Joe LoVetri
Dept. of Electrical &
Computer Engineering
University of Manitoba
204-474-6295
lovetri@ee.umanitoba.ca

On the Velocity of Transient Plane Waves

When discussing the mathematical representation of a transient plane wave some confusion is apt to result on the definition of velocity. The reason for this confusion is that there is much greater familiarity with time-harmonic representations of plane waves. In this short exposition the nature of a transient plane wave will be described in a relatively general framework. We want to describe transient waves which, if we take a snap-shot in time, are constant along planes in 3-D space and which travel undeformed with a speed given by the speed of light in the medium, say c_0. Therefore, we are assuming a non-dispersive medium having a constant permittivity, ε, and permeability, μ, over frequency and with the speed of light in the medium being given as $c_0 = 1/\sqrt{\varepsilon\mu}$ (also constant over frequency). (We will use speed to denote a scalar quantity and the term velocity to denote a vector quantity. This usage is consistent with standard usage in physics.)

If the plane-wave is propagating in the $\hat{\boldsymbol{a}}$ direction, where $\hat{\boldsymbol{a}}$ is a constant unit vector, then planes perpendicular to $\hat{\boldsymbol{a}}$ can be described by the equation

$$\hat{\boldsymbol{a}} \cdot \boldsymbol{r} = \mathrm{d} \tag{1}$$

where $\boldsymbol{r}$ is the position vector in 3-D space with respect to a set origin, and d is the perpendicular distance from the origin to a particular plane. This relationship is depicted in Fig. 1.

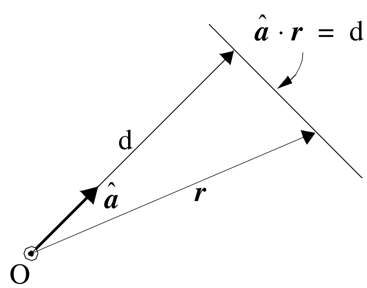

Figure 1. General orientation of a transient plane wave propagating in the $\hat{\boldsymbol{a}}$ direction.

We can represent planes moving in the direction $\hat{\boldsymbol{a}}$ by planes whose perpendicular distance from the origin changes with time, that is

$$\hat{\boldsymbol{a}} \cdot \boldsymbol{r} = \mathrm{d}(t)$$

and we see that the distance from the origin is given by the function of time, $\mathrm{d}(t)$. Notice that the unit vector $\hat{\boldsymbol{a}}$ is dimensionless. Since the planes are travelling with speed c_0 in the direction $\hat{\boldsymbol{a}}$ the velocity is defined as

$$\boldsymbol{v} = c_0\hat{\boldsymbol{a}} \tag{2}$$

and has units of meters per second, m/s, or whatever units c_0 is given in.

Now since the plane wave propagates a distance $c_0 t$ in an amount of time t, the expression

$$c_0 t - \mathrm{d}(t) = c_0 t - \hat{\boldsymbol{a}} \cdot \boldsymbol{r}$$

is a constant which can be used as the argument in the functional description of the transient plane wave. That is, the function

$$f(c_0 t - \hat{\boldsymbol{a}} \cdot \boldsymbol{r})$$

describes a plane wave which at any location $\boldsymbol{r}_0$ has the functional description $f(c_0 t - \hat{\boldsymbol{a}} \cdot \boldsymbol{r}_0)$ and at the origin will be a function of time given by $f(c_0 t)$.

Alternatively, if we want the functional behaviour at the origin to be specified by $f(t)$, then we can use the argument $t - \hat{\boldsymbol{a}} \cdot \boldsymbol{r}/c_0$ or equivalently $t - \boldsymbol{v} \cdot \boldsymbol{r}/c_0^2$. Therefore, the function

$$f(t - \boldsymbol{v} \cdot \boldsymbol{r}/c_0^2) \tag{3}$$

represents plane waves propagating in the direction $\hat{\boldsymbol{a}}$, with speed c_0, velocity $\boldsymbol{v} = c_0\hat{\boldsymbol{a}}$, and having a functional behaviour of $f(t)$ at the origin.

I. Relationship to phase speed

It is traditional to represent time-harmonic plane waves using the functional argument $\psi(t, \boldsymbol{r}) = \omega t - \boldsymbol{k} \cdot \boldsymbol{r}$, also called the *instantaneous phase*, in any of the harmonic functions. For example, it is customary to use the complex function

$$e^{j(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})} = \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) + j\sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) \tag{4}$$

as the prototypical time-harmonic plane wave. Letting $\boldsymbol{k} = k_0\hat{\boldsymbol{a}}$ we see that this represents a plane wave propagating in the $\hat{\boldsymbol{a}}$ direction, *i.e.* the direction of $\boldsymbol{k}$ which is the so called *vector propagation constant*. If we re-write this argument as

$$\omega t - \boldsymbol{k} \cdot \boldsymbol{r} = \omega\left(t - \frac{\boldsymbol{k} \cdot \boldsymbol{r}}{\omega}\right)$$

we can identify $k_0 = \omega/c_0$ as the propagation constant and the vector propagation constant is related to the velocity as $\boldsymbol{k} = (k_0/c_0)\boldsymbol{v}$. Note that in terms of the individual cartesian components we have

$$k_x = \frac{v_x}{c_0}k_0,\ k_y = \frac{v_y}{c_0}k_0,\ k_z = \frac{v_z}{c_0}k_0 \tag{5}$$

with

$$k_0 = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{k_0}{c_0}\sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{k_0}{c_0}c_0 . \tag{6}$$

The components of the velocity are written in terms of the components of the vector propagation constant as

$$v_x = c_0\frac{k_x}{k_0},\ v_y = c_0\frac{k_y}{k_0},\ v_z = c_0\frac{k_z}{k_0} . \tag{7}$$

Now, following standard phasor notation, the general *complex wave function* is represented as

$$e^{-j(\boldsymbol{k}\cdot\boldsymbol{r})} = e^{-j\Phi(\boldsymbol{r})}$$

where any of the time-harmonic functions can be recovered by multiplying with $e^{j\omega t}$ and taking the real or imaginary part as required. The spatial function $\Phi(\boldsymbol{r})$ is called the *phase*. For a change in position given by

$$d\boldsymbol{r} = dx\hat{\boldsymbol{a}}_x + dy\hat{\boldsymbol{a}}_y + dz\hat{\boldsymbol{a}}_z$$

the corresponding change in phase is

$$d\Phi = \nabla\Phi\cdot d\boldsymbol{r} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz .$$

A surface of constant phase is defined by $d\Phi = 0$, which means that $d\boldsymbol{r}$ must be perpendicular to $\nabla\Phi$. For the case $\Phi(\boldsymbol{r}) = \boldsymbol{k}\cdot\boldsymbol{r}$ we have $\nabla\Phi = \boldsymbol{k}$ and therefore on plane surfaces perpendicular to $\boldsymbol{k}$ the phase will be constant.

For a change in time and space given by dt and $d\boldsymbol{r}$, respectively, the corresponding change in the instantaneous phase is given by

$$d\psi = \omega dt - \nabla\Phi\cdot d\boldsymbol{r} . \tag{8}$$

This will be zero along space-time surfaces defined by

$$0 = \omega dt - \nabla\Phi\cdot d\boldsymbol{r}$$

or when

$$\omega = \nabla\Phi\cdot\frac{d\boldsymbol{r}}{dt} . \tag{9}$$

The term $d\boldsymbol{r}/dt$ defines the rate of change of position along which the instantaneous phase is zero and can therefore be defined as the velocity of the constant phase surfaces, $\boldsymbol{v} = d\boldsymbol{r}/dt$, when $\boldsymbol{r}(t)$ satisfies the above equation. For example, for plane waves $\nabla\Phi = \boldsymbol{k}$ and we have

$$\omega = \boldsymbol{k}\cdot\frac{d\boldsymbol{r}}{dt} = \boldsymbol{k}\cdot\boldsymbol{v} = k_0c_0(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{a}}) = k_0c_0 \tag{10}$$

as before. Equation (9) should be thought of as the equation which defines the surfaces of constant instantaneous phase.

Another interpretation of the meaning of (9) can be made by re-writing it as

$$\omega = \nabla\Phi \cdot \frac{d\boldsymbol{r}}{dt} = |\nabla\Phi| c_\theta \cos\theta \tag{11}$$

where c_θ is the speed of the surface of constant phase in a direction which subtends an angle θ with $\nabla\Phi$. Thus we have

$$c_\theta = \frac{\omega}{|\nabla\Phi|\cos\theta} \tag{12}$$

which is minimum in the direction of $\nabla\Phi$ where $\theta = 0$. For plane waves, $|\nabla\Phi| = k_0$ and $c_0 = \omega/k_0$ is the magnitude of the velocity, but at a direction perpendicular to $\nabla\Phi$ we have the speed $c_{\pi/2} = \infty$.

Now it is traditional (see Harrington p. 86 [1]) to define *phase speed* as follows:

> "The *phase velocity*[1] of a wave in a given direction is defined as the velocity of surfaces of constant phase in that direction."

which is just a definition which follows the interpretation given by (12). Thus, in our present notation, the phase speeds along the cartesian coordinates are defined by

$$v_{xp} = \frac{\omega}{\partial\Phi/\partial x} = \frac{\omega}{k_x},\ v_{yp} = \frac{\omega}{\partial\Phi/\partial y} = \frac{\omega}{k_y},\ v_{zp} = \frac{\omega}{\partial\Phi/\partial z} = \frac{\omega}{k_z} \tag{13}$$

but these are *not* the components of a vector and should not be associated with the components of the velocity $\boldsymbol{v} = c_0\hat{\boldsymbol{a}}$. All these tell us are the rates of change of the instantaneous phase along the cartesian coordinates. Note that for the case where one of the components of the wave vector goes to zero the corresponding phase speed goes to *infinity* using this definition! For example, if $k_x = 0$ then $v_{xp} = \infty$ whereas the true x component of the velocity is $v_x = 0$. Note also that

$$\sqrt{v_{xp}^2 + v_{yp}^2 + v_{zp}^2} \neq c_0$$

confirming the assertion that the phase speed is not a vector. This does *not* imply that we cannot define a velocity which is indeed a vector as we have above.

II. References

[1] R.F. Harrington, "Time-Harmonic Electromagnetic Fields," McGraw-Hill Book Company, 1961.

---

1. Note that Harrington is using the term velocity inappropriately since the term *velocity* is reserved for a vector quantity while the term *speed* is used for a scalar quantity.