One Way Wave Equation

consider: $\begin{cases} u_t + a u_x = 0 & \text{(P.D.E.)} \\ u(x,0) = u_0(x) & \text{(I.C.)} \end{cases}$

where $a = a(x,t)$ is the speed of the wave and we want to solve for $u(x,t)$ on the interval $-\infty < x < \infty$ $t > 0$.

This is called an initial value problem.

now a differential change in u can be written as

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \quad \text{(total differential)}$$

or $$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \quad \left(\text{integration along line define by } \frac{dx}{dt}\right)$$

Comparing this to the P.D.E. we see that the rate of change of u along the line defined by $\frac{dx}{dt} = a(x,t)$ is zero

i.e. $u(x,t) = \text{const.}$ on characteristic lines:

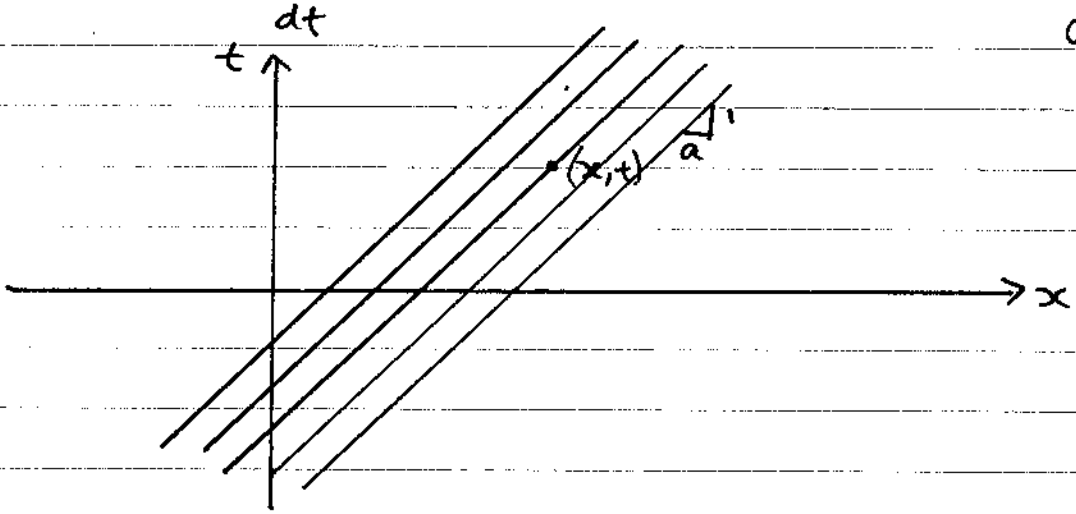
$$\Gamma: \quad \frac{dx}{dt} = a(x,t)$$

these are in general curves in the x-t plane.

IF $\quad a(x,t) = \text{const} = a \quad$ then

$$\Gamma^1: \quad \frac{dx}{dt} = a \quad \Longrightarrow \quad x = at + c$$

assuming $a > 0$



$\therefore$ u is constant on this familly of lines but since u is given on $t=0$ line

$$\Longrightarrow \quad \boxed{u(x,t) = u_0(x - at)}$$

is the solution to our problem this represents a distortionless wave travelling in the +ve x direction with speed a.

Time Domain Maxwell's Equations

starting with Maxwell's curl equations

$$\bar{\nabla} \times \bar{E} = \frac{-\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J}$$

in a source-free, homogeneous, isotropic, linear, and stationary medium:

$$\bar{\nabla} \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \qquad \bar{\nabla} \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t}$$

which represent 6 scalar equations in 6 scalar unknowns.

In rectangular coordinates these can be written out component-wize as

$$\bar{\nabla} \times \bar{A} = (\partial_y A_z - \partial_z A_y)\hat{a}_x + (\partial_z A_x - \partial_x A_z)\hat{a}_y + (\partial_x A_y - \partial_y A_x)\hat{a}_z$$

$$\begin{cases} -\mu\, \partial_t H_x = \partial_y E_z - \partial_z E_y \\ -\mu\, \partial_t H_y = \partial_z E_x - \partial_x E_z \\ -\mu\, \partial_t H_z = \partial_x E_y - \partial_y E_x \end{cases}$$

$$\begin{cases} \epsilon\, \partial_t E_x = \partial_y H_z - \partial_z H_y \\ \epsilon\, \partial_t E_y = \partial_z H_x - \partial_x H_z \\ \epsilon\, \partial_t E_z = \partial_x H_y - \partial_y H_x \end{cases}$$

these can be written in general vector form as

$$A\, \partial_t \underline{u} + \partial_x \underline{E} + \partial_y \underline{F} + \partial_z \underline{G} = \underline{0}$$

$$\underline{u} = \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} \qquad \underline{E} = \begin{pmatrix} 0 \\ H_z \\ -H_y \\ 0 \\ -E_z \\ E_y \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} -H_z \\ 0 \\ H_x \\ E_z \\ 0 \\ -E_x \end{pmatrix} \qquad \underline{G} = \begin{pmatrix} H_y \\ -H_x \\ 0 \\ -E_y \\ E_x \\ 0 \end{pmatrix}$$

$$A = \mathrm{diag}\{\epsilon\ \epsilon\ \epsilon\ \mu\ \mu\ \mu\} = \begin{bmatrix} \epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

$$A^{-1} = \mathrm{diag}\{e\ e\ e\ m\ m\ m\} \qquad e = \frac{1}{\epsilon} \qquad m = \frac{1}{\mu}$$

$$\therefore \boxed{\partial_t \underline{u} + A^{-1}\partial_x \underline{E} + A^{-1}\partial_y \underline{F} + A^{-1}\partial_z \underline{G} = \underline{0}}$$

One dimensional waves:

if we have no variation of $\underline{u}$ with respect to the y and z directions then $\partial_y = \partial_z = 0$ and our system of equations becomes:

$$\partial_t \underline{u} + A^{-1} \partial_x \underline{E} = \underline{0}$$

$$\partial_t \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} + A^{-1} \partial_x \begin{pmatrix} 0 \\ H_z \\ -H_y \\ 0 \\ -E_z \\ E_y \end{pmatrix} = \underline{0}$$

From this we see that we have 4 independent systems:

$$\partial_t E_x = 0 \qquad (1)$$

$$\partial_t H_x = 0 \qquad (2)$$

$$\left.\begin{aligned} \partial_t E_y + e\,\partial_x H_z &= 0 \\ \partial_t H_z + m\,\partial_x E_y &= 0 \end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} \partial_t E_z - e\,\partial_x H_y &= 0 \\ \partial_t H_y - m\,\partial_x E_z &= 0 \end{aligned}\right\} \qquad (4)$$

solutions of ① + ② :

$$\partial_t \psi(x,t) = 0 \quad \Rightarrow \quad \psi = \psi(x)$$

$$\therefore \begin{cases} E_x(x,t) = E_0(x) & \text{(initial condition)} \\ H_x(x,t) = H_0(x) & \text{(initial condition)} \end{cases}$$

"static" fields, no variations in time

<u>Solution of ③</u>:

$$\boxed{\partial_t \begin{pmatrix} E_y \\ H_z \end{pmatrix} + \begin{bmatrix} 0 & e \\ m & 0 \end{bmatrix} \partial_x \begin{pmatrix} E_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

consider:

$$\partial_t \underline{u} + A\,\partial_x \underline{u} = \underline{0}$$

if A has any off-diagonal components then we are required to solve coupled P.D.E.'S.

now say we could diagonalize A by the similarity transformation:

$$L A R = \Lambda = \text{diag}\{\lambda_i\} \qquad \lambda_i \text{ - eigenvalue}$$

$$R = L^{-1} \qquad R^{-1} = L$$

then set $\quad R\underline{v} = \underline{u}$

$$\partial_t (R\underline{v}) + A\,\partial_x (R\underline{v}) = \underline{0}$$

Since $A, R, L,$ are constant:

$$R\,\partial_t \underline{v} + AR\,\partial_x \underline{v} = \underline{0}$$

pre-multiplying by $L = R^{-1}$:

$$LR\,\partial_t \underline{v} + LAR\,\partial_x \underline{v} = 0$$

$$\boxed{\partial_t \underline{v} + \Lambda\,\partial_x \underline{v} = 0}$$

this is a system of uncoupled P.D.E's since Λ is diagonal.

For $n=2$: $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$

$$\begin{cases} \partial_t v_1 + \lambda_1\,\partial_x v_1 = 0 \\ \partial_t v_2 + \lambda_2\,\partial_x v_2 = 0 \end{cases}$$

the general solution can easily be found as

$$\left.\begin{array}{l} v_1 = f_1(x - \lambda_1 t) \\ v_2 = f_2(x - \lambda_2 t) \end{array}\right\}$$

where f_1 and f_2 are arbitrary functions
this "arbitrariness" is removed by imposing initial conditions:

$$v_1(x,0) = v_{10}(x) = f_1(x)$$
$$v_2(x,0) = v_{20}(x) = f_2(x)$$

$$\therefore \left.\begin{array}{l} v_1(x,t) = v_{10}(x - \lambda_1 t) \\ v_2(x,t) = v_{20}(x - \lambda_2 t) \end{array}\right\}$$

Once $\underline{v}(x,t)$ has been found, the original solution vector $\underline{u}$ is found as

$$\boxed{\underline{u}(x,t) = R\,\underline{v}(x,t)}$$

For our specific matrix:

$$A = \begin{bmatrix} 0 & e \\ m & 0 \end{bmatrix}$$

<u>eigenvalues</u>: $\lambda_1 = -\sqrt{em} = \frac{-1}{\sqrt{\epsilon\mu}} = -c$

$\lambda_2 = \sqrt{em} = \frac{+1}{\sqrt{\epsilon\mu}} = c$

<u>right eigenvectors</u>: $(AR = R\Lambda)$ $\quad R = \begin{bmatrix} c & c \\ -m & m \end{bmatrix}$

<u>left eigenvectors</u>: $(LA = \Lambda L)$ $\quad L = \begin{bmatrix} \frac{1}{2c} & -\frac{1}{2m} \\ \frac{1}{2c} & \frac{1}{2m} \end{bmatrix}$

note: $RL = LR = I$

$$LAR = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} = \Lambda$$

$\therefore$ Given initial conditions on E_y and H_z:

$$E_y(x,0) = E_{y_0}(x)$$
$$H_z(x,0) = H_{z_0}(x)$$

$$\underline{v}(x,0) = L\,\underline{u}_0 = \begin{pmatrix} \frac{1}{2c} & -\frac{1}{2m} \\ \frac{1}{2c} & \frac{1}{2m} \end{pmatrix} \begin{pmatrix} E_{y_0}(x) \\ H_{z_0}(x) \end{pmatrix} = \underline{v}_0(x)$$

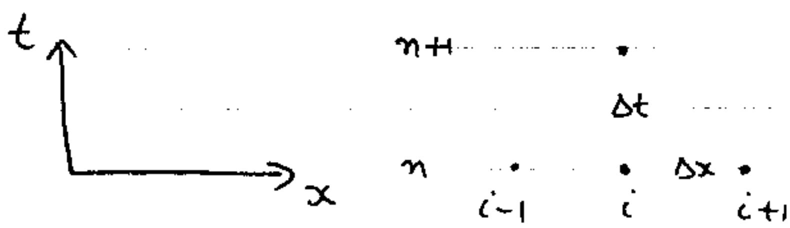
$$\underline{v}(x,t) = \begin{pmatrix} v_{10}(x+ct) \\ v_{20}(x-ct) \end{pmatrix} = \begin{pmatrix} \frac{1}{2c} E_{y_0}(x+ct) - \frac{1}{2m} H_{z_0}(x+ct) \\ \frac{1}{2c} E_{y_0}(x-ct) + \frac{1}{2m} H_{z_0}(x-ct) \end{pmatrix}$$

$$\underline{u}(x,t) = R\,\underline{v}(x,t)$$

$$\begin{pmatrix} E_y(x,t) \\ H_z(x,t) \end{pmatrix} = \begin{pmatrix} c & c \\ -m & m \end{pmatrix} \begin{pmatrix} v_{10}(x+ct) \\ v_{20}(x-ct) \end{pmatrix}$$

$$\boxed{\begin{aligned} E_y(x,t) &= \frac{1}{2}\Big[E_{y_0}(x+ct) + E_{y_0}(x-ct)\Big] + \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\Big[H_{z_0}(x-ct) - H_{z_0}(x+ct)\Big] \\ H_z(x,t) &= \frac{1}{2}\Big[H_{z_0}(x+ct) + H_{z_0}(x-ct)\Big] + \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\Big[E_{y_0}(x-ct) - E_{y_0}(x+ct)\Big] \end{aligned}}$$

Discretization of Exact 1-D Solution



Grid Fcns:

$$E_i^n = E_y(i\Delta x, n\Delta t) \qquad H_i^n = H_z(i\Delta x, n\Delta t)$$

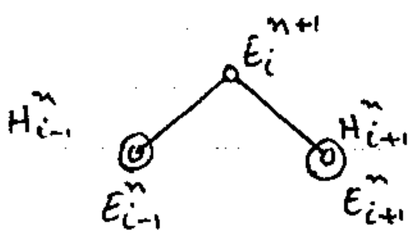
initial conditions

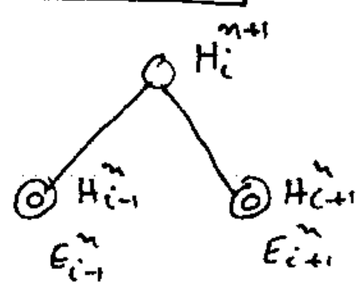
$$E_i^0 = E_{y0}(i\Delta x) = E_y(i\Delta x, 0)$$

$$H_i^0 = H_{z0}(i\Delta x) = H_z(i\Delta x, 0)$$

For the discrete exact solution, we <u>must</u> pick $\boxed{\Delta x/\Delta t = c}$:

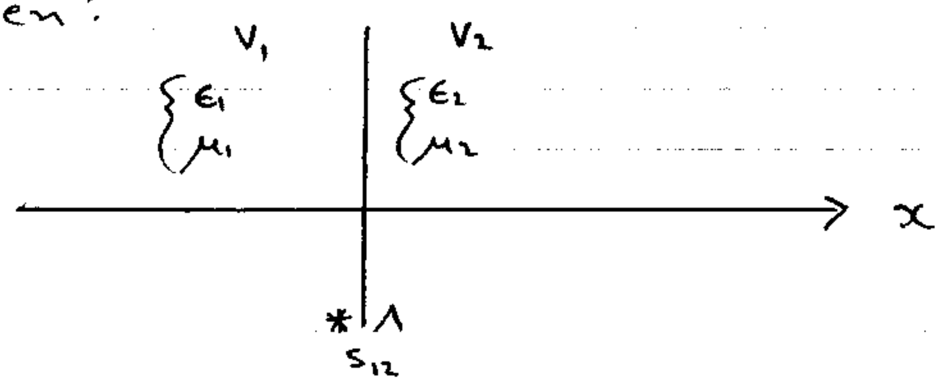
$$\boxed{\begin{aligned} E_i^{n+1} &= \tfrac{1}{2}\left(E_{i+1}^n + E_{i-1}^n\right) + \tfrac{1}{2}\sqrt{\tfrac{\mu}{\epsilon}}\left(H_{i-1}^n - H_{i+1}^n\right) \\ H_i^{n+1} &= \tfrac{1}{2}\left(H_{i+1}^n + H_{i-1}^n\right) + \tfrac{1}{2}\sqrt{\tfrac{\epsilon}{\mu}}\left(E_{i-1}^n - E_{i+1}^n\right) \end{aligned}}$$





Now what happens when a material boundary exists?

consider two regions V_1 and V_2 shown below with material constants given:



In region V_1 and V_2 the PDE

$$\partial_t \underline{u} + A_i \partial_x \underline{u} = \underline{0}$$

must be satisfied where $A_i = A_1$ or A_2. at the surface S_{12} the electromagnetic boundary conditions must be satisfied:

$$E_* = E_\wedge \qquad H_* = H_\wedge \qquad (1)$$

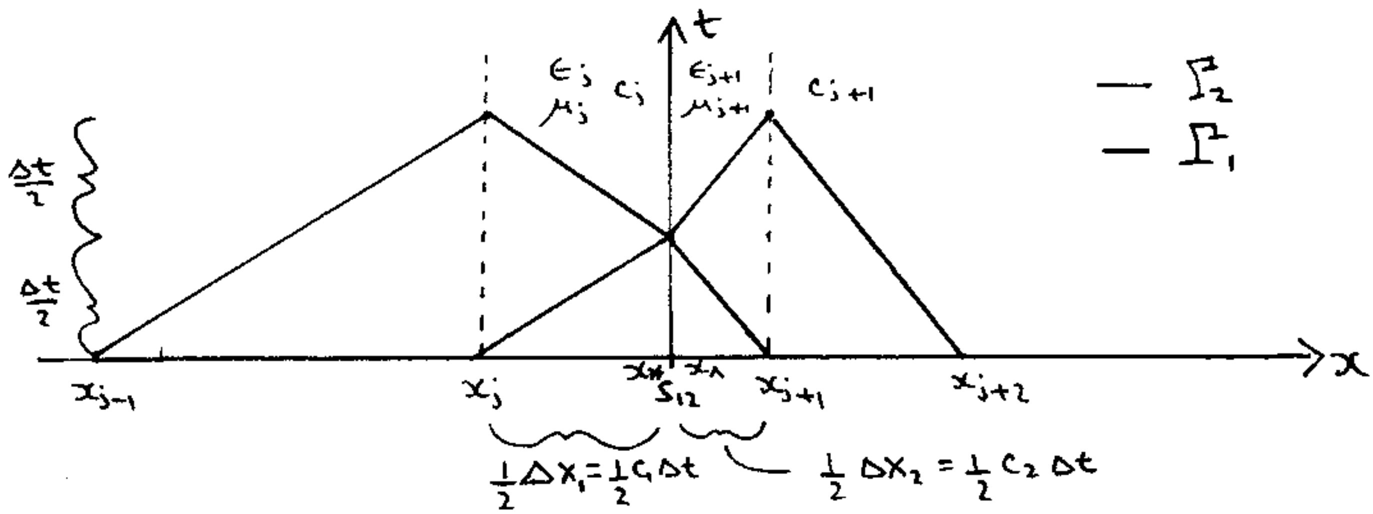
i.e. tangential components of E and H are continuous.

now drawing the characteristic curves

$$\Gamma_1: \quad \frac{dx}{dt} = -c \Rightarrow x = -ct \quad \text{(to left)}$$

$$\Gamma_2: \quad \frac{dx}{dt} = c \Rightarrow x = ct \quad \text{(to right)}$$

and assuming $\underline{c_1 > c_2}$ we choose a grid such that the characteristics meet at the surface S_{12}:



From the solution of the PDE we know that v_1 is constant on Γ_1 and v_2 is constant on Γ_2.

$\therefore$ on Γ_2: $v_{2j} = v_{2*} \Rightarrow \frac{1}{2c_j}E_j + \frac{1}{2m_j}H_j = \frac{1}{2c_j}E_* + \frac{1}{2m_j}H_*$

$$\boxed{m_j(E_j - E_*) + c_j(H_j - H_*) = 0} \qquad (2)$$

on Γ_1^2: $\quad v_{1\,j+1} = v_{1\,\wedge} \Rightarrow \frac{1}{2c_{j+1}} E_{j+1} - \frac{1}{2m_{j+1}} H_{j+1} = \frac{1}{2c_{j+1}} E_{\wedge} - \frac{1}{2m_{j+1}} H_{\wedge}$

$$\boxed{m_{j+1}\left(E_{j+1} - E_{\wedge}\right) - c_{j+1}\left(H_{j+1} - H_{\wedge}\right) = 0} \qquad (3)$$

using (1) in (3):

$$m_{j+1}\left(E_{j+1} - E_{*}\right) - c_{j+1}\left(H_{j+1} - H_{*}\right) = 0 \qquad (4)$$

$$\frac{m_{j+1}}{c_{j+1}}\left(E_{j+1} - E_{*}\right) - \left(H_{j+1} - H_{*}\right) = 0$$

From (2):

$$\frac{m_j}{c_j}\left(E_j - E_{*}\right) + \left(H_j - H_{*}\right) = 0$$

adding these last two:

$$\left(\frac{m_{j+1}}{c_{j+1}} + \frac{m_j}{c_j}\right) E_{*} = \frac{m_{j+1}}{c_{j+1}} E_{j+1} + \frac{m_j}{c_j} E_j - H_{j+1} + H_j$$

$$E_{*} = \frac{\frac{m_{j+1}}{c_{j+1}} E_{j+1} + \frac{m_j}{c_j} E_j + H_j - H_{j+1}}{\frac{m_{j+1}}{c_{j+1}} + \frac{m_j}{c_j}}$$

but $\quad \frac{m}{c} = \sqrt{\frac{m}{e}} = Y = \frac{1}{Z} \qquad (5)$

$$\therefore \boxed{E_* = \frac{Y_{j+1} E_{j+1} + Y_j E_j + H_j - H_{j+1}}{Y_{j+1} + Y_j}} \quad (6)$$

From (4):

$$\left(E_{j+1} - E_*\right) - \frac{c_{j+1}}{m_{j+1}} \left(H_{j+1} - H_*\right) = 0$$

From (2):

$$\left(E_j - E_*\right) + \frac{c_j}{m_j} \left(H_j - H_*\right) = 0$$

subtracting these:

$$E_j - E_{j+1} + \frac{c_{j+1}}{m_{j+1}} H_{j+1} + \frac{c_j}{m_j} H_j = \left(\frac{c_j}{m_j} + \frac{c_{j+1}}{m_{j+1}}\right) H_*$$

$$\boxed{H_* = \frac{E_j - E_{j+1} + Z_{j+1} H_{j+1} + Z_j H_j}{Z_{j+1} + Z_j}}$$

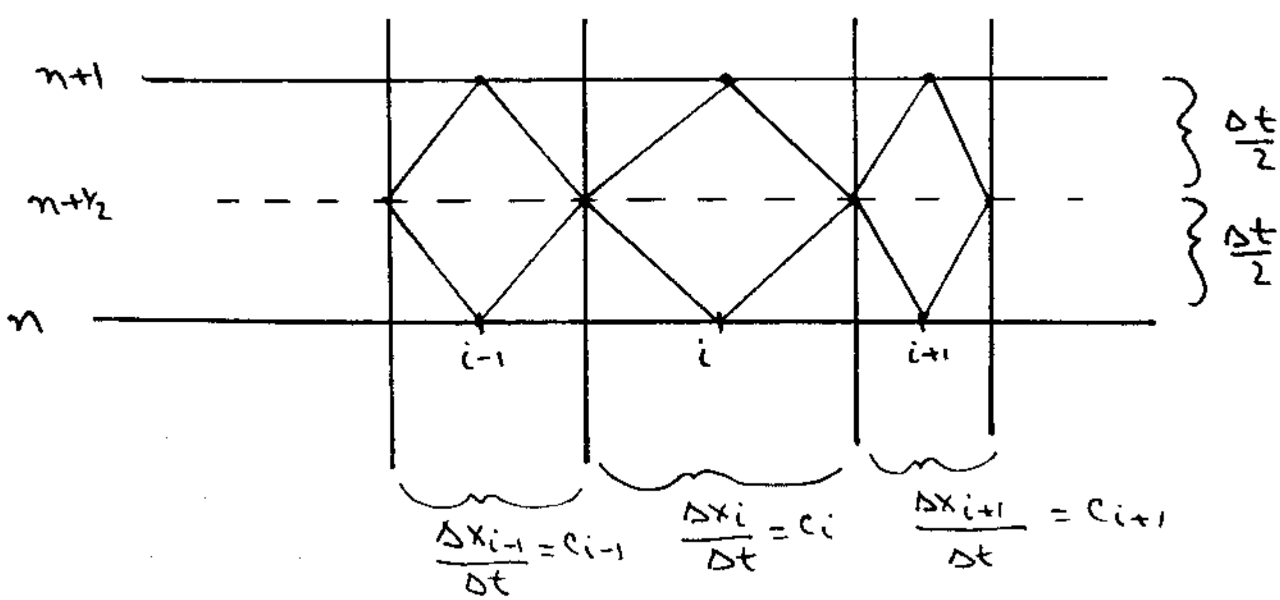
notice that we can write

$$H_\wedge = H_* = H^{n+1/2}_{j+1/2} \qquad E_\wedge = E_* = E^{n+1/2}_{j+1/2}.$$

$\therefore$ the <u>exact</u> numerical scheme becomes:

$$\left.\begin{aligned} E_i^{n+1} &= \tfrac{1}{2}\left(E^{n+1/2}_{i+1/2} + E^{n+1/2}_{i-1/2}\right) + \tfrac{1}{2} Z_i \left(H^{n+1/2}_{i-1/2} - H^{n+1/2}_{i+1/2}\right) \\ H_i^{n+1} &= \tfrac{1}{2}\left(H^{n+1/2}_{i+1/2} + H^{n+1/2}_{i-1/2}\right) + \tfrac{1}{2} Y_i \left(E^{n+1/2}_{i-1/2} - E^{n+1/2}_{i+1/2}\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} E^{n+1/2}_{i+1/2} &= \frac{Y_{i+1} E^n_{i+1} + Y_i E^n_i + H^n_i - H^n_{i+1}}{Y_{i+1} + Y_i} \\ H^{n+1/2}_{i+1/2} &= \frac{E^n_i - E^n_{i+1} + Z_{i+1} H^n_{i+1} + Z_i H^n_i}{Z_{i+1} + Z_i} \end{aligned}\right\}$$



Finite Difference Methods for Time Domain Maxwell's equations

starting with the 1-D system:

$$\left.\begin{aligned} &\partial_t \underline{u} + A\partial_x \underline{u} = \underline{0} \\ &\underline{u} = \begin{pmatrix} E_y \\ H_z \end{pmatrix} \qquad A = \begin{bmatrix} 0 & e \\ m & 0 \end{bmatrix} \end{aligned}\right\}$$

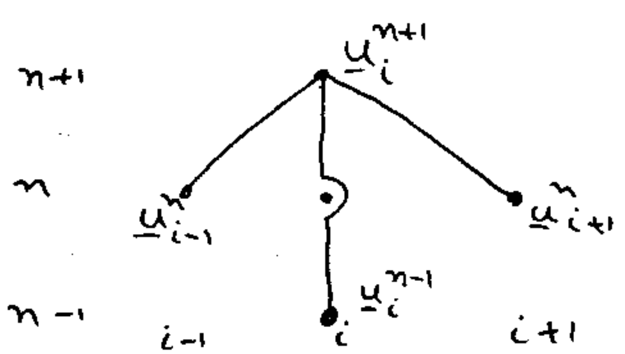
use the center-difference approximation to the partial derivative operators

$$\partial_t \underline{u} = \frac{\underline{u}_i^{n+1} - \underline{u}_i^{n-1}}{2\Delta t} + O(\Delta t^2)$$

$$\partial_x \underline{u} = \frac{\underline{u}_{i+1}^{n} - \underline{u}_{i-1}^{n}}{2\Delta x} + O(\Delta x^2)$$

$$\frac{\underline{u}_i^{n+1} - \underline{u}_i^{n-1}}{2\Delta t} + A\,\frac{\underline{u}_{i+1}^{n} - \underline{u}_{i-1}^{n}}{2\Delta x} = \underline{0}$$

$$\boxed{\underline{u}_i^{n+1} = \underline{u}_i^{n-1} - \frac{\Delta t}{\Delta x} A \left(\underline{u}_{i+1}^{n} - \underline{u}_{i-1}^{n} \right)}$$



" computational " molecule

the above is called <u>Leap - Frog</u> scheme

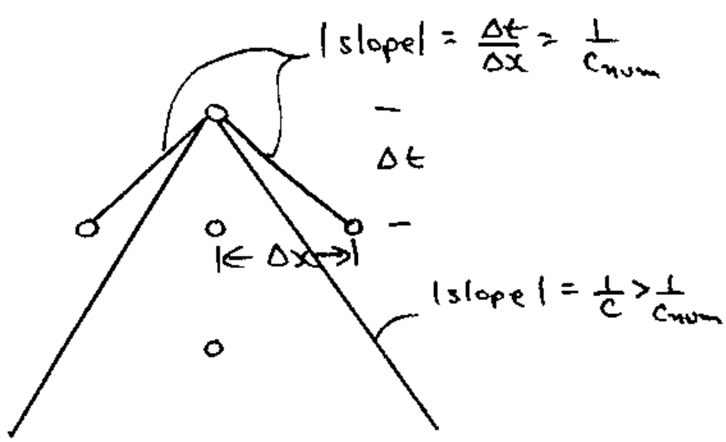
and is a two-step scheme since we require $\underline{u}^0$ and $\underline{u}^1$ to start the scheme.

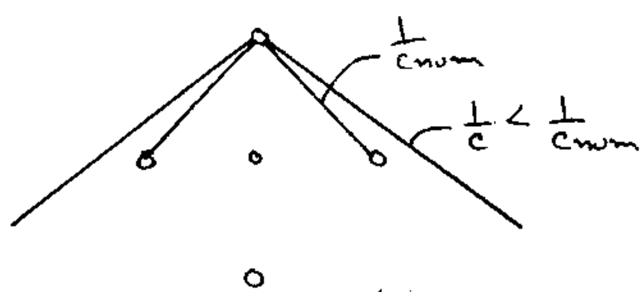
It can be shown that this scheme is stable for:

$$\boxed{\frac{c\,\Delta t}{\Delta x} \leq 1}$$ Courant - Friedrichs - Lewy Condition (CFL)

$\frac{\Delta x}{\Delta t} = c_{num}$ — numerical speed

$\therefore$ CFL condition implies that the numerical speed must be greater than the actual speed.





Stable
<u>$c_{num} > c$</u>

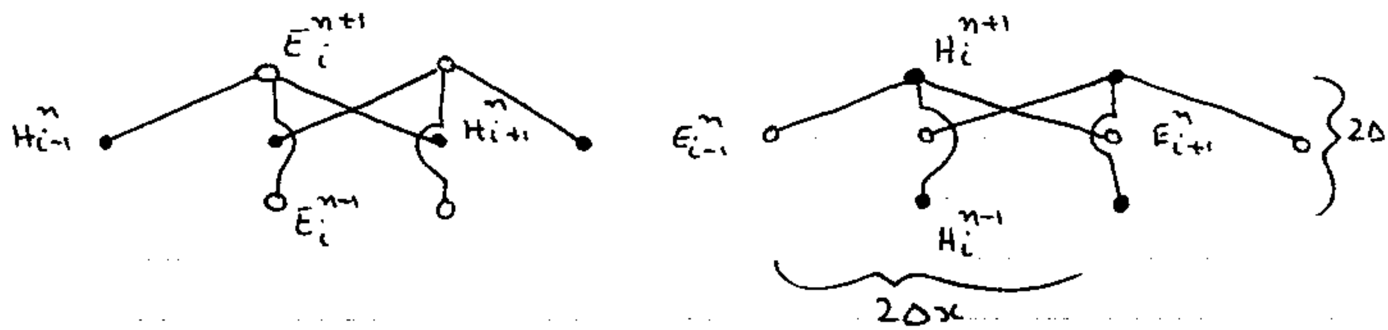
Unstable.
<u>$c > c_{num}$</u>

now writting these component wize

let $\begin{pmatrix} E_y \\ H_z \end{pmatrix} = \begin{pmatrix} E \\ H \end{pmatrix}$

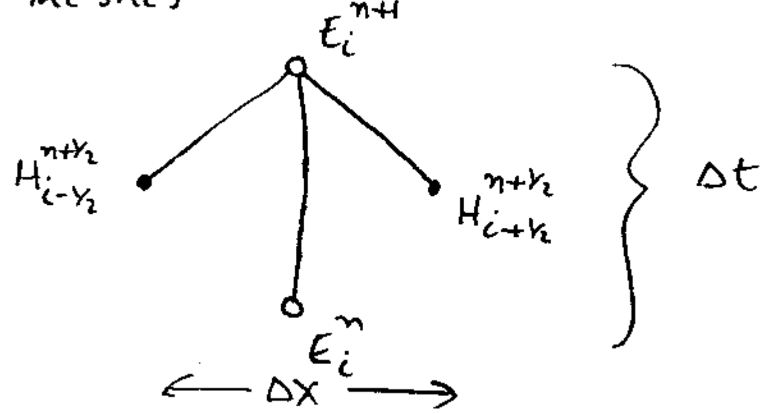
$$\left.\begin{aligned} E_i^{n+1} &= E_i^{n-1} - \frac{\Delta t}{\Delta x}\, e \left(H_{i+1}^{n} - H_{i-1}^{n} \right) \\ H_i^{n+1} &= H_i^{n-1} - \frac{\Delta t}{\Delta x}\, m \left(E_{i+1}^{n} - E_{i-1}^{n} \right) \end{aligned}\right\}$$

<u>computational molecules:</u>



a close look reveals <u>4 independent</u> meshes which are interlaced in space and time.

we only need to keep one of these meshes:



$$\frac{\Delta x}{\Delta t} > c$$

$$\boxed{\begin{aligned} E_i^{n+1} &= E_i^n - \frac{\Delta t}{\Delta x} e \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2} \right) \\ H_{i+1/2}^{n+1/2} &= H_{i+1/2}^{n-1/2} - \frac{\Delta t}{\Delta x} m \left(E_{i+1}^n - E_i^n \right) \end{aligned}}$$

1-D Yee Algorithm.

Given an initial $\bar{E}$ field $- E(x,0) = E_0(x)$

we set $E_i^0 = E_0(i\Delta x)$

the first set of H fields are then computed as.

$$\boxed{H_{i+1/2}^{n+1/2} = -\frac{\Delta t}{2\Delta x} m \left(E_{i+1}^n - E_i^n \right)}$$

if we choose $\frac{\Delta t}{\Delta x} = \frac{1}{c} = (em)^{-1/2}$

then $\frac{\Delta t}{\Delta x} e = e(em)^{-1/2} = \sqrt{\frac{e}{m}} = \sqrt{\frac{\mu}{\epsilon}} \doteq Z$

$\frac{\Delta t}{\Delta x} m = m(em)^{-1/2} = \sqrt{\frac{m}{e}} = \sqrt{\frac{\epsilon}{\mu}} = Y = \frac{1}{Z}$

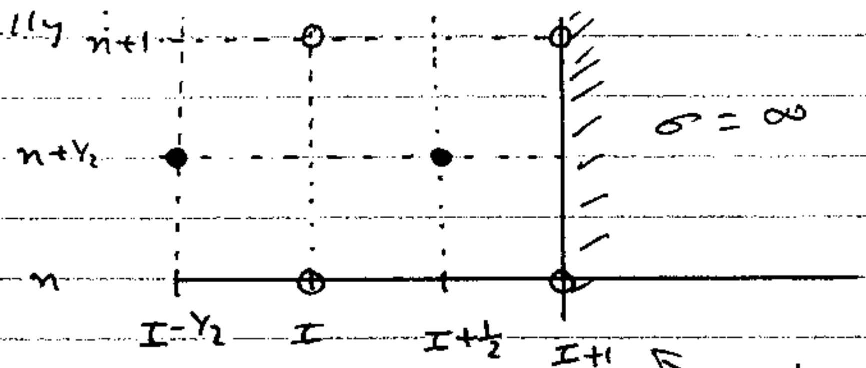
in free space $Z \approx 376.734 \approx 376.7$

$(\mu_0 = 4\pi \times 10^{-7} \quad \epsilon_0 = 8.854 \times 10^{-12})$

Implementing Boundary Conditions & Inhomogeneous Medium

1-D Yee:

at a perfectly conducting we know that the tangential electric field must be zero. Since the magnetic field is specified at interlaced points in space they can be handled normally



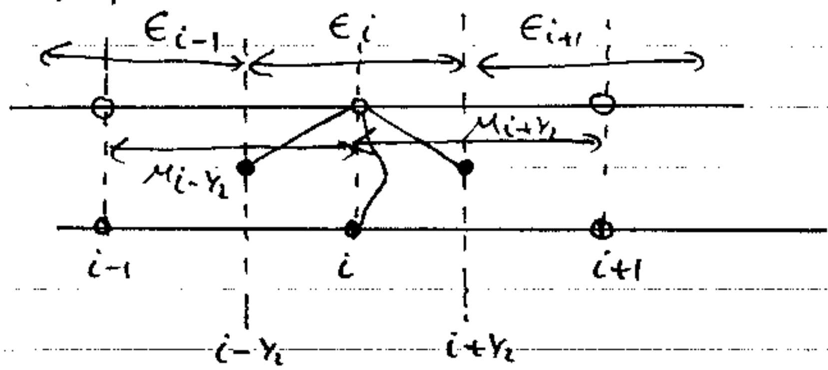
end of mesh. Since nothing gets through

$$\begin{cases} E^{n}_{I+1} = 0 \\ H^{n+1/2}_{I+1/2} = H^{n-1/2}_{I+1/2} - \dfrac{\Delta t}{\Delta x} m_{I+1/2} \Big(\underbrace{E^{n}_{I+1}}_{0} - E^{n}_{I} \Big) \end{cases}$$

∴ Rule is simple:

> Put your perfect conducting boundary at an "E" point and keep E = 0 at that point

inhomogeneous μ and ϵ are also simply handled:



$$\begin{cases} E_i^{n+1} = E_i^n - \dfrac{\Delta t}{\Delta x} e_i \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2} \right) \\[2ex] H_{i+1/2}^{n+1/2} = H_{i+1/2}^{n-1/2} - \dfrac{\Delta t}{\Delta x} m_{i+1/2} \left(E_{i+1}^n - E_i^n \right) \end{cases}$$

unfortunately boundaries of μ and boundaries of ϵ <u>cannot</u> coincide!

Non-pefectly conducting Media

$\sigma \neq 0$ $\quad\quad \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu \frac{\partial \bar{H}}{\partial t}$

Ohms Law $(\bar{J} = \sigma \bar{E})$ $\quad\quad \bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} = \epsilon \frac{\partial E}{\partial t} + \sigma \bar{E}$

in 1-D the curl equations become:

$$\begin{cases} \partial_t \underline{u} + A \partial_x \underline{u} = \underline{s} \\ \underline{u} = \begin{pmatrix} E_y \\ H_z \end{pmatrix} = \begin{pmatrix} E \\ H \end{pmatrix} \quad A = \begin{pmatrix} 0 & e \\ m & 0 \end{pmatrix} \quad \underline{s} = \begin{pmatrix} -\sigma e E \\ 0 \end{pmatrix} \end{cases}$$

Show this!

Now how do we difference this equation?

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + A \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = s_i^{n+1}$$

must use $n+1$ for stability.

using Yee version component wize:

$$E_i^{n+1} + \Delta t \sigma e E_i^{n+1} = E_i^n - \frac{\Delta t}{\Delta x} e_i \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2} \right)$$

$$E_i^{n+1} = \left(\frac{1}{1 + \Delta t \sigma_i e_i} \right) \left[E_i^n - \frac{\Delta t}{\Delta x} e_i \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2} \right) \right]$$

$H_{i+1/2}^{n+1/2}$ — same as before.

2-D FIELDS. $\partial_z = 0$

$$\partial_t \underline{u} + A^{-1}\partial_x \underline{E} + A^{-1}\partial_y \underline{F} = \underline{0}$$

$$\partial_t \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} + A^{-1}\partial_x \begin{pmatrix} 0 \\ H_z \\ -H_y \\ 0 \\ -E_z \\ E_y \end{pmatrix} + A^{-1}\partial_y \begin{pmatrix} -H_z \\ 0 \\ H_x \\ E_z \\ 0 \\ -E_x \end{pmatrix} = \underline{0}$$

we notice that there are two independent systems of equation

Transverse Electric to z (TE_z) (E_x, E_y, H_z)

$$\partial_t \begin{pmatrix} E_x \\ E_y \\ H_z \end{pmatrix} + \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & m \end{bmatrix} \partial_x \begin{pmatrix} 0 \\ H_z \\ E_y \end{pmatrix} + \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & m \end{bmatrix} \partial_y \begin{pmatrix} -H_z \\ 0 \\ -E_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Transverse Magnetic to z (TM_z) (H_x, H_y, E_z)

$$\partial_t \begin{pmatrix} E_z \\ H_x \\ H_y \end{pmatrix} + \begin{bmatrix} e & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \partial_x \begin{pmatrix} -H_y \\ 0 \\ -E_z \end{pmatrix} + \begin{bmatrix} e & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \partial_y \begin{pmatrix} H_x \\ E_z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we can now apply the Leap-Frog scheme as:

$$\frac{\underline{u}_{ij}^{n+1} - \underline{u}_{ij}^{n-1}}{2\Delta t} + \vec{A}\left(\frac{\underline{E}_{i+1j}^{n} - \underline{E}_{i-1j}^{n}}{2\Delta x} + \frac{\underline{F}_{ij+1}^{n} - \underline{F}_{ij-1}^{n}}{2\Delta y}\right) = \underline{0}$$

$$\underline{u}_{ij}^{n+1} = \underline{u}_{ij}^{n-1} - \frac{\Delta t A'}{\Delta x}\left(\underline{E}_{i+1j}^{n} - \underline{E}_{i-1j}^{n}\right) - \frac{\Delta t}{\Delta y} A'\left(\underline{F}_{ij+1}^{n} - \underline{F}_{ij-1}^{n}\right)$$

let $\Delta x = \Delta y = h$, $\Delta t = k$

$$\boxed{\underline{u}_{ij}^{n+1} = \underline{u}_{ij}^{n-1} - \frac{k}{h} A'\left(\underline{E}_{i+1j}^{n} - \underline{E}_{i-1j}^{n} + \underline{F}_{ij+1}^{n} - \underline{F}_{ij-1}^{n}\right)}$$

specific A', $\underline{E}$, $\underline{F}$ depends on TE_z, or TM_z.

This scheme is stable for (3-D)

$$\Delta t \le \frac{1}{\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right)^{1/2} c_{max}}$$

in 2-D , $\Delta x = \Delta y = \Delta h$

$$\boxed{\frac{\Delta t}{\Delta h} c_{max} \le \frac{1}{\sqrt{2}}}$$

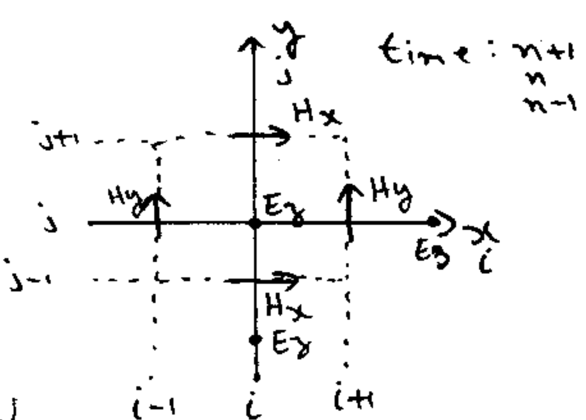
Application of the Leap-Frog Formula:

<u>TM waves:</u> $(\Delta x = \Delta y = h \qquad \Delta t = k)$

$$\underline{u} = \begin{pmatrix} E_z \\ H_x \\ H_y \end{pmatrix} \qquad \underline{E} = \begin{pmatrix} -H_y \\ 0 \\ -E_z \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} H_x \\ E_z \\ 0 \end{pmatrix}$$

$$A^{-1} = \begin{bmatrix} e & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$\begin{cases} E_{z_{ij}}^{n+1} = E_{z_{ij}}^{n-1} - \dfrac{k e_{ij}}{h}\left(-H_{y_{i+1j}}^{n} + H_{y_{i-1j}}^{n} + H_{x_{ij+1}}^{n} - H_{x_{ij-1}}^{n}\right) \\[2ex] H_{x_{ij}}^{n+1} = H_{x_{ij}}^{n-1} - \dfrac{k}{h} m_{ij}\left(E_{z_{ij+1}}^{n} - E_{z_{ij-1}}^{n}\right) \\[2ex] H_{y_{ij}}^{n+1} = H_{y_{ij}}^{n-1} - \dfrac{k}{h} m_{ij}\left(-E_{z_{i+1j}}^{n} + E_{z_{i-1j}}^{n}\right) \end{cases}$$



"computational molecule."

It is a bit harder to see now but this scheme is also interlaced in both time and space.

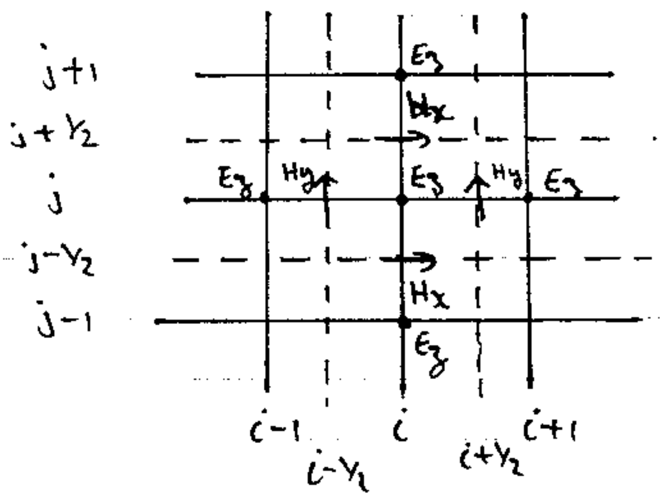
example: $\left\{ \begin{array}{l} E_z \text{ at even time points} \\ \quad n = 0, 2, 4, \ldots \\ \\ \text{depend on:} \\ \\ \text{other } E_z \text{ at even time points} \\ \text{and } H_x, H_y \text{ at odd time points.} \end{array} \right.$

this results in independent interlaced meshes.

Keeping only one mesh:

$$\left\{ \begin{array}{llll} E_z & \text{at} & n = 0, 1, 2, \ldots & \\ & & i, j = \pm 1, \pm 2, \ldots & \\ & & & \\ H_x & \text{at} & n = \frac{1}{2}, 1+\frac{1}{2}, 2+\frac{1}{2} & n+\frac{1}{2} \\ & & i = \pm 1, \pm 2, \ldots & i \\ & & j = \pm\frac{1}{2}, \pm(1+\frac{1}{2}), \pm(2\pm\frac{1}{2}) & j+\frac{1}{2} \\ & & & \\ H_y & \text{at} & n = \frac{1}{2}, 1+\frac{1}{2}, \ldots & n+\frac{1}{2} \\ & & i = \pm\frac{1}{2}, \ldots & i+\frac{1}{2} \\ & & j = \pm 1, \pm 2 & j \end{array} \right.$$

now the computational molecule becomes.

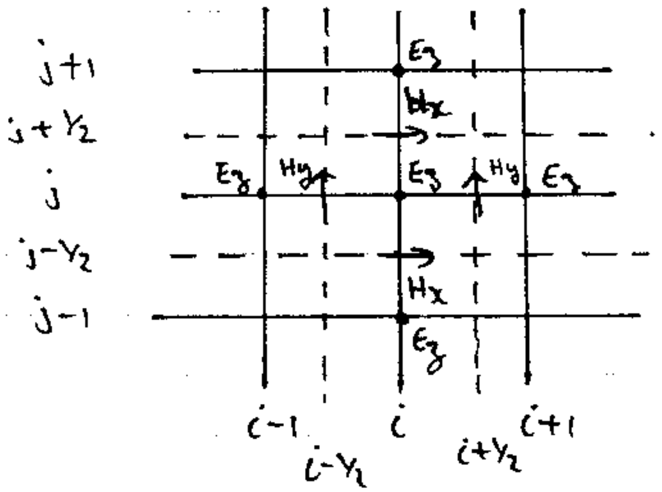


(Yee Algorithm)

$$
\begin{cases}
E_{z_{ij}}^{n+1} = E_{z_{ij}}^{n} - \dfrac{k}{h} e_{ij} \left(-H_{y_{i+\frac12 j}}^{n+\frac12} + H_{y_{i-\frac12 j}}^{n+\frac12} + H_{x_{i j+\frac12}}^{n+\frac12} - H_{x_{i j-\frac12}}^{n+\frac12} \right) \\[2ex]
H_{x_{ij+\frac12}}^{n+\frac12} = H_{x_{ij+\frac12}}^{n-\frac12} - \dfrac{k}{h} m_{ij+\frac12} \left(E_{z_{ij+1}}^{n} - E_{z_{ij}}^{n} \right) \\[2ex]
H_{y_{i+\frac12 j}}^{n+\frac12} = H_{y_{i+\frac12 j}}^{n-\frac12} - \dfrac{k}{h} m_{i+\frac12 j} \left(-E_{z_{i+1 j}}^{n} + E_{z_{ij}}^{n} \right)
\end{cases}
$$

We would initialize the E_{ij}^{0} with the initial conditions and $H^{-\frac12} = 0$ or I.C. if they are known.

now the computational molecule becomes.



(Yee Algorithm)

$$\begin{cases} E_{z_{ij}}^{n+1} = E_{z_{ij}}^{n} - \frac{k}{h} e_{ij} \left(-H_{y_{i+\frac12 j}}^{n+\frac12} + H_{y_{i-\frac12 j}}^{n+\frac12} + H_{x_{ij+\frac12}}^{n+\frac12} - H_{x_{ij-\frac12}}^{n+\frac12} \right) \\ H_{x_{ij+\frac12}}^{n+\frac12} = H_{x_{ij+\frac12}}^{n-\frac12} - \frac{k}{h} m_{ij+\frac12} \left(E_{z_{ij+1}}^{n} - E_{z_{ij}}^{n} \right) \\ H_{y_{i+\frac12 j}}^{n+\frac12} = H_{y_{i+\frac12 j}}^{n-\frac12} - \frac{k}{h} m_{i+\frac12 j} \left(-E_{z_{i+1 j}}^{n} + E_{z_{ij}}^{n} \right) \end{cases}$$

We would initialize the E_{ij}^{0} with the initial conditions and $H^{-\frac12} = 0$ or I.C. if they are known.

Full 3-D plus Time Difference Methods

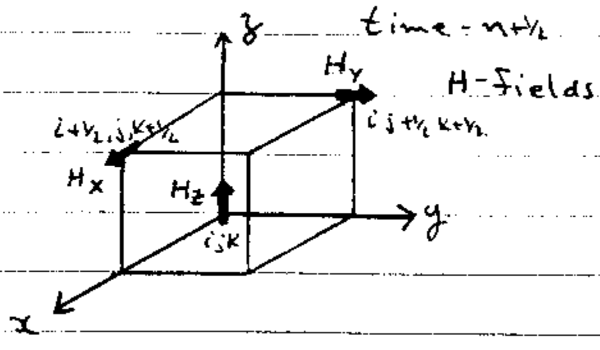
starting with the conservation law:

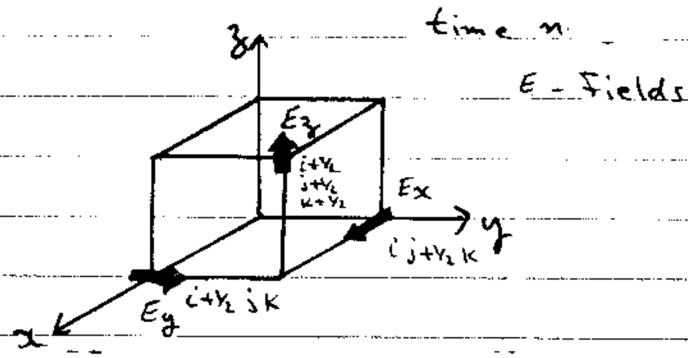
$$\partial_t \underline{u} + A^{-1}\partial_x \underline{E} + A^{-1}\partial_y \underline{F} + A^{-1}\partial_z \underline{G} = \underline{0}$$

expanding:

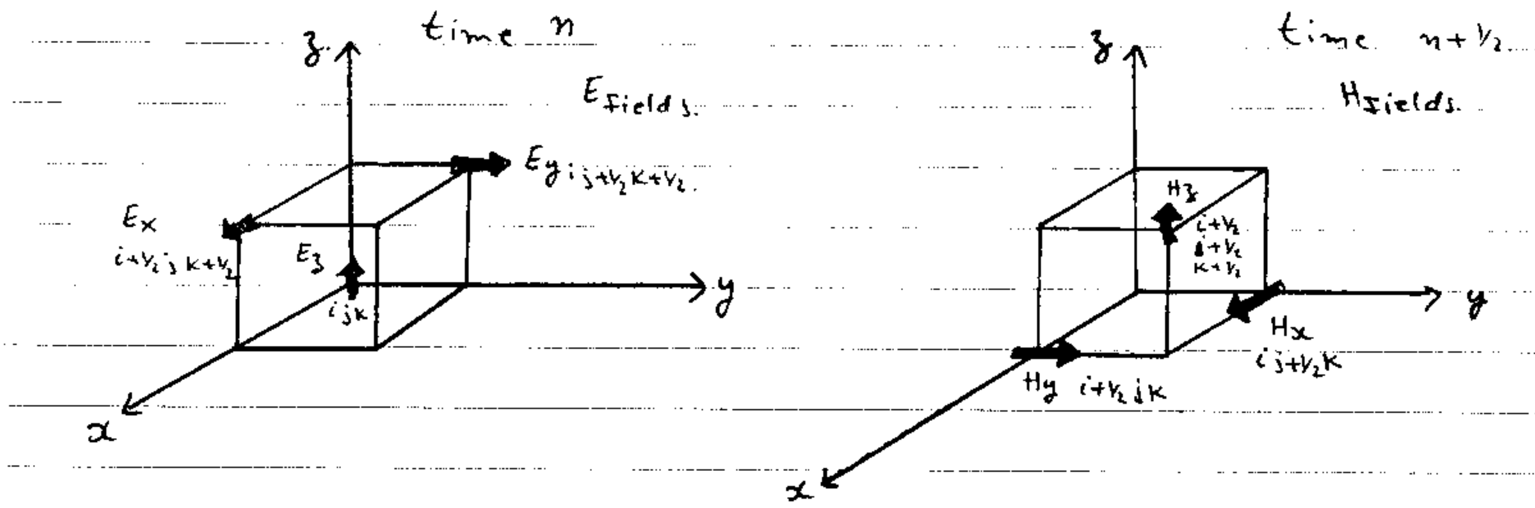
$$\begin{cases} \partial_t E_x + e\,\partial_x(0) + e\,\partial_y(-H_z) + e\,\partial_z(H_y) = 0 \\ \partial_t E_y + e\,\partial_x(H_z) + e\,\partial_y(0) + e\,\partial_z(-H_x) = 0 \\ \partial_t E_z + e\,\partial_x(-H_y) + e\,\partial_y(H_x) + e\,\partial_z(0) = 0 \\ \partial_t H_x + m\,\partial_x(0) + m\,\partial_y(E_z) + m\,\partial_z(-E_y) = 0 \\ \partial_t H_y + m\,\partial_x(-E_z) + m\,\partial_y(0) + m\,\partial_z(E_x) = 0 \\ \partial_t H_z + m\,\partial_x(E_y) + m\,\partial_y(-E_x) + m\,\partial_z(0) = 0 \end{cases}$$

if we difference these equations using the leap-frog method (i.e. centered time and centered space) then again we end up with independent meshes which are interlaced in space and time. We retain only one mesh by "putting" E's on integer time steps and H's on $1/2$ time steps





alternatively :



Note: in first set:

bottom of cubes : TE_z

top of cubes : TM_z

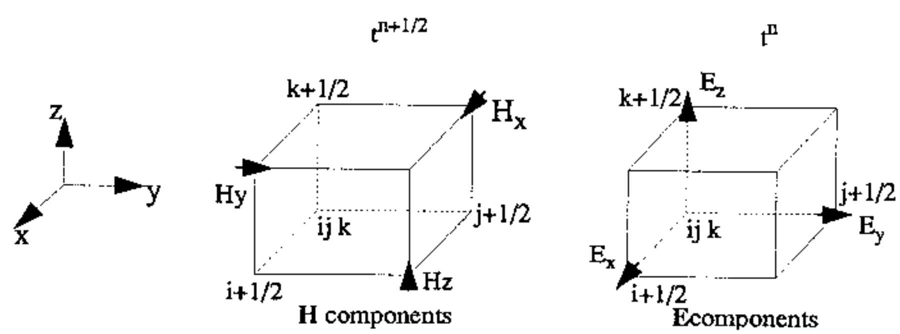
in second set:

bottom of cubes : TM_z

top of cubes : TE_z

it doesn't really make any difference whether we choose one set or the other

a set which will simplify notation is a slight modification of the first:



say we take this last set and formulate the leap-frog scheme with $\Delta x = \Delta y = \Delta z = h \quad \Delta t = k$:

$$
\left\{
\begin{aligned}
E_{x_{i+\frac{1}{2} j k}}^{n+1} &= E_{x_{i+\frac{1}{2} j k}}^{n} - \frac{k}{h} e \Big(H_{z_{i+\frac{1}{2} j-\frac{1}{2} k}}^{n+\frac{1}{2}} - H_{z_{i+\frac{1}{2} j+\frac{1}{2} k}}^{n+\frac{1}{2}} + \\
&\qquad H_{y_{i+\frac{1}{2} j k+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{y_{i+\frac{1}{2} j k-\frac{1}{2}}}^{n+\frac{1}{2}} \Big) \\
E_{y_{i j+\frac{1}{2} k}}^{n+1} &= E_{y_{i j+\frac{1}{2} k}}^{n} - \frac{ke}{h} \Big(H_{z_{i+\frac{1}{2} j+\frac{1}{2} k}}^{n+\frac{1}{2}} - H_{z_{i-\frac{1}{2} j+\frac{1}{2} k}}^{n+\frac{1}{2}} + \\
&\qquad H_{x_{i j+\frac{1}{2} k-\frac{1}{2}}}^{n+\frac{1}{2}} - H_{x_{i j+\frac{1}{2} k+\frac{1}{2}}}^{n+\frac{1}{2}} \Big) \\
E_{z_{i j k+\frac{1}{2}}}^{n+1} &= E_{z_{i j k+\frac{1}{2}}}^{n} - \frac{ke}{h} \Big(H_{y_{i-\frac{1}{2} j k+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{y_{i+\frac{1}{2} j k+\frac{1}{2}}}^{n+\frac{1}{2}} + \\
&\qquad H_{x_{i j+\frac{1}{2} k+\frac{1}{2}}}^{n+\frac{1}{2}} - H_{x_{i j-\frac{1}{2} k+\frac{1}{2}}}^{n+\frac{1}{2}} \Big)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
H_{x_{i j+\frac{1}{2} k+\frac{1}{2}}}^{n+\frac{1}{2}} &= H_{x_{i j+\frac{1}{2} k+\frac{1}{2}}}^{n-\frac{1}{2}} - \frac{km}{h} \Big(E_{z_{i j+1 k+\frac{1}{2}}}^{n} - E_{z_{i j k+\frac{1}{2}}}^{n} + \\
&\qquad E_{y_{i j+\frac{1}{2} k}}^{n} - E_{y_{i j+\frac{1}{2} k+1}}^{n} \Big) \\
H_{y_{i+\frac{1}{2} j k+\frac{1}{2}}}^{n+\frac{1}{2}} &= H_{y_{i+\frac{1}{2} j k+\frac{1}{2}}}^{n-\frac{1}{2}} - \frac{km}{h} \Big(E_{z_{i j k+\frac{1}{2}}}^{n} - E_{z_{i+1 j k+\frac{1}{2}}}^{n} + \\
&\qquad E_{x_{i+\frac{1}{2} j k+1}}^{n} - E_{x_{i+\frac{1}{2} j k}}^{n} \Big) \\
H_{z_{i+\frac{1}{2} j+\frac{1}{2} k}}^{n+\frac{1}{2}} &= H_{z_{i+\frac{1}{2} j+\frac{1}{2} k}}^{n-\frac{1}{2}} - \frac{km}{h} \Big(E_{y_{i+1 j+\frac{1}{2} k}}^{n} - E_{y_{i j+\frac{1}{2} k}}^{n} + \\
&\qquad E_{x_{i+\frac{1}{2} j k}}^{n} - E_{x_{i+\frac{1}{2} j+1 k}}^{n} \Big)
\end{aligned}
\right.
$$

since the $\bar{E}$ and $\bar{H}$ fields exist at different time steps (i.e. n and $n+\frac{1}{2}$) we can reduce the complexity of the above equations by introducing some new notation.
we first split the conservation law into two systems:

$$\begin{cases} \partial_t \underline{u} + A\,\partial_x \underline{E}(\underline{v}) + A\,\partial_y \underline{F}(\underline{v}) + A\,\partial_z \underline{G}(\underline{v}) = \underline{0} \\ \partial_t \underline{v} + B\,\partial_x \underline{E}(\underline{u}) + B\,\partial_y \underline{F}(\underline{u}) + B\,\partial_z \underline{G}(\underline{u}) = \underline{0} \end{cases}$$

$$\begin{cases} \underline{u} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \quad \underline{E}(\underline{v}) = \underline{E} = \begin{pmatrix} 0 \\ H_z \\ -H_y \end{pmatrix} \quad \underline{F}(\underline{v}) = \underline{F} = \begin{pmatrix} -H_z \\ 0 \\ H_x \end{pmatrix} \quad \underline{G}(\underline{v}) = \underline{G} = \begin{pmatrix} H_y \\ -H_x \\ 0 \end{pmatrix} \\ \underline{v} = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} \quad \underline{E}(\underline{u}) = \underline{H} = \begin{pmatrix} 0 \\ -E_z \\ E_y \end{pmatrix} \quad \underline{F}(\underline{u}) = \underline{I} = \begin{pmatrix} E_z \\ 0 \\ -E_x \end{pmatrix} \quad \underline{G}(\underline{u}) = \underline{J} = \begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix} \\ A = \mathrm{diag}\{e\ e\ e\} \qquad \underline{B} = \mathrm{diag}\{m\ m\ m\} \end{cases}$$

discretized notation becomes:

$$\underline{u}^n_{ijk} = \begin{pmatrix} E x^n_{i+\frac{1}{2}\,j\,k} \\ E y^n_{i\,j+\frac{1}{2}\,k} \\ E z^n_{i\,j\,k+\frac{1}{2}} \end{pmatrix} \qquad \underline{v}^n_{ijk} = \begin{pmatrix} H x^{n+\frac{1}{2}}_{i\,j+\frac{1}{2}\,k+\frac{1}{2}} \\ H y^{n+\frac{1}{2}}_{i+\frac{1}{2}\,j\,k+\frac{1}{2}} \\ H z^{n+\frac{1}{2}}_{i+\frac{1}{2}\,j+\frac{1}{2}\,k} \end{pmatrix}$$

$$\underline{E}^n_{ijk} = \underline{E}(\underline{v}^n_{ijk}) \qquad \underline{F}^n_{ijk} = \underline{F}(\underline{v}^n_{ijk}) \qquad \underline{G}^n_{ijk} = \underline{G}(\underline{v}^n_{ijk})$$

$$\underline{H}^n_{ijk} = \underline{E}(\underline{u}^n_{ijk}) \qquad \underline{I}^n_{ijk} = \underline{F}(\underline{u}^n_{ijk}) \qquad \underline{J}^n_{ijk} = \underline{G}(\underline{u}^n_{ijk})$$

- The update procedure is a <u>two-step form</u>:

$$\begin{cases} \underline{u}^{n+1}_{ijk} = \underline{u}^{n}_{ijk} - A\dfrac{k}{h}\left(\underline{E}^{n}_{ijk} - \underline{E}^{n}_{i-1jk} + \underline{F}^{n}_{ijk} - \underline{F}^{n}_{ij-1k} + \underline{G}^{n}_{ijk} - \underline{G}^{n}_{ijk-1} \right) \\[2ex] \underline{v}^{n+1}_{ijk} = \underline{v}^{n}_{ijk} - B\dfrac{k}{h}\left(\underline{H}^{n+1}_{i+1jk} - \underline{H}^{n+1}_{ijk} + \underline{I}^{n+1}_{ij+1k} - \underline{I}^{n+1}_{ijk} + \underline{J}^{n+1}_{ijk+1} - \underline{J}^{n+1}_{ijk} \right) \end{cases}$$

or

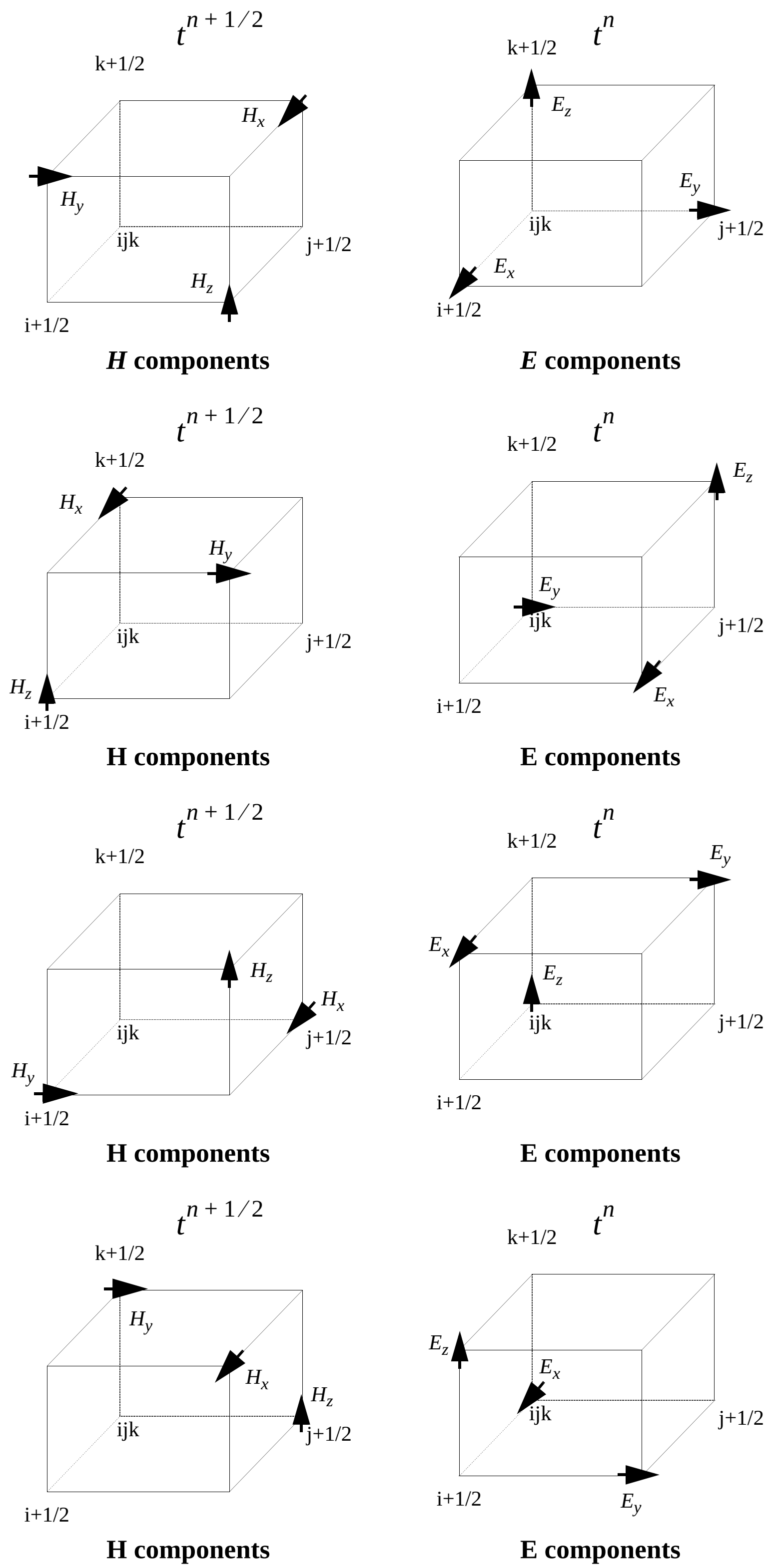
$$\begin{cases} \underline{u}^{n+1} = \underline{u}^{n} - \dfrac{k}{h} A \left(\nabla_x \underline{E}^n + \nabla_y \underline{F}^n + \nabla_z \underline{G}^n \right) \\[2ex] \underline{v}^{n+1} = \underline{v}^{n} - \dfrac{k}{h} B \left(\Delta_x \underline{H}^{n+1} + \Delta_y \underline{I}^{n+1} + \Delta_z \underline{J}^{n+1} \right) \end{cases}$$

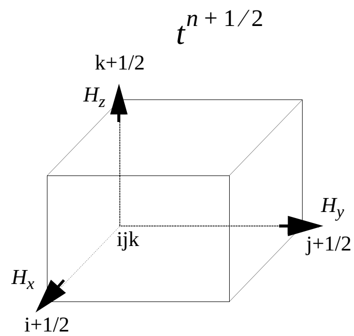
where $\nabla \rightarrow$ backward difference operator

$\Delta \rightarrow$ forward difference operator

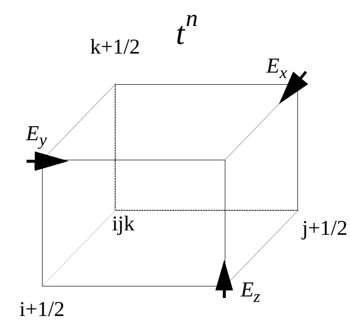
eg: $\nabla_x \underline{E}^n = \underline{E}^n_{ijk} - \underline{E}^n_{i-1jk}$

The Leap-frog scheme can be applied to the Maxwell curl equations and the result is sixteen independent sets of space-time interleaved discretized electric and magnetic fields. Eight of these are shown below. In order to get the remaining 8 the time interlacing between electric and magnetic fields is simply interchanged.

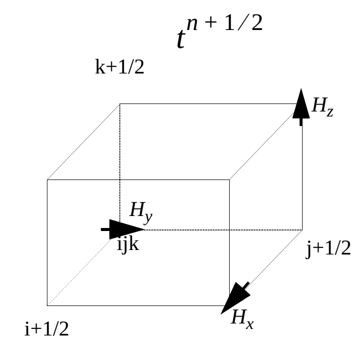




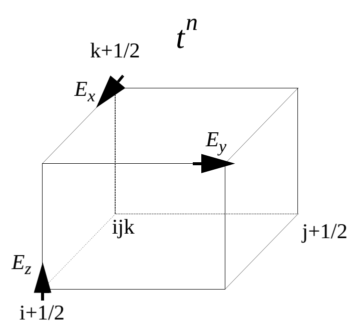
H components



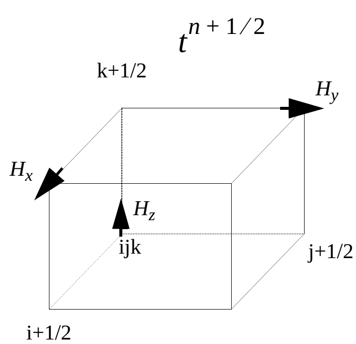
E components



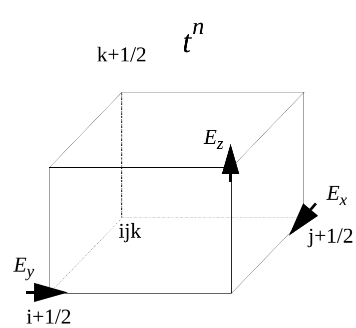
H components



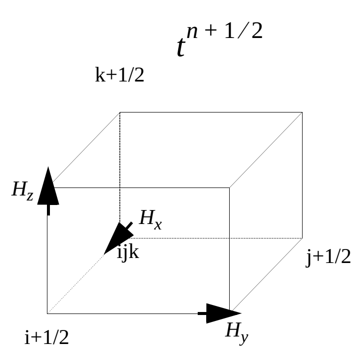
E components



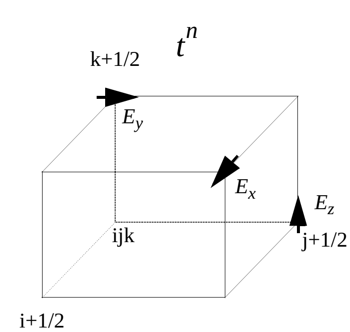
H components



E components



H components



E components