Linear Programming

- Linear programming is the simplest form of constrained optimization because the objective function is linear.

- This implies that the minimum must lie on the boundary of the feasible region.

- We can specify a linear programming problem in so-called ***normal form*** as:

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & f = c^T x \\
\text{subject to:} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the vector of decision variables,

$b \in \mathbb{R}^m$ is a vector of constants which define the objective function $f(x)$,

$A \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix with rank$(A) = m \leq n$.

- $x \geq 0$ constrains each of the decision variables to be positive;

- $Ax = b$ constrains x to one of the possibly infinite solutions to the linear system of equations.

- The x which satisfy both of these equations define the ***feasible region*** from which we must choose x in order to maximize $f(x)$.

- If $m = n$, then only ***one feasible solution*** $x = A^{-1}b$ exists.

- If one also has ***inequality constraints*** given by

$$a_i^T x \le b_i$$

  then one can introduce "*slack variables*" such that:

$$a_i^T x + x_{n+1} = b_i$$

$$x_{n+1} \ge 0$$

  This will add one more row to the matrix A, that is a_i^T

  one more element b to the vector $\boldsymbol{b}$, and

  one more decision variable x_{n+1} to the vector of decision variables $\boldsymbol{x}$.

- In this way we can always convert a linear program with inequality constraints into normal form by the addition of slack variables.

- Also, if we were interested in the

$$\textit{minimization of } f(\boldsymbol{x})$$

  instead, then we could

$$\textit{maximize } -f(\boldsymbol{x})$$

  That is, we would substitute $\boldsymbol{c}' = -\boldsymbol{c}$

Example:

Consider a factory in which there are three machines which we will denote as: M_1, M_2, M_3 and which are used to make two products, P_1, P_2. For each unit of P_1 made, machines M_1, M_2 and M_3 have to be run for 5, 3, and 4 minutes respectively. For each unit of P_2 the numbers are 1, 4, and 3 minutes. Every unit of P_1 produces a net profit of \$30.00 while every unit of P_2 produces \$20.00 of profit. We want to determine which production plan will give us the most profit?

Solution:

Suppose we produce x_1 units of P_1 and x_2 units of P_2 per hour.

Then the profit we want to maximize can be written as

$$\underset{x}{\text{maximize}} \ \ F = 30x_1 + 20x_2$$

The constraints are that we cannot use any particular machine for more than 60 minutes per hour, that is:

$$5x_1 + x_2 \le 60 \qquad \text{(M1)}$$

$$3x_1 + 4x_2 \le 60 \qquad \text{(M2)}$$

$$4x_1 + 3x_2 \le 60 \qquad \text{(M3)}$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$

We can now solve the problem graphically by first sketching the feasibility region in the (x_1, x_2) plane.

- The feasibility region is shown shaded in the figure.

- The level or contour lines for the objective functions are straight lines.

- The maximum value will be attained at one of the corners of the boundary of the feasible region.

- The maximum of the objective function is attained at point B where its value is $F = 436.36$. Found at the point where the two lines, labelled M1 and M3 intersect.

$$5x_1 + x_2 = 60$$
$$4x_1 + 3x_2 = 60$$

from which we get $x^* = (10.9, 5.5)$.
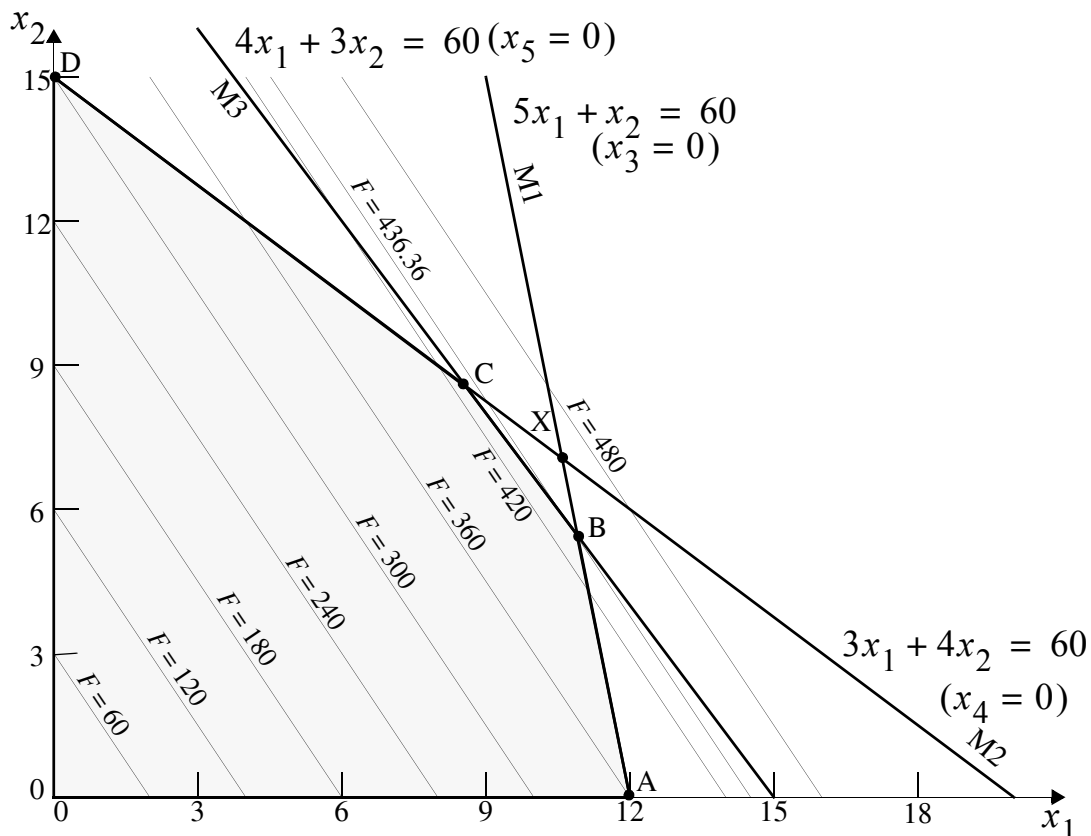
Graphical sketch of the problem.

- This same problem can be written in normal form by introducing three slack variables x_3, x_4, x_5 (*i.e.* one for every inequality constraint).

- Then we have:

$$\underset{x}{\text{maximize}} \ \ F = 30x_1 + 20x_2$$

subject to the new equality constrains:

$$\begin{cases} 5x_1 + x_2 + x_3 = 60 \\ 3x_1 + 4x_2 + x_4 = 60 \\ 4x_1 + 3x_2 + x_5 = 60 \end{cases}$$

$$x_i \geq 0, \ i = 1, \ ..., \ 5$$

- The three equality constraints define a 2-D subspace of the 5-D space of the problem (*i.e.* 5 - 3 = 2).

  Therefore, picking values for 2 of the variables, or coordinates, uniquely determines the remaining ones.

- Each side of the feasible region in the original problem with the inequality constraints has an equation of the form $x_i = 0$. Specifically:

$$x_1 = 0 \ \text{(left side)} \ x_2 = 0 \ \text{(bottom)}$$

$$x_3 = 0 \ (M_1) \qquad x_4 = 0 \ (M_2) \qquad x_5 = 0 \ (M_3)$$

- Since a vertex of the boundary of the feasible region is the intersection of two sides, therefore two of the variables are zero at a vertex point.

- The vertices in the figure are defined as follows:

$$\text{vertex 0: } x_1 = x_2 = 0$$

$$\text{vertex A: } x_2 = x_3 = 0$$

$$\text{vertex B: } x_3 = x_5 = 0$$

$$\text{vertex C: } x_4 = x_5 = 0$$

$$\text{vertex D: } x_1 = x_4 = 0$$

The vertex labeled X is defined by setting the two variables: $x_3 = x_4 = 0$ but it is not a feasible point.

- We can generalize these ideas by again considering a general linear programming problem written in normal form

$$\underset{x}{\text{maximize}} \quad f = c^T x, \quad x \in \mathbb{R}^n, c \in \mathbb{R}^n$$

$$\text{subject to:} \quad Ax = b, \quad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$x \geq 0$$

- The m equations, $A\boldsymbol{x} = \boldsymbol{b}$, define an $(n-m)$ dimensional subspace in the n dimensional space of the coordinates $\boldsymbol{x}$.

- This implies that $(n-m)$ coordinates may be chosen arbitrarily and these determine the remaining m coordinates.

- A point satisfying the equations $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$ is called a feasible point or *feasible vector*. The feasible vectors define a "***polyhedron***" in the $(n-m)$ dimensional space.

- If $(n-m)$ of the n coordinates of a feasible vector are zero, then the vector is called a **basic feasible vector**.
  (These correspond to the vertices of the $(n-m)$-dimensional polyhedron).

- If more than $(n-m)$ coordinates are zero at a feasible vector, then the vector is called a ***degenerate feasible vector***.

- The ***optimal feasible vector*** (optimal solution) is the feasible vector at which the function takes on the largest possible value in the feasible region.

  There may exist more than one optimal feasible vector; this is the case when the level lines of the function are parallel to one of the sides of the polyhedron. This implies that some optimal solution is not *basic* (*i.e.* not a vertex).

Theorem:

Some optimal feasible vector is also a basic feasible vector - *i.e.* at least $n - m$ of its coordinates are zero. (Geometrically, this means that an optimal feasible vector exists at one of the vertices of the polyhedron). The column vectors of the equation $A\boldsymbol{x} = \boldsymbol{b}$ corresponding to the non-zero coordinates are linearly independent. ❑

- Using this theorem, we can search for the optimal basic feasible vector by trying all the

$$\binom{n}{n-m} = \frac{n!}{(n-m)!m!}$$

combination of choosing $(n - m)$ coordinates out of n to be zero. This is very inefficient!

Simplex Method

- A more efficient method is to use what is called the ***simplex method***.

  It is an iterative search technique which, starting at one vertex, moves the search along the lines defining the polyhedron to another vertex only if the function increases at that new vertex. If no movement can be made, then the algorithm stops.

- We explain the method by example. Consider the previous example in normal form:

$$\underset{x}{\text{maximize}} \quad F = 30x_1 + 20x_2, \qquad x \in \mathbb{R}^5$$

$$\text{subject to:} \quad \begin{cases} 5x_1 + x_2 + x_3 = 60 \\ 3x_1 + 4x_2 + x_4 = 60 \\ 4x_1 + 3x_2 + x_5 = 60 \end{cases}$$

$$x_i \geq 0, \, i = 1, \ldots, 5$$

Step I: Look for a basic feasible vector at which to begin the search.

  We do this by choosing $(n - m)$ *right-hand variables* which will be given a value of zero and m *left-hand variables* whose value we will calculate using only the right-hand variables.

  We also express the objective function using only the right-hand variables.

In our example, we have:

$$x_3 = 60 - 5x_1 - x_2$$

$$x_4 = 60 - 3x_1 - 4x_2$$

$$x_5 = 60 - 4x_1 - 3x_2$$

$$f = 30x_1 + 20x_2$$

Here we have chosen $x_1 = x_2 = 0$ and we note that at this point $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$.

If these inequalities were not satisfied then we would have to choose a *another* vertex.

Here, $x_1 = x_2 = 0 \Rightarrow x_3 = x_4 = x_5 = 60 > 0$ which means that we have a feasible point.

Step II: Check the following *maximum criterion.*

If none of the coefficients in the expression of f as a function of the present right-hand variables is positive, then the maximum criterion is satisfied.

If any of the coefficients are positive, then increasing the right hand variable associated with that co-efficient increases $f(x)$ which means that the maximum has not been found.

If the coefficients are negative, this implies that the value of the function cannot be increased since the variables must remain positive, $x_i \geq 0$.

Step III: If maximum criterion is not fulfilled, determine the right hand variable which, when increased produces the greatest increase in $f(x)$ while keeping *all* the left hand variables ≥ 0.

For our example both right-hand variables x_1, x_2 have positive coefficients. Looking at the first three equations and varying x_1 and leaving $x_2 = 0$ we have:

$$x_3 \geq 0 \text{ if } \Delta x_1 \leq \frac{60}{5} = 12$$

$$x_4 \geq 0 \text{ if } \Delta x_1 \leq \frac{60}{3} = 20$$

$$x_5 \geq 0 \text{ if } \Delta x_1 \leq \frac{60}{4} = 15$$

Thus, we can not increase x_1 more than $\Delta x_1 = 12$. Varying x_2 while keeping $x_1 = 0$

$$x_3 \geq 0 \text{ if } \Delta x_2 \leq 60$$

$$x_4 \geq 0 \text{ if } \Delta x_2 \leq \frac{60}{4} = 15$$

$$x_5 \geq 0 \text{ if } \Delta x_2 \leq \frac{60}{3} = 20$$

therefore we cannot increase x_2 more than $\Delta x_2 = 15$.

- This procedure can be shortened as follows: identify the right-hand variable in the equality constraint equations which has a positive coefficient in $f(\boldsymbol{x})$ and is allowed to increase the least.

$$x_3 = 60 - 5x_1 + -x_2 \Rightarrow \Delta x_1 = \frac{60}{5} = 12 \qquad \Delta f = (30)(12) = 360$$

$$x_4 = 60 - 3x_1 - 4x_2 \Rightarrow \Delta x_2 = \frac{60}{4} = 15 \qquad \Delta f = (20)(15) = 300$$

$$x_5 = 60 - 4x_1 - 3x_2$$

$$f = 30x_1 + 20x_2$$

- Therefore, changing x_1 by $\Delta x_1 = 12$ will give $x_3 = 0$ and produce the biggest increase in f.

- This variable will now be the new right-hand variable which implies that we have moved to a new vertex.

Step IV: Exchange the right-hand variable, say x_R, which gave the greatest value for Δf with the left-hand variable in the constraint equation where x_R is identified.

Back to our example, we have

$$x_1 = \frac{60 - x_3 - x_2}{5}$$

which we substitute into the remaining equations:

$$x_1 = 12 + -\frac{1}{5}x_2 + -\frac{1}{5}x_3$$

$$x_4 = 60 - (3)(12) + \frac{3}{5}x_2 + \frac{3}{5}x_3 - 4x_2$$

$$x_5 = 60 - 4(12) + \frac{4}{5}x_2 + x_3 - 3x_2$$

$$f = 30\left(12 - \frac{1}{5}x_2 - \frac{1}{5}\right)x_3 + 20x_2$$

- Now repeat steps II, III, and IV until the maximum criterion is satisfied.

- Continuing with our example, we have

$$x_1 = 12 - \frac{1}{5}x_2 + -\frac{1}{5}x_3$$

$$x_4 = 24 - \frac{17}{5}x_2 + \frac{3}{5}x_3$$

$$x_5 = 12 - \frac{11}{5}x_2 + \frac{4}{5}x_3 \quad \Delta x_2 = \frac{60}{11} \quad \Delta f = \frac{(14)(60)}{11} = \frac{840}{11}$$

$$f = 360 + 14x_2 - 6x_3$$

Note, we only need to check x_2 since the coefficient of x_3 is negative. Now exchange x_5 with x_2:

$$x_2 = \frac{5}{11}\left(12 + \frac{4}{5}x_3 - x_5\right) = \frac{60 + 4x_3 - 5x_5}{11}$$

$$x_4 = 24 - \frac{17}{5}\left[\frac{60 + 4x_3 - 5x_5}{11}\right] + \frac{3}{5}x_3 = \frac{60 - 7x_3 + 17x_5}{11}$$

$$x_1 = \frac{120 - 3x_3 + x_5}{11}$$

$$f = \frac{4800 - 10x_3 - 70x_5}{11}$$

- Since none of the coefficients of the new right-hand variables in $f(x)$ are positive, this implies that the maximum criterion is fulfilled, which implies

$$f_{max} = \frac{4,800}{11}$$

$$x^* = \left(\frac{120}{11}, \frac{60}{11}, 0, \frac{60}{11}, 0\right)$$

which is the vertex labeled B in our figure.

Some Special Cases

Degenerate feasible vector

- Recall that at a degenerate feasible vector, more than $n - m$ of its coordinates are zero.

- Consider the following example:

$$\text{maximize } f = 2x_1 + 2x_2 + 3x_3$$

$$\text{subject to:} \quad x_1 + x_3 \le 1$$

$$x_2 + x_3 \le 1$$

$$x_i \ge 0 \qquad i = 1, 2, 3$$

- The feasible region is pyramid shown and is defined by 5 vertices labelled 0, A, B, C and D.

Feasible region is inside a pyramid in three dimensional space.

- We now introduce the slack variables x_4, x_5 which in normal form gives us 5 variables and 2 equality constraints.

  Therefore we have

  $$n = 5 \left.\right\} \\ m = 2 \left.\right\} n - m = 3$$

  degrees of freedom.

- The equality constraints become

  $$x_1 + x_3 + x_4 = 1$$

  $$x_2 + x_3 + x_5 = 1$$

  Therefore, setting $x_4 = 0$ implies that $x_1 + x_3 = 1$ which is the plane defined by the vertices ABC.

  Setting $x_5 = 0$ means that $x_2 + x_3 = 1$ and this represents the plane defined by the vertices ABD.

- The vertices are defined as follows:

  $$\text{at vertex 0: } x_1 = x_2 = x_3 = 0$$

  $$\text{at vertex A: } x_1 = x_2 = x_4 = x_5 = 0 \text{ (degenerate)}$$

  $$\text{at vertex B: } x_3 = x_4 = x_5 = 0$$

  $$\text{at vertex C: } x_2 = x_3 = x_4 = 0$$

  $$\text{at vertex D: } x_1 = x_3 = x_5 = 0$$

- Consider the initial basic feasible vector $x_1 = x_2 = x_3 = 0$ which implies the origin. If we choose these as the right-hand variables, we have

$$x_4 = 1 - x_1 - x_3 \quad \Delta x_1 = 1, \Delta f = 2 \quad \Delta x_3 = 1, \Delta f = 3$$

$$x_5 = 1 - x_2 - x_3 \quad \Delta x_2 = 1, \Delta f = 2 \quad \Delta x_3 = 1, \Delta f = 3$$

$$f = 2x_1 + 2x_2 + 3x_3$$

  Now x_3 gives us the largest increase in f but it can be interchanged with either x_4 or x_5 since either interchange allows us to increase f by the same amount.

- Say we choose to interchange x_3 with x_4. Then (x_1, x_2, x_4) will be the new right-hand variables which implies that we are at the vertex labelled A.

- Algebraically, we have

$$x_3 = 1 - x_1 - x_4$$

$$x_5 = 1 - x_2 - (1 - x_1 - x_4) = x_1 - x_2 + x_4 \quad \Delta x_2 = 0, \Delta f = 0$$

$$f = 2x_1 + 2x_2 + 3(1 - x_1 - x_4) = 3 - x_1 + 2x_2 - 3x_4$$

  in which only x_2 has a positive coefficient, but we are not allowed to increase x_2 at all.

- Therefore we exchange x_2 and x_5 and (x_1, x_4, x_5) become the new right-hand variables, but we are still at vertex A. This is because point A is a degenerate feasible vector: $(0, 0, 1, 0, 0)$.

- We now have:

$$x_2 = x_1 + x_4 - x_5$$

$$x_3 = 1 - x_1 - x_4 \quad \Delta x_1 = 1, \Delta f = 1$$

$$f = 3 - x_1 + 2(x_1 + x_4 - x_5) - 3x_4 = 3 + x_1 - x_4 - 2x_5$$

and we exchange x_1 and x_3 making (x_3, x_4, x_5) the new right-hand variables which implies that we are now at vertex **B**.

- This gives us the algebraic system:

$$x_1 = 1 - x_3 - x_4$$

$$x_2 = 1 - x_3 - x_4 + x_4 - x_5 = 1 - x_3 - x_5$$

$$f = 3 + 1 - x_3 - x_4 - x_4 - 2x_5 = 4 - x_3 - 2x_4 - 2x_5$$

and since all the coefficients are negative the maximum criterion is fulfilled.

- We see that the optimal solution is given by

$$x^* = (1, 1, 0, 0, 0)$$

for which we have $f_{max} = 4$.

- Thus, we have seen that once we reach a degenerate feasible vector, it may take more than one exchange of right-hand variables to left-hand variables in order to reach another basic feasible vector.

Difficulty in Finding Starting Points

- A second difficulty may arise when the origin is not feasible. Now we may not know how to find a basic feasible vector as a starting point. We demonstrate how to overcome this difficulty by example.

Example: Consider the following problem:
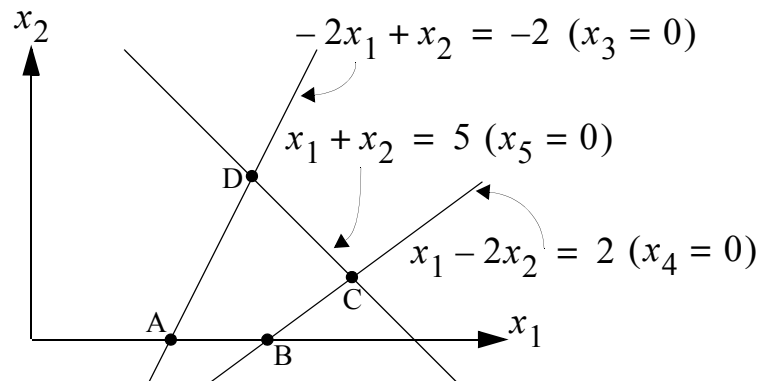
$$\text{maximize } f = x_1 - x_2$$

$$\text{subject to:} \quad -2x_1 + x_2 \leq -2$$

$$x_1 - 2x_2 \leq 2$$

$$x_1 + x_2 \leq 5$$

$$x_i \geq 0 \qquad i = 1, 2$$

The graph of this problem is shown below.

Feasible region is polygon enclosed by ABCD

- Introducing the slack variables x_3, x_4, x_5 we see that $n = 5$, $m = 3$ and therefore we have $n - m = 2$ degrees of freedom (*i.e.* 2 right-hand variables). In normal form the problem is written as:

$$f = x_1 - x_2$$

$$-2x_1 + x_2 + x_3 = -2$$

$$x_1 - 2x_2 + x_4 = 2$$

$$x_1 + x_2 + x_5 = 5$$

$$x_i \geq 0 \qquad (i = 1, 2, 3, 4, 5)$$

Now we notice that the origin $x_1 = 0$, $x_2 = 0$ is not feasible!

- A general technique to find a starting point is the introduction of a new *artificial variable*. Notice that if we tried to use the origin as the first basic feasible vector then we would get the system

$$x_3 = -2 + 2x_1 - x_2$$

$$x_4 = 2 - x_1 + 2x_2$$

$$x_5 = 5 - x_1 - x_2$$

and since the right-hand variables are set to zero, that is $x_1 = x_2 = 0$, we would have $x_3 = -2 < 0$ which violates the $x_i \geq 0$ constraint.

- We now introduce the artificial variable $x_6 \geq 0$ in the offending equation. The number of equations remains the same but the number of variables has increased by 1, therefore we can set $(n + 1) - m = (n - m) + 1$ variables equal to zero. We add the artificial variable to the equation

$$x_3 = -2 + 2x_1 - x_2 + x_6$$

  and move the new variable to the left

$$x_6 = 2 - 2x_1 + x_2 + x_3$$

- Now choosing (x_1, x_2, x_3) as the right-hand variables, we will have $x_6 \geq 0$ which is feasible. In the final solution we require that $x_6 = 0$ and this will imply that $-2 + 2x_1 - x_2 = x_3$. To ensure this we modify the objective function as:

$$\hat{f} = x_1 - x_2 - Mx_6$$

  where M is a very large positive number.

- Therefore, in order to maximize this, we require $x_6 \to 0$, and in terms of the right-hand variables:

$$x_6 = 2 - 2x_1 + x_2 + x_3 \; \Delta x_1 = 1, \Delta f = 1 + 2M$$

$$x_4 = 2 - x_1 + 2x_2$$

$$x_5 = 5 - x_1 - x_2$$

$$\hat{f} = x_1 - x_2 - M(2 - 2x_1 + x_2 + x_3) = (1 + 2M)x_1 - (1 + M)x_2 - Mx_3 - 2M$$

- Now we see that only the variable x_1 has a positive coefficient. We exchange x_1 and x_6 and get

$$x_1 = \frac{2 + x_2 + x_3 - x_6}{2}$$

$$x_4 = 2 - \left[1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_6\right] + 2x_2 = 1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_6$$

$$x_5 = 5 - \left[1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_6\right] - x_2 = 4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_6$$

$$\hat{f} = (1 + 2M)\left[1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_6\right] - (1 + M)x_2 - Mx_3 - 2M$$

$$= 1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_6 - Mx_6$$

- Now since x_6 has become the right hand variable, it can be set to zero and removed. The new right-hand variables are x_2 and x_3 which implies point A. We have

$$x_1 = 1 + \frac{x_2}{2} + \frac{x_3}{2}$$

$$x_4 = 1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 \quad \Delta x_3 = 2, \Delta f = 1$$

$$x_5 = 4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

$$f = 1 - \frac{1}{2}x_2 + \frac{1}{2}x_3$$

- We exchange x_3 and x_4: right hand variables x_2, x_4 which implies *Point B.*

$$x_3 = 2 + 3x_2 - 2x_4$$

$$x_1 = 1 + \frac{1}{2}x_2 + \frac{1}{2}[2 + 3x_2 - 2x_4] = 2 + 2x_2 - x_4$$

$$x_5 = 4 - \frac{3}{2}x_2 - \frac{1}{2}[2 + 3x_2 - 2x_4] = 3 - 3x_2 + x_4 \quad \Delta x_2 = 1, \Delta f = 1$$

$$f = 1 - \frac{1}{2}x_2 + \frac{1}{2}[2 + 3x_2 - 2x_4] = 2 + x_2 - x_4$$

- We exchange x_2 and x_5 and the right-hand variables become x_4, x_5 which implies point C.

$$x_2 = 1 - \frac{1}{3}x_4 - \frac{1}{3}x_5$$

$$x_3 = 2 + 3\left[1 - \frac{1}{3}x_4 - \frac{1}{3}x_5\right] - 2x_4 = 5 - 3x_4 - x_5$$

$$x_1 = 2 + 2\left[1 - \frac{1}{3}x_4 - \frac{1}{3}x_5\right] - x_4 = 4 - \frac{5}{3}x_4 - \frac{2}{3}x_5$$

$$f = 2 + \left[1 - \frac{1}{3}x_4 - \frac{1}{3}x_5\right] - x_4 = 3 - \frac{4}{3}x_4 - \frac{1}{3}x_5$$

- Since all the coefficients are negative, the maximum criterion is satisfied and we have

$$f_{max} = 3 \text{ at } (4, 1, 5, 0, 0)$$

which is point C.

Multiple Optimal Solutions:

- If the level lines of the objective function are parallel to one of the sides of the feasible region, then a multiple optimal solution will occur along that side. Of course, there is still only one value for f_{max} and f_{min}.

- This condition will manifest itself algebraically by the absence of one or more of the right-hand variables in the final form of the objective function.

Example: Optimal Utilization of a Communications Network

- It is quite remarkable to find such a large number of applications which can be formulated as a linear program.

- Consider a communications network in which we have:

  1) N stations in which messages are **sent**, **relayed**, and **received**;

  2) a set of *links* connecting the stations over which the messages travel; and

  3) a set of messages.

- The following notation and terminology is now used:

  T is the interval of time over which messages are sent, received and relayed.

  a_{ij} are the demand coefficients which represent the number of messages *originating* at station i per time T and destined for another station j.

  c_{ij} is the capacity of the direct link from station i to j during period T.

  s_i is the switching capacity at station i during the time interval T. (*i.e.* number of messages entering and leaving the ith node during T cannot exceed s_i.

  x_{ijk} is the number of messages sent from station i (not necessarily originating at i) over one link to another station j during T and destined for $k \neq i$.

$$x_{iik} = 0 \qquad \forall ik \text{ combinations}$$

$$x_{iji} = 0 \qquad \forall ij \text{ combinations}$$

  P_1 is the ratio of the number of delivered messages to the number of originating messages during T.

- The **optimal routing** or utilization problem tries to optimize P_1

Therefore we can write the objective function as

$$\underset{\boldsymbol{x}}{\text{maximize}} \ P_1 \ = \ \frac{\sum\limits_{i,j} x_{ijj}}{\sum\limits_{i,j} a_{ij}}$$

where x_{ijj} is the number of messages which have been delivered.

- We have an **inequality constraint** related to the demand for messages

$$\sum_{m} x_{imj} - \sum_{m} x_{mij} \leq a_{ij} \qquad \forall i,j \qquad i \neq j$$

where x_{imj} are the messages sent from i to m and destined for j and x_{mij} are the messages sent from m to i and destined for j.

The **first term** on the left of the inequality represents the total number of messages sent out from i and destined for j.

The **second term** represents the total number of messages coming into i destined for j from all other nodes.

- The total number of messages sent out from i and destined for j must be less than the total number of messages originating at i plus the messages using i as an intermediate node with j as the destination node.
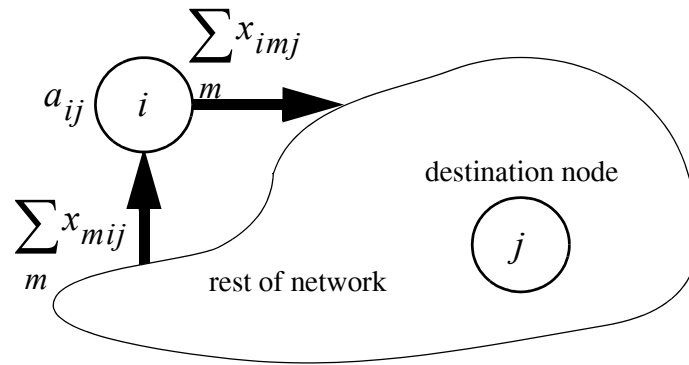
Demand limitation constraint.

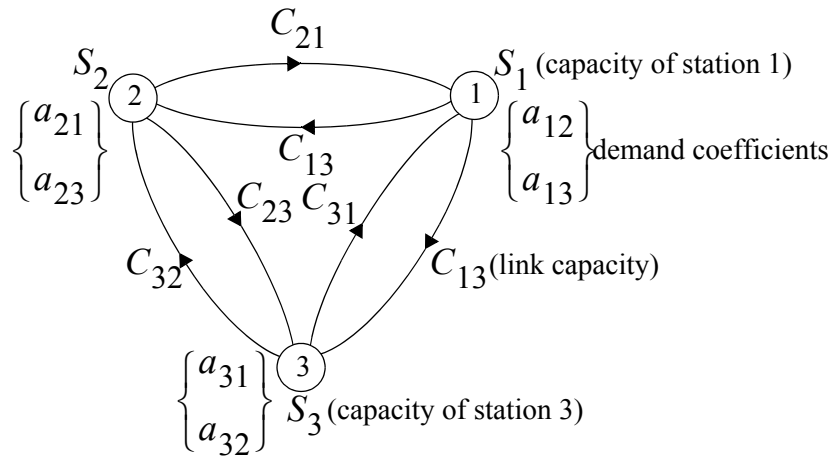- The next set of inequalities are related to link capacity:

$$\sum_k x_{ijk} \le c_{ij} \qquad \forall i, j \qquad i \ne j$$

- switching centre capacity for messages **leaving** node i:

$$\sum_{j,k} x_{ijk} \le s_i \qquad \forall i$$

- and switching centre capacity for messages **entering** node i:

$$\sum_{j,k} x_{kij} \le s_i \qquad \forall i.$$

Example network with three nodes.

- The linear program associated with this network can be written as

$$\underset{x}{\text{maximize}} \ \ P_1 \ = \ \frac{x_{122} + x_{133} + x_{211} + x_{233} + x_{311} + x_{322}}{a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}}$$

subject to the demand constraints:

$$x_{122} + x_{132} - x_{312} \leq a_{12}$$

$$x_{123} + x_{133} - x_{213} \leq a_{13}$$

$$x_{231} + x_{211} - x_{321} \leq a_{21}$$

$$x_{233} + x_{213} - x_{123} \leq a_{23}$$

$$x_{311} + x_{321} - x_{231} \leq a_{31}$$

$$x_{312} + x_{322} - x_{132} \leq a_{32}$$

- the link capacity constraints:

$$x_{122} + x_{123} \leq C_{12}$$

$$x_{132} + x_{133} \leq C_{13}$$

$$x_{211} + x_{213} \leq C_{21}$$

$$x_{231} + x_{233} \leq C_{23}$$

$$x_{311} + x_{312} \leq C_{31}$$

$$x_{321} + x_{322} \leq C_{32}$$

- switch capacity constraints for messages *leaving* each node

$$x_{122} + x_{123} + x_{132} + x_{133} \leq S_1$$

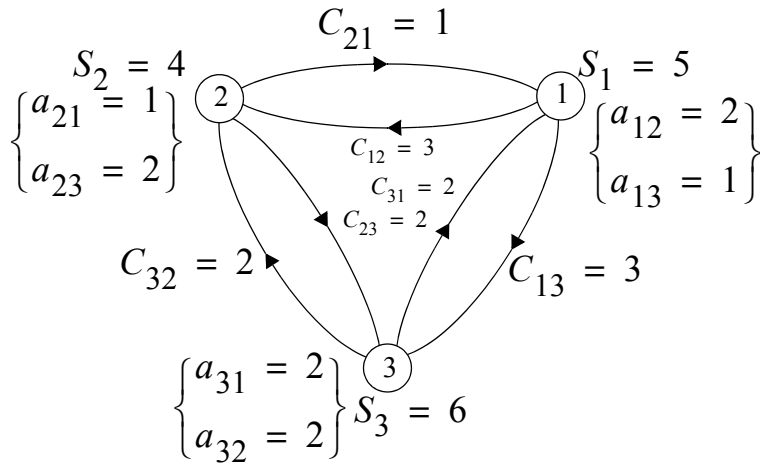$$x_{211} + x_{213} + x_{231} + x_{233} \leq S_2$$

$$x_{311} + x_{312} + x_{321} + x_{322} \leq S_3$$

- and switch capacity constraints for messages *entering* each node

$$x_{211} + x_{213} + x_{311} + x_{312} \leq S_1$$

$$x_{122} + x_{123} + x_{321} + x_{322} \leq S_2$$

- The message demands, a_{ij}, may change depending on the time period (business hours, night time, *etc.*) therefore, the decision on how to choose x_{ijk} so as to maximize P_1 may have to be determined many times a day from estimates of the demands.

$$C_{21} = 1$$

$$S_2 = 4$$
$$\begin{Bmatrix} a_{21} = 1 \\ a_{23} = 2 \end{Bmatrix}$$

$$S_1 = 5$$
$$\begin{Bmatrix} a_{12} = 2 \\ a_{13} = 1 \end{Bmatrix}$$

$$C_{12} = 3$$
$$C_{31} = 2$$
$$C_{23} = 2$$

$$C_{32} = 2$$

$$C_{13} = 3$$

$$\begin{Bmatrix} a_{31} = 2 \\ a_{32} = 2 \end{Bmatrix} S_3 = 6$$

Specific network with three nodes.

- The variables are:

$$x = \begin{bmatrix} x_{122} & x_{132} & x_{123} & x_{133} \\ x_{211} & x_{231} & x_{233} & x_{213} \\ x_{311} & x_{321} & x_{322} & x_{312} \end{bmatrix}$$

and the linear program is written as

$$\underset{x}{\text{maximize}} \ P_1 = \frac{x_{122} + x_{133} + x_{211} + x_{233} + x_{311} + x_{322}}{a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}}$$

that is

$$\underset{x}{\text{maximize}} \ P_1 = \frac{1}{10}(x_{122} + x_{133} + x_{211} + x_{233} + x_{311} + x_{322})$$

- The next equations are the demand limitation constraints:

$$i = 1, j = 2, x_{122} + x_{132} - x_{312} \leq 2$$

$$i = 1, j = 3, x_{123} + x_{133} - x_{213} \leq 1$$

$$i = 2, j = 1, x_{211} + x_{231} - x_{321} \leq 1$$

$$i = 2, j = 3, x_{213} + x_{233} - x_{123} \leq 2$$

$$i = 3, j = 1, x_{311} + x_{321} - x_{231} \leq 2$$

$$i = 3, j = 2, x_{312} + x_{322} - x_{132} \leq 2$$

the link capacity constraints

$$x_{122} + x_{123} \leq C_{12} = 3$$

$$x_{132} + x_{133} \leq C_{13} = 3$$

$$x_{211} + x_{213} \leq C_{21} = 1$$

$$x_{231} + x_{233} \leq C_{23} = 2$$

$$x_{311} + x_{312} \leq C_{31} = 2$$

$$x_{321} + x_{322} \leq C_{32} = 2$$

the switch capacity constraints for messages *leaving* each node

$$x_{122} + x_{123} + x_{132} + x_{133} \leq S_1 = 5$$

$$x_{211} + x_{213} + x_{231} + x_{233} \leq S_2 = 4$$

$$x_{311} + x_{312} + x_{321} + x_{322} \leq S_3 = 6$$

and the switch capacity constraints for messages *entering* each node

$$x_{211} + x_{213} + x_{311} + x_{312} \leq S_1 = 5$$

$$x_{122} + x_{123} + x_{321} + x_{322} \leq S_2 = 4$$

$$x_{131} + x_{132} + x_{231} + x_{233} \leq S_3 = 6$$

Another Look at Basic Solutions

- Recall that a linear program can be written in normal or standard form.

$$\text{minimize}_{\boldsymbol{x}} \quad f = \boldsymbol{c}^T\boldsymbol{x}, \quad\quad \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{c} \in \mathbb{R}^n$$

$$\text{subject to:} \quad A\boldsymbol{x} = \boldsymbol{b}, \quad\quad A \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$$

$$\boldsymbol{x} \geq 0$$

with $m < n$, rank $A = m$, and $\boldsymbol{b} \geq 0$. Now consider the equality constraints $A\boldsymbol{x} = \boldsymbol{b}$.

- Let B be an $m \times m$ matrix whose columns are linearly independent, and re-write:

$$A = [B, D]$$

where D is a $m \times (m-n)$ matrix. Since the matrix B is non-singular we can write

$$B\boldsymbol{x}_B = \boldsymbol{b}, \boldsymbol{x}_B \in \mathbb{R}^m$$

and

$$\boldsymbol{x}_B = B^{-1}\boldsymbol{b}.$$

- If we let

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{0} \end{bmatrix}$$

then $\boldsymbol{x}$ is a solution of $A\boldsymbol{x} = \boldsymbol{b}$. We now make the following definitions:

Definitions

- $x = \begin{bmatrix} x_B & \mathbf{0} \end{bmatrix}^T \in \mathbb{R}^n$ is a *basic solution* of $Ax = b$ with respect to the basis B. The components of x_B are called *basic variables*, (or left hand variables). The columns of B are called *basic columns.*❑

- if some basic variables of a basic solution are zero then the solution is called a *degenerate basic solution.*❑

- if x satisfies $Ax = b$ and $x \geq 0$ then x is called a *feasible solution.*❑

- if a feasible solution is also basic it is called a *basic feasible solution.*❑

- if a basic feasible solution is also degenerate then we call it a *degenerate basic feasible solution.*❑

Consider the problem defined by

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

and form the augmented matrix $[A, b]$:

$$[A, b] = \begin{bmatrix} 2 & 3 & -1 & -1 & -1 \\ 4 & 1 & 1 & -2 & 9 \end{bmatrix}$$

we now perform elementary row operations to get

$$\begin{bmatrix} 1 & 0 & 2/5 & -1/2 & 14/5 \\ 0 & 1 & -3/5 & 0 & -11/5 \end{bmatrix}.$$

- This implies that the general solution can be written with two parameters, $s, t \in \mathbb{R}$, as

$$x_1 = \frac{14}{5} - \frac{2}{5}s + \frac{1}{2}t$$

$$x_2 = -\frac{11}{5} + \frac{3}{5}s$$

$$x_3 = s$$

$$x_4 = t$$

$$\boldsymbol{x} = \begin{pmatrix} 14/5 \\ -11/5 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2/5 \\ 3/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Note that $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{h}$ where

$$\boldsymbol{v} = \begin{pmatrix} 14/5 \\ -11/5 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{h} = s\begin{pmatrix} -2/5 \\ 3/5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $A\boldsymbol{v} = \boldsymbol{b}$, $A\boldsymbol{h} = \boldsymbol{0}$. We say that $\boldsymbol{h}$ is in the *null space* of A. We now can set the parameters so as to make any two elements of $\boldsymbol{x}$ equal to 0. There are six ways we can do this.

1) If we choose $s = t = 0$ then we have $x_3 = x_4 = 0$ which implies the basis is

$B = [a_1, a_2]$ which is the same as solving:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14/5 \\ -11/5 \end{bmatrix}$$

which is a basic solution, but since $x_2 < 0$ this is not a basic *feasible* solution.

2) If we choose s and t such as to set $x_2 = x_4 = 0$ then this implies that the basis

is $B = [a_1, a_3]$ which is the same as solving:

$$\begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 11/3 \end{bmatrix}$$

which is a basic feasible solution.

3) If we choose s and t such as to set $x_2 = x_3 = 0$ then this implies that

$$B = \begin{bmatrix} a_1 & a_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

which is singular and therefore B is <u>not</u> a basis.

4) If we choose s and t such as to set $x_1 = x_4 = 0 \Rightarrow B = \begin{bmatrix} a_2, a_3 \end{bmatrix}$ then we get

$$x = \begin{pmatrix} 0 \\ 2 \\ 7 \\ 0 \end{pmatrix}$$

which is basic feasible.

5) If we choose s and t such as to set $x_1 = x_3 = 0 \Rightarrow B = [a_2, a_4]$, then we get

$$x = \begin{pmatrix} 0 \\ -11/5 \\ 0 \\ -28/5 \end{pmatrix}$$

which is basic but not feasible.

6) If we choose s and t such as to set $x_1 = x_2 = 0 \Rightarrow B = [a_3, a_4]$, then we get

$$x = \begin{pmatrix} 0 \\ 0 \\ 11/3 \\ -8/3 \end{pmatrix}$$

which is basic but not feasible.

- Notice that the number of ways we can choose 2 elements of x to be zero given

  that it has 4 elements is $\binom{4}{2} = \dfrac{4!}{2!2!} = 6$ choices; or more generally

$\binom{n}{n-m} = \binom{n}{m}$.

Definitions

- any x which minimizes the objective function $c^T x$ and satisfies the constraints

  $Ax = b$, $A \in \mathbb{R}^{m \times n}$ and $x \geq 0$ is said to be an *optimal feasible solution.*□

- an optimal feasible solution which is basic is said to be an *optimal basic feasible*
  solution.□

- **Theorem**: fundamental theorem of linear program.

  given a linear program in standard form:

- if there exists a feasible solution, then there exists is a basic feasible solution.

- if there exists an optimal feasible solution, then there exists an optimal basic feasible solution.☐