \quad (33)$$

Here $H_n(x)$ is the Hermitian polynomial of order $n$ [12]. The last expression should not come as a surprise to us. If $p_I(i)$ is an exponential distribution ($\chi^2$ of two degrees of freedom), then $\alpha_0 = 1$ and $\alpha_k = 0$ for $k > 0$. This reduces (33) to the Gaussian pdf as it must. At the same time, (33) is nothing but the Gram–Charlier expansion of the pdf $p_{\xi_R}(x)$. The odd order terms are missing due to the fact that $p_{\xi_R}(x)$ is a symmetrical pdf.

Although analytical expressions have been provided herein for some relatively complicated probability distributions, the detailed discussion of questions related to numerical generation of such non-Gaussian random processes is out of the scope of this paper. Gaussian-correlated random processes can be generated relatively easily by the linear filtering of white Gaussian noise (WGN) [18]–[23]. This approach has been used in [3]–[9] to couple Gaussian correlated random electromagnetic fields, which were postulated as a good model for the fields existing in complex enclosures, to transmission lines. In [6], the authors introduce a complex approximate procedure in order to simulate the more demanding Lehman distribution.

In contrast to these efforts, some recently suggested techniques [24]–[26] allow us to obtain a random process with any probability density function and exponential correlation. This can be achieved by nonlinear filtering of WNG. A useful algorithm based on the Markov chain approximation of continuous random processes can be found in [27]. It is worthwhile to note here that, using the approach suggested in [27], one can generate a random envelope (magnitude) $A(t)$ of the random electromagnetic field as well as its uniformly distributed (and correlated) phase $\phi(t)$. The field $E(t)$ can then be obtained as

$$E(t) = A(t)\exp(j\phi(t)). \tag{34}$$

## V. Examples

### A. $K$ Distribution

The $K$ distribution

$$p_A(a) = \frac{2b}{\Gamma(\nu)}\left(\frac{ba}{2}\right)^{\nu} K_{\nu-1}(ba), \qquad \nu > 0, \quad x \ge 0 \tag{35}$$

which describes the distribution of the envelope $A(t)$ is a rare case when the Blanc–Lapierre transformation can be calculated analytically. The corresponding distribution of the intensity is given by

$$p_I(i) = \frac{b^2}{2\Gamma(\nu)}\left(\frac{b\sqrt{i}}{2}\right)^{\nu-1} K_{\nu-1}\left(b\sqrt{i}\right). \tag{36}$$

Let us first find the characteristic function of the pdf (35) using the Bessel transform [15]

$$\begin{aligned}\theta_{\xi_R}(s) &= \int_0^\infty p_A(a)J_0(sa)\,da \\ &= \frac{b^{\nu+1}}{2^{\nu-1}\Gamma(\nu)}\int_0^\infty A^\nu K_{\nu-1}(ba)J_0(sa)\,da \\ &= \frac{b^{2\nu}}{(s^2+b^2)^\nu}.\end{aligned} \tag{37}$$

The pdf $p_{\xi_R}(x)$ itself can be found by using an inverse Fourier transform of its characteristic function (37) [11]

$$\begin{aligned}p_{\xi_R}(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \theta_{\xi_R}(s)\exp(-isx)\,ds \\ &= \frac{b^\nu}{2\pi}\int_{-\infty}^\infty \frac{\exp(-isx)}{(s^2+b^2)^\nu}\,ds \\ &= \frac{2^{-\nu+1/2}b}{\Gamma(\nu)\sqrt{\pi}}\,(b|x|)^{\nu-1/2}K_{\nu-1/2}(b|x|).\end{aligned} \tag{38}$$

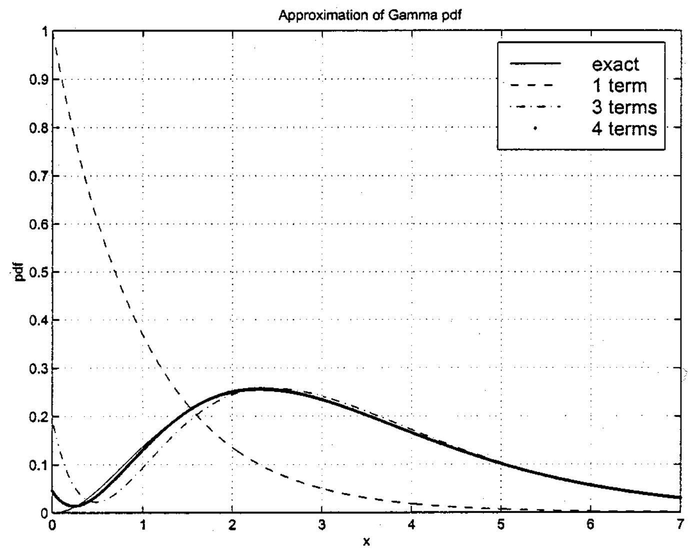


Fig. 1. Approximation of the Gamma distribution by its Gram–Charlier series. The approximation using four terms (dotted line) almost completely coincides with the exact pdf (solid line).

Thus, the distribution of instantaneous values corresponding to the $K$ distribution of the magnitude is again expressed in terms of the modified Bessel function $K$.

### B. Gamma Distribution

Letting the pdf of the intensity be the Gamma distribution

$$p_I(i) = \frac{\beta^{\alpha+1}i^\alpha\exp(-\beta i)}{\Gamma(\alpha+1)} \tag{39}$$

the coefficients $\alpha_k$ of the expansion (24) can be found as

$$\begin{aligned}\alpha_k &= \int_0^\infty p_I(i)L_k(\beta i)\,di \\ &= \int_0^\infty \frac{\beta^{\alpha+1}i^\alpha\exp(-\beta i)}{\Gamma(\alpha+1)}L_k(\beta i)\,di = \frac{\Gamma(k-\alpha)}{k!\Gamma(-\alpha)}.\end{aligned} \tag{40}$$

Substituting (40) into (33) results in the following expression for the pdf of the amplitude of the distribution:

$$\begin{aligned}P_{\xi_R}(x) &= \frac{\sqrt{\beta}}{\sqrt{\pi}}\exp[-\beta x^2] \\ &\quad \cdot\sum_{k=0}^\infty \frac{(-1)^k}{2^{2k}k!}\,\frac{\Gamma(k-\alpha)}{k!\Gamma(-\alpha)}\,H_{2k}\left(\sqrt{\beta}x\right).\end{aligned} \tag{41}$$

A few examples showing the accuracy of the approximation of the Gamma pdf can be found in Fig. 1. Since the coefficients of the expansion (41) decay as fast as $\alpha_k$, one can expect that the approximation (41) converges as fast to the exact pdf $p_{\xi_R}(x)$ as the corresponding approximation of $p_i(I)$ converges to the exact Gamma pdf.

### C. Log-Normal Distribution

The log-normal distribution is also frequently used in modeling random electromagnetic fields [7], [8]. In this case, we have

$$p_I(i) = \frac{1}{i\sigma\sqrt{2\pi}}\exp\left[-\frac{(\ln i - \alpha)^2}{2\sigma^2}\right] \tag{42}$$

with the moments, given by

$$m_{Ln} = \exp\left(an + \frac{n^2\sigma^2}{2}\right). \tag{43}$$

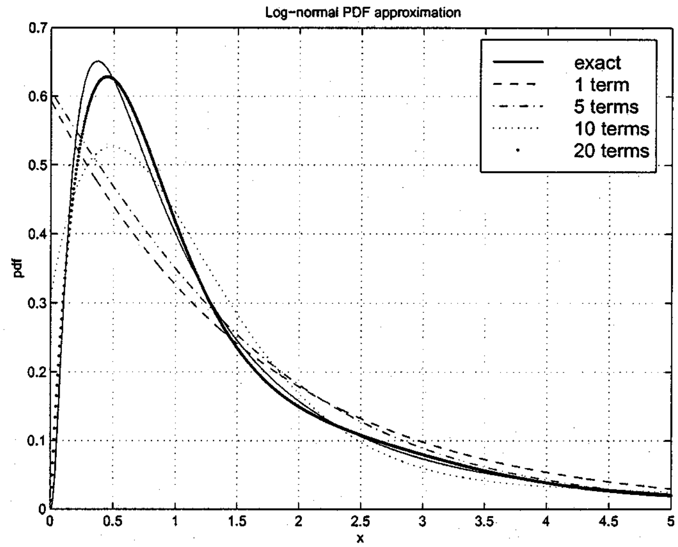


Fig. 2. Approximation of log-normal PDF by its Gram–Charlier series.

It is impossible to obtain an expression similar to (40) in this case. However, the fact that the moments of this distribution are analytically known can be used as follows.

Let us rewrite $L_k(\beta i)$ as a power series of $(\beta i)$

$$L_k(\beta i) = \sum_{l=0}^{k} a_l^{(k)} (\beta i)^l \tag{44}$$

where the coefficients $a_l^{(k)}$ can be recursively calculated using [12, p. 775, eq. 22.3.9]

$$\alpha_l^{(k)} = (-1)^l \binom{k}{k-l} \frac{1}{l!} = (-1)^l \frac{k!}{(k-l)!l!} \frac{1}{l!}. \tag{45}$$

Plugging (44) into (25), one can obtain a recursive algorithm for the calculation of coefficients in expansion (33) as

$$\begin{aligned} \alpha_k &= \int_0^\infty p_I(i) \left( \sum_{l=0}^{k} a_l^{(k)} (\beta i)^l \right) dI = \sum_{l=0}^{k} a_l^{(k)} \beta^l m_l \\ &= \sum_{l=0}^{k} a_l^{(k)} \beta^l \exp\left( al + \frac{l^2\sigma^2}{2} \right). \end{aligned} \tag{46}$$

Substituting (46) into (33) thus leads to the final expression for $p_{\xi_R}(x)$ in the form

$$\begin{aligned} p_{\xi_R}(x) = \frac{\sqrt{\beta}}{\sqrt{\pi}} \exp[-\beta x^2] \sum_{k=0}^{\infty} \frac{1}{2^{2k}} H_{2k}\left(\sqrt{\beta} x\right) \\ \cdot \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k-l)!l!l!} \beta^l \exp\left( al + \frac{l^2\sigma^2}{2} \right). \end{aligned} \tag{47}$$

A few examples showing the accuracy of the approximation of the log-normal PDF can be found in Fig. 2.

## VI. Conclusion

In this paper, we have derived relations between the distributions of the amplitude $\xi_R(t)$, magnitude $A(t)$ and intensity $I(t)$ of a random electromagnetic field. We have also provided a detailed derivation of the relations between the moments of the intensity of an electric field which is made up of a sum of many harmonic components, and the moments of the magnitude of the individual plane-wave components themselves. Convenient expansions have also been derived which allow us to calculate the distribution of the amplitude of the electromagnetic field from the distribution of its intensity in the case when the integrals in Blank–Lapierre transformation cannot be analytically calculated.

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