# Spectral collocation in space and time for linear PDEs 

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## A R T I C L E I N F O

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#### Abstract

Spectral methods solve elliptic partial differential equations (PDEs) numerically with errors bounded by an exponentially decaying function of the number of modes when the solution is analytic. For time dependent problems, almost all focus has been on low-order finite difference schemes for the time derivative and spectral schemes for spatial derivatives. Spectral methods that converge spectrally in both space and time have appeared recently. This paper is a continuation of the authors' previous works on Legendre and Chebyshev space-time methods for the heat equation. Here space-time spectral collocation methods for the Schrodinger, wave, Airy and beam equations are proposed and analyzed. In particular, a condition number estimate of each global Chebyshev space-time operator is shown. The analysis requires new estimates of eigenvalues of some spectral derivative matrices. In particular, it is shown that the real part of every eigenvalue of the thirdorder Chebyshev derivative matrix is positive and bounded away from zero, settling a twenty-year-old conjecture. Similarly, the real part of every eigenvalue of the fourthorder Chebyshev derivative matrix with Dirichlet boundary conditions is shown to be also positive and bounded away from zero. Numerical results verify the theoretical results, and demonstrate that the space-time methods also work well for some common nonlinear PDEs.


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## 1. Introduction

Spectral methods have been used successfully to solve elliptic PDEs for many decades. If the solution is analytic, the numerical solution converges exponentially as a function of the number of spectral modes. For time dependent PDEs, the most common approach is to use low-order finite difference approximation of the time derivative and spectral approximation of the spatial derivatives. This is not ideal since the time discretization error overwhelms the spatial discretization error. In two earlier papers [1] and [2], a Legendre and Chebyshev collocation method in both space and time based on the work of Tang and Xu [3] were analyzed for the heat equation. The methods were shown to converge spectrally when the solution is analytic. A condition number estimate of $O\left(N^{4}\right)$ for the global space-time operators was derived, where $N$ is the number of spectral modes in each direction. A second space-time method which is easier to implement and has similar performance was also proposed and studied. Two nonlinear PDEs, viscous Burgers' and Allen-Cahn were successfully solved numerically, hinting that these methods are also effective solvers for nonlinear PDEs.

[^0]The purpose of this paper is to propose and analyze a space-time Chebyshev collocation method for other canonical linear PDEs including the Schrodinger, wave, Airy and beam equations. A condition number estimate of the global spacetime operator will be given for each PDE. In the course of the analysis, we have been able to show that the real part of every eigenvalue of the third-order and fourth-order Chebyshev derivative operators is positive and bounded away from zero. The estimate for the third-order derivative has been open for more than 20 years. See Propositions 4.10 and 4.13. The estimate for the condition number of the space-time Chebyshev method for the beam equation is still incomplete because of our inability to show that the spectrum of the fourth-order Chebyshev derivative matrix is real. This problem has remained open also for 20 years. Space-time spectral convergence for the Schrodinger and wave equations are also shown in this paper. Numerical experiments verify the theoretical results, and further demonstrate that these methods can also solve common nonlinear PDEs such as a nonlinear reaction diffusion equation in combustion, nonlinear Schrodinger, Sine-Gordon, KdV, Kuramoto-Shivashinsky and Cahn-Hilliard equations. [1], [2] and the present paper are the only ones in the literature to address the condition number of discrete global space-time operators.

One drawback of these methods is that time stepping is no longer possible. The unknowns for all times must be solved at the same time. This presents a serious problem for PDEs in three spatial dimensions and is particularly onerous for nonlinear PDEs. It should be made clear that due to the spectral convergence, many fewer unknowns are needed compared to finite difference/element schemes for the same error tolerance.

An early work on spectrally accurate ordinary differential equation (ODE) solvers is [4]. Among the first works on spacetime spectral methods for PDEs with periodic boundary conditions include [5] and [6]. Other references include [7-16] and the references therein. Of course, this list is incomplete. See [1] for additional papers that address space-time spectral methods and papers that attempt to couple space and time components for faster computations. We wish to add one more reference [17] that gives a good survey of algorithms that are parallel in time.

In the next section, we give the notation used in this paper and recall some basic estimates used in the analysis. Following that, a space-time Chebyshev collocation method, the, so-called, second method in [1] and [2], for the 1D Schrodinger, wave, Airy and beam equations are introduced. A condition number estimate of the method for each PDE is shown in Section 4. Basically, the condition number is bounded by a multiple of the condition number of the spectral approximation of the associated spatial differential operator. In Section 5, space-time spectral convergence for the Schrodinger and wave equations are discussed. After that section, some simple iterative schemes for five nonlinear PDEs (nonlinear reaction diffusion from combustion, Sine-Gordon, KdV, Kuramoto-Sivashinsky and Cahn-Hilliard equations) are briefly discussed. Numerical experiments in MATLAB are shown in Section 7, confirming the theoretical results. In the final section, a conclusion and some future work are outlined.

## 2. Notation and basic estimates

Below, we summarize our matrix notation to be followed by notation pertaining to spectral methods. Let $I_{n}$ denote the $n \times n$ identity matrix. For an $n \times n$ matrix $M$, let [ $M$ ] denote the $(n-1) \times(n-1)$ matrix obtained from $M$ by deleting the last column and row, while $\llbracket M \rrbracket$ denotes the $(n-2) \times(n-2)$ matrix obtained from $M$ by deleting the first and last columns and rows. For any complex number $a$, its complex conjugate is denoted by $\bar{a}$ and its real and imaginary parts are denoted by $\operatorname{Re} a$ and $\operatorname{Im} a$, respectively. For any matrix $M$, let $M^{T}$ and $M^{*}$ denote the transpose and complex conjugate transpose of $M$, respectively. Let $|\cdot|_{2}$ denote the vector/matrix 2 -norm and $|\cdot|_{\infty}$ denote the vector $\infty$-norm. For positive integers $m, n$ and vector $a \in \mathbb{C}^{m n}$, let $A \in \mathbb{C}^{m \times n}$ be the matrix representation of $a$, that is, the columns of $A$ stacked on top of one another form $a$. The notation is $a=\operatorname{vec}(A)$. Finally, $\otimes$ denotes tensor product. For matrices $X \in \mathbb{C}^{N \times N}, Y \in \mathbb{C}^{M \times M}$ and $z \in \mathbb{C}^{M N}$, recall that $(X \otimes Y) z=\operatorname{vec}\left(Y Z X^{T}\right)$, where $\operatorname{vec}(Z)=z$, the vector representation of $Z$. For any vector $v$, denote by $\operatorname{diag}(v)$ the diagonal matrix whose diagonal entries are elements of $v$. Throughout, $C, c$ denote positive constants whose values may differ at different occurrences, but are independent of $N$, the spatial and temporal dimension.

Fix a positive integer $N$. Let $P_{N}$ denote the space of polynomials of degree at most $N$. For polynomials in two variables $x$ and $t, P_{N}$ denotes polynomials in $x$ of degree at most $N$ for a fixed $t$, and in $t$ of degree at most $N$ for a fixed $x$. Let $x_{0}, \ldots, x_{N}$ denote the Chebyshev Gauss-Lobatto nodes with $x_{0}=1, x_{N}=-1$ and $x_{j}$ descending zeros of $T_{N}^{\prime}(x)$, where $1 \leq j \leq N-1$ and $T_{N}$ is the $N$ th Chebyshev polynomial. The Chebyshev Gauss-Lobatto nodes along the $t$ axis are denoted by $\left\{t_{k}\right\}$. Let

$$
x_{h}=\left[\begin{array}{c}
x_{1} \\
\vdots \\
x_{N-1}
\end{array}\right], \quad t_{h}=\left[\begin{array}{c}
t_{0} \\
\vdots \\
t_{N-1}
\end{array}\right]
$$

Note that $x_{h}$ excludes both boundary points, while $t_{h}$ excludes only the initial point -1 . For $0 \leq j \leq N$, let $\ell_{j}$ be the Lagrange interpolant, a polynomial of degree $N$, of $x_{j}$ so that $\ell_{j}\left(x_{k}\right)=\delta_{j k}$. Recall that the Chebyshev pseudospectral derivative matrix $D \in \mathbb{R}^{(N+1) \times(N+1)}$ has entries

$$
D_{j k}=\frac{d \ell_{k}\left(x_{j}\right)}{d x}, \quad 0 \leq j, k \leq N
$$

Let $d_{h}=D(0: N-1, N)$, the first $N$ entries of the last column of $D$. Define the Chebyshev interpolation operator as usual: for any continuous $u$,

$$
\begin{equation*}
\mathcal{I}_{N} u=\sum_{j=0}^{N} u\left(x_{j}\right) \ell_{j} \tag{2.1}
\end{equation*}
$$

The following is an important property of Chebyshev quadrature: for any polynomial $v$ of degree at most $2 N-1$,

$$
\begin{equation*}
\int_{-1}^{1} v(x) w(x) d x=\sum_{k=0}^{N} v\left(x_{k}\right) \rho_{k}, \quad w(x)=\frac{1}{\sqrt{1-x^{2}}} \tag{2.2}
\end{equation*}
$$

where $\left\{\rho_{k}\right\}$ is the set of weights associated with Chebyshev Gauss-Lobatto quadrature. Let $W_{h}$ be the $(N+1) \times(N+1)$ diagonal matrix whose diagonal entries are $\left\{\rho_{k}\right\}$ and $W=\llbracket W_{h} \rrbracket \otimes\left[W_{h}\right]$.

Denote the weighted $L^{2}$ norm of a continuous function $v$ on $\Omega:=(-1,1)^{2}$ by

$$
\|v\|:=\left(\int_{\Omega}|v(x, t)|^{2} w(x) w(t) d x d t\right)^{1 / 2}
$$

Also, define the corresponding discrete norm

$$
\|v\|_{N}:=\left(\sum_{j, k=0}^{N} \rho_{j} \rho_{k}\left|v\left(x_{j}, t_{k}\right)\right|^{2}\right)^{1 / 2}
$$

It is well known (inequality (5.3.2) in [18], for instance) that the weighted $L^{2}$ and discrete norms are equivalent for all polynomials $v$ of degree at most $N$ :

$$
\begin{equation*}
\|v\| \leq\|v\|_{N} \leq 2\|v\| \tag{2.3}
\end{equation*}
$$

In case $v$ is a function of one variable, we also write

$$
\|v\|=\left(\int_{-1}^{1}|v(x)|^{2} w(x) d x\right)^{1 / 2}
$$

The space of all $v$ for which $\|v\|^{2}+\left\|v^{\prime}\right\|^{2}<\infty$ is denoted by $H_{w}^{1}(-1,1)$, while $H_{0, w}^{1}(-1,1)$ is the closure of $C_{0}^{\infty}(-1,1)$ with respect to the $H_{w}^{1}(-1,1)$ norm. Similarly define $H_{0, w}^{2}(-1,1)$ as the closure of $C_{0}^{\infty}(-1,1)$ with respect to the $H_{w}^{2}(-1,1)$ norm. The following are well known inequalities for all $v \in P_{N}$ and $z \in H_{0, w}^{1}(-1,1)$ :

$$
\begin{aligned}
& \left\|v^{\prime}\right\| \leq c N^{2}\|v\| \quad \text { (inverse estimate) } \\
& \sup _{x \in[-1,1]}|v(x)| \leq c N^{1 / 2}\|v\| \quad \text { (trace inequality) }
\end{aligned}
$$

and

$$
\|z\| \leq c\left\|z^{\prime}\right\| \quad \text { (weighted Poincaré inequality). }
$$

[19], [20], [18] and [21] are four excellent references on spectral methods.

## 3. Linear PDEs

In [1] and [2], space-time spectral methods for the heat equation were examined. Now we consider other common linear PDEs in applications: Schrodinger, wave, Airy and beam equations. We treat the simplest case where the spatial and temporal domains are both $(-1,1)$. This is no loss of generality since this can always be accomplished by a simple change of variables. In cases where this may not be appropriate, the method can be repeatedly applied over several time intervals, for instance. Since the purpose of the paper is an analysis of the method, we shall not dwell on these and other refinements.

### 3.1. Schrodinger equation

The linear Schrodinger equation is

$$
u_{t}=i u_{x x}+f(x, t), \text { on }(-1,1)^{2}
$$

with boundary conditions $u( \pm 1, t)=0$ and initial condition $u(x,-1)=u_{0}(x)$. Here $i=\sqrt{-1}$. We seek a numerical solution in $P_{N}$ at $t=1$. The following space-time Chebyshev collocation method is analogous to the method for the heat equation studied in [1]:

$$
\left(I_{N+1} \otimes D\right) u_{h}=i\left(D^{2} \otimes I_{N+1}\right) u_{h}+f_{h}
$$

where $f_{h}$ is the vector of $f$ evaluated at the (spatial and temporal) collocation points. Of course, since $u_{h}$ vanishes at the boundary $x= \pm 1$ and the initial value of $u$ is known at $t=-1$, it is sufficient to solve for the unknowns $\hat{u}_{h}$, which is $u_{h}$ deleting the components corresponding to boundary points and initial points. The spectral equations become

$$
A_{s} \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where

$$
\begin{equation*}
A_{s}=\left(I_{N-1} \otimes[D]\right)-i\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right) \tag{3.1}
\end{equation*}
$$

### 3.2. Wave equation

Consider the linear wave equation

$$
u_{t t}=u_{x x}+f(x, t), \text { on }(-1,1)^{2}
$$

with boundary conditions $u( \pm 1, t)=0$ and initial conditions $u(x,-1)=u_{0}(x)$ and $u_{t}(x,-1)=u_{1}(x)$. We seek a numerical solution in $P_{N}$ at $t=1$. First write the PDE as a first order system for $v=\left[v_{1}, v_{2}\right]^{T}:=\left[u, u_{t}\right]^{T}$

$$
v_{t}=\left[\begin{array}{cc}
0 & I \\
\partial_{x x} & 0
\end{array}\right] v+\left[\begin{array}{l}
0 \\
f
\end{array}\right], \quad v( \pm 1, t)=0, v(x,-1)=\left[\begin{array}{l}
u_{0}(x) \\
u_{1}(x)
\end{array}\right]
$$

For $j=1,2$, let $v_{j h}$ be the vector of $v_{j}$ evaluated at the collocation points. The spectral equations in matrix form are

$$
\left[\begin{array}{cc}
I_{N+1} \otimes D & 0 \\
0 & I_{N+1} \otimes D
\end{array}\right]\left[\begin{array}{l}
v_{1 h} \\
v_{2 h}
\end{array}\right]=\left[\begin{array}{cc}
0 & I_{N+1} \otimes I_{N+1} \\
D^{2} \otimes I_{N+1} & 0
\end{array}\right]\left[\begin{array}{l}
v_{1 h} \\
v_{2 h}
\end{array}\right]+\left[\begin{array}{c}
0 \\
f_{h}
\end{array}\right]
$$

where $f_{h}$ is $f$ evaluated at the collocation points. Again, since the solution vanishes at the boundary and the initial values are known, it is only necessary to solve for a subset of those values. Using the ^ notation to denote vectors stripping away those corresponding to boundary and initial points, the spectral equations are

$$
\left[\begin{array}{cc}
I_{N-1} \otimes[D] & 0 \\
0 & I_{N-1} \otimes[D]
\end{array}\right]\left[\begin{array}{l}
\hat{v}_{1 h} \\
\hat{v}_{2 h}
\end{array}\right]=\left[\begin{array}{cc}
0 & I_{N-1} \otimes I_{N} \\
\llbracket D^{2} \rrbracket \otimes I_{N} & 0
\end{array}\right]\left[\begin{array}{c}
\hat{v}_{1 h} \\
\hat{v}_{2 h}
\end{array}\right]+\left[\begin{array}{c}
0 \\
\hat{f}_{h}
\end{array}\right]-\left[\begin{array}{l}
u_{0 h} \otimes d_{h} \\
u_{1 h} \otimes d_{h}
\end{array}\right],
$$

where $u_{0 h}$ and $u_{1 h}$ are $u_{0}$ and $u_{1}$ evaluated at the interior collocation points. From the first equation, it follows that

$$
\hat{v}_{2 h}=\left(I_{N-1} \otimes[D]\right) \hat{v}_{1 h}+\left(u_{0 h} \otimes d_{h}\right)
$$

Substitute this into the second equation to get, after some algebra, the final spectral equation

$$
A_{w} \hat{v}_{1 h}=\hat{f}_{h}-\left(u_{0 h} \otimes\left([D] d_{h}\right)\right)-\left(u_{1 h} \otimes d_{h}\right)
$$

where

$$
\begin{equation*}
A_{w}=\left(I_{N-1} \otimes[D]^{2}\right)-\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right) \tag{3.2}
\end{equation*}
$$

### 3.3. Airy equation

Consider the Airy equation

$$
u_{t}+u_{x x x}=f(x, t), \text { on }(-1,1)^{2}
$$

with boundary conditions $u( \pm 1, t)=0=u_{x}(1, t)$ and initial condition $u(x,-1)=u_{0}(x)$. We seek a numerical solution in $P_{N}$ at $t=1$. The spectral equations are

$$
\left(I_{N+1} \otimes D\right) u_{h}+\left(D^{3} \otimes I_{N+1}\right) u_{h}=f_{h}
$$

where $f_{h}$ is the vector of $f$ evaluated at the collocation points. Let us define the spectral approximation of the third derivative, taking into account the boundary conditions.

Let $Y=Y(x)$ be a polynomial so that $Y( \pm 1)=0=Y^{\prime}(1)$. Let $Z$ vanish at $\pm 1$ so that $Y(x)=(1-x) Z(x)$. Note that $Y$ clearly satisfies all the boundary conditions. A simple calculation leads to

$$
\begin{equation*}
Y^{\prime \prime \prime}(x)=(1-x) Z^{\prime \prime \prime}(x)-3 Z^{\prime \prime}(x) \tag{3.3}
\end{equation*}
$$

It should be clear now that a spectral approximation of the third derivative satisfying the three boundary conditions is

$$
\begin{equation*}
B_{1}:=\left(M \llbracket D^{3} \rrbracket-3 \llbracket D^{2} \rrbracket\right) M^{-1} \tag{3.4}
\end{equation*}
$$

where $M$ is an $(N-1) \times(N-1)$ diagonal matrix whose diagonal entries are $1-x_{j}, 1 \leq j \leq N-1$. The resulting spectral equations are

$$
\left(I_{N-1} \otimes[D]\right) \hat{u}_{h}+\left(B_{1} \otimes I_{N}\right) \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where $u_{0 h}$ is $u_{0}$ evaluated at the (interior spatial) collocation points, $\hat{u}_{h}$ is $u_{h}$ deleting the components corresponding to boundary points and initial points and $\hat{f}_{h}$ is $f_{h}$ removing the components corresponding to boundary points and initial points. The linear equation to be solved becomes

$$
A_{a} \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where

$$
\begin{equation*}
A_{a}=\left(I_{N-1} \otimes[D]\right)+\left(B_{1} \otimes I_{N}\right) \tag{3.5}
\end{equation*}
$$

See [22] for an alternative Petrov-Galerkin formulation of the third derivative operator.

### 3.4. Beam equation

Finally, consider the fourth-order beam equation

$$
u_{t t}+u_{x x x x}=f(x, t), \text { on }(-1,1)^{2}
$$

with boundary conditions $u( \pm 1, t)=0=u_{x}( \pm 1, t)$ and initial conditions $u(x,-1)=u_{0}(x)$ and $u_{t}(x,-1)=u_{1}(x)$. We seek a numerical solution in $P_{N}$ at $t=1$. As in the wave equation, we write the PDE as a first-order system for $v=\left[v_{1}, v_{2}\right]^{T}:=$ $\left[u, u_{t}\right]^{T}$,

$$
v_{t}=\left[\begin{array}{cc}
0 & I \\
-\partial_{x x x x} & 0
\end{array}\right] v+\left[\begin{array}{l}
0 \\
f
\end{array}\right], \quad v( \pm 1, t)=0, v(x,-1)=\left[\begin{array}{l}
u_{0}(x) \\
u_{1}(x)
\end{array}\right] .
$$

Using the same notation as before, the spectral equations in matrix form are

$$
\left[\begin{array}{cc}
I_{N+1} \otimes D & 0 \\
0 & I_{N+1} \otimes D
\end{array}\right]\left[\begin{array}{c}
v_{1 h} \\
v_{2 h}
\end{array}\right]=\left[\begin{array}{cc}
0 & I_{N+1} \otimes I_{N+1} \\
-D^{4} \otimes I_{N+1} & 0
\end{array}\right]\left[\begin{array}{c}
v_{1 h} \\
v_{2 h}
\end{array}\right]+\left[\begin{array}{c}
0 \\
f_{h}
\end{array}\right]
$$

Again, the components of $v_{j h}$ along the boundary and initial points must be removed. However, it is not as simple as before since Neumann boundary conditions must also be imposed. There are at least three ways to do this. One is to impose the boundary conditions explicitly as constraints, as in spectral tau methods. A second approach to approximate the fourth derivative is to write $Y(x)=\left(1-x^{2}\right) Z(x)$, so that $Y$ automatically satisfies the boundary conditions if $Z$ vanishes at the boundary. Then

$$
\begin{equation*}
Y^{\prime \prime \prime \prime}(x)=\left(1-x^{2}\right) Z^{\prime \prime \prime \prime}(x)-8 x Z^{\prime \prime \prime}(x)-12 Z^{\prime \prime}(x) \tag{3.6}
\end{equation*}
$$

The spectral approximation of the fourth derivative, taking into account of the boundary conditions, is

$$
\begin{equation*}
B_{2}:=\left(M \llbracket D^{4} \rrbracket-8 X \llbracket D^{3} \rrbracket-12 \llbracket D^{2} \rrbracket\right) M^{-1} \tag{3.7}
\end{equation*}
$$

where $M$ and $X$ are $(N-1) \times(N-1)$ diagonal matrices with diagonal entries $1-x_{j}^{2}$ and $x_{j}$, respectively. See, for instance, [20]. Another approach, suggested in [23], gives a symmetric matrix approximation of the fourth derivative accommodating the boundary conditions. There is no particular advantage in the current application since the discrete time derivative is not symmetric. This last approach appears to only work for Legendre collocation and not for Chebyshev collocation. For these reasons, we apply the second approach.

The spectral equations for $\hat{v}_{j h}$, which is $v_{j h}$ removing the variables corresponding to boundary and initial points, become

$$
\left[\begin{array}{cc}
I_{N-1} \otimes[D] & 0 \\
0 & I_{N-1} \otimes[D]
\end{array}\right]\left[\begin{array}{l}
\hat{v}_{1 h} \\
\hat{v}_{2 h}
\end{array}\right]=\left[\begin{array}{cc}
0 & I_{N-1} \otimes I_{N} \\
-B_{2} \otimes I_{N} & 0
\end{array}\right]\left[\begin{array}{l}
\hat{v}_{1 h} \\
\hat{v}_{2 h}
\end{array}\right]+\left[\begin{array}{c}
0 \\
f_{h}
\end{array}\right]-\left[\begin{array}{l}
u_{0 h} \otimes d_{h} \\
u_{1 h} \otimes d_{h}
\end{array}\right]
$$

From the first equation, it follows that

$$
\hat{v}_{2 h}=\left(I_{N-1} \otimes[D]\right) \hat{v}_{1 h}+\left(u_{0 h} \otimes d_{h}\right)
$$

Substitute this into the second equation to get the final spectral equations:

$$
A_{b} \hat{v}_{1 h}=\hat{f}_{h}-\left(u_{0 h} \otimes\left([D] d_{h}\right)\right)-\left(u_{1 h} \otimes d_{h}\right)
$$

where

$$
\begin{equation*}
A_{b}=\left(I_{N-1} \otimes[D]^{2}\right)+\left(B_{2} \otimes I_{N}\right) \tag{3.8}
\end{equation*}
$$

## 4. Condition number estimates

In this section, we estimate the condition number of the spectral approximations of the differential operators of the previous section. As is commonly done, we estimate the, so-called, spectral condition number, defined by

$$
\kappa(M)=\frac{\max _{\lambda \in \Lambda(M)}|\lambda|}{\min _{\lambda \in \Lambda(M)}|\lambda|},
$$

where $\Lambda(M)$ is the spectrum of matrix $M$.
In numerical analysis, the 2-norm condition number $\kappa_{2}(M):=|M|\left|M^{-1}\right|$ quantities the well-posedness of any linear system with coefficient matrix $M$. For a general matrix $M$, of course $\kappa(M) \leq \kappa_{2}(M)$. If $M$ is symmetric, then $\kappa_{2}(M)=\kappa(M)$. For all matrices appearing in spectral methods known to the authors, $\kappa(M)=O\left(\kappa_{2}(M)\right)$. Because the spectral condition number is usually easier to estimate, this explains its popularity. The spectral condition number is also useful for the analysis of preconditioned systems. Consider the simplest case of SPD matrices $A$ and $M$. It is not difficult to show (see Theorem 4.10 in [23], for instance) that $\kappa_{M}\left(M^{-1} A\right)=\kappa\left(M^{-1} A\right)$, where $\kappa_{M}(B)=|B|_{M}\left|B^{-1}\right|_{M}$ with $|x|_{M}:=\sqrt{x^{T} M x}$. Hence the spectral condition number of the preconditioned matrix $M^{-1} A$ is the same as its $M$-norm condition number. See [24] for the use of the spectral condition number for preconditioning the third-order spectral derivative matrix.

As we shall see, the spectral condition number of each discrete spectral operator in this paper scales like that of the corresponding elliptic part of the operator.

We state some lemmas (see [2]) which will be needed in the analysis. The first improves upon an earlier result of [25], while the second one is mostly well known. Both deal with spectral properties of $[D]$.

Lemma 4.1. Let $N \geq 1$. Then the real part of every eigenvalue of $[D]$ is larger than some positive constant independent of $N$.
Lemma 4.2. Let $N \geq 1$ and $\lambda$ be an eigenvalue of $[D]$. Then $|\lambda| \leq c N^{2}$.
The next lemma is well known; see [26] or inequality (7.3.5) in [18], for instance.
Lemma 4.3. Let $N \geq 2$. Then the eigenvalues of $-\llbracket D^{2} \rrbracket$ are real, bounded below by $c$ and above by $C N^{4}$, where $c$ and $C$ are positive and independent of $N$.

### 4.1. Schrodinger equation

Theorem 4.4. Let $N \geq 2$. Let $A_{s}$ be the Chebyshev spectral collocation matrix defined by (3.1) and $\lambda$ be any eigenvalue of $A_{s}$. Then

$$
c \leq|\lambda| \leq C N^{4}
$$

Consequently

$$
\kappa\left(A_{s}\right) \leq C N^{4}
$$

Proof. From (3.1), $\lambda=\gamma+i \mu$, where $\gamma$ is some eigenvalue of [ $D$ ] and $\mu$ is some eigenvalue of $-\llbracket D^{2} \rrbracket$. Write $\gamma=\gamma_{r}+i \gamma_{i}$, where $\gamma_{r}$ and $\gamma_{i}$ are real. From Lemmas 4.1, 4.2 and 4.3, $\gamma_{r} \geq c,|\gamma| \leq c N^{2}$ and $c \leq \mu \leq C N^{4}$ for some positive constants $c, C$ independent of $N$. Thus

$$
|\lambda|^{2}=\gamma_{r}^{2}+\left(\gamma_{i}+\mu\right)^{2} \leq c^{2}+\left(c N^{2}+C N^{4}\right)^{2} \leq C_{1} N^{8}
$$

and

$$
|\lambda|^{2} \geq c^{2}
$$

or equivalently,

$$
c \leq|\lambda| \leq C N^{4}, \quad \kappa\left(A_{s}\right) \leq C N^{4} .
$$

### 4.2. Wave equation

Theorem 4.5. Let $N \geq 2$. Let $A_{w}$ be the Chebyshev spectral collocation matrix defined by (3.2) and $\lambda$ be any eigenvalue of $A_{w}$. Then

$$
c \leq|\lambda| \leq C N^{4} .
$$

Consequently

$$
\kappa\left(A_{w}\right) \leq C N^{4} .
$$

Proof. From (3.2), it follows that

$$
\lambda=\gamma^{2}+\mu,
$$

where $\gamma=\gamma_{r}+i \gamma_{i}$ is an eigenvalue of $[D]$ and $\mu$ is an eigenvalue of $-\llbracket D^{2} \rrbracket$. A calculation yields

$$
|\lambda|^{2}=\gamma_{r}^{4}+2 \mu \gamma_{r}^{2}+2 \gamma_{r}^{2} \gamma_{i}^{2}+\left(\mu-\gamma_{i}^{2}\right)^{2} \geq \gamma_{r}^{4}+2 \mu \gamma_{r}^{2} \geq c,
$$

by Lemmas 4.1 and 4.3. Using Lemmas 4.2 and 4.3, it follows that

$$
|\lambda|^{2} \leq C N^{8}
$$

Thus

$$
c \leq|\lambda| \leq C N^{4}, \quad \kappa\left(A_{w}\right) \leq C N^{4}
$$

### 4.3. Airy equation

We begin with a couple of useful technical results followed by another one which is directly needed for an estimate of the condition number of the Airy spectral operator. The first one is well known. See, for instance, Lemma 5.31 in [23].

Lemma 4.6. Let $u \in H_{w}^{1}(-1,1)$ and $v \in H_{0, w}^{1}(-1,1)$. Then

$$
\left|\int_{-1}^{1} u^{\prime}(v w)^{\prime}\right| \leq 2\left\|u^{\prime}\right\|\left\|v^{\prime}\right\|
$$

Lemma 4.7. Let $v \in H_{0, w}^{1}(-1,1)$. Then

$$
\int_{-1}^{1} v^{2} w^{5} \leq \frac{2}{3}\left\|v^{\prime}\right\|^{2}, \quad \int_{-1}^{1} \frac{v^{2} w}{(1-x)^{2}} \leq \frac{8}{3}\left\|v^{\prime}\right\|
$$

Proof. The first inequality of this lemma is a Hardy-type inequality (inequality (13.4) in [19], for instance). For $x \in(-1,1)$, it is easy to see that $(1+x)^{2} \leq 4$, leading to

$$
\frac{1}{\sqrt{1+x}} \leq \frac{4}{(1+x)^{5 / 2}}
$$

Using the above inequality and Hardy-type inequality, it follows that

$$
\int_{-1}^{1} \frac{v^{2} w}{(1-x)^{2}}=\int_{-1}^{1} \frac{v^{2}}{(1-x)^{5 / 2}(1+x)^{1 / 2}} \leq 4 \int_{-1}^{1} \frac{v^{2}}{\left(1-x^{2}\right)^{5 / 2}} \leq \frac{8}{3}\left\|v^{\prime}\right\|^{2}
$$

Proposition 4.8. Let $N \geq 2$ and $B_{1}$ be defined in (3.4). Suppose $\lambda$ is any eigenvalue of $B_{1}$. Then

$$
|\lambda| \leq C N^{6} .
$$

Proof. Let $u_{h}$ be an eigenvector of $B_{1}$ corresponding to $\lambda$. Let $v \in P_{N}$ so that $v( \pm 1)=0$ and $v\left(x_{h}\right)=M^{-1} u_{h}$, where $M$ is diagonal with diagonal entries of the form $1-x_{j}$. Define $u(x)=(1-x) v(x) \in P_{N+1}$. Note that $u\left(x_{h}\right)=M v\left(x_{h}\right)=u_{h}$. Now

$$
\lambda u\left(x_{h}\right)=\lambda u_{h}=B_{1} u_{h}=\left(M \llbracket D^{3} \rrbracket-3 \llbracket D^{2} \rrbracket\right) v\left(x_{h}\right)=u^{\prime \prime \prime}\left(x_{h}\right)
$$

by (3.3). Observe that $u( \pm 1)=0=u^{\prime}(1)$. Therefore

$$
\lambda \sum_{j=1}^{N} \frac{\left|u\left(x_{j}\right)\right|^{2}}{\left(1-x_{j}\right)^{2}} \rho_{j}=\sum_{j=1}^{N} u^{\prime \prime \prime}\left(x_{j}\right) \frac{\overline{u\left(x_{j}\right)}}{\left(1-x_{j}\right)^{2}} \rho_{j}
$$

Since $u^{\prime \prime \prime}=-3 v^{\prime \prime}+(1-x) v^{\prime \prime \prime}$, the above equation becomes

$$
\begin{aligned}
\lambda \sum_{j=1}^{N}\left|v\left(x_{j}\right)\right|^{2} \rho_{j} & =\sum_{j=1}^{N}\left(-3 v^{\prime \prime}\left(x_{j}\right)+\left(1-x_{j}\right) v^{\prime \prime \prime}\left(x_{j}\right)\right) \frac{\overline{v\left(x_{j}\right)}}{1-x_{j}} \rho_{j} \\
& =-3 \sum_{j=1}^{N} \frac{v^{\prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}} \rho_{j}+\sum_{j=1}^{N} v^{\prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)} \rho_{j} \\
& =-3 \sum_{j=0}^{N} \frac{v^{\prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}} \rho_{j}-3 v^{\prime \prime}(1) \overline{v^{\prime}(1)} \rho_{0}+\sum_{j=0}^{N} v^{\prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)} \rho_{j}
\end{aligned}
$$

In the first sum of the last equality on the right-hand side, the term $j=0$ is taken in the sense of a limit since there is division by zero.

Next we estimate the boundary term. Let $v(x)=(1-x) \xi(x)$ with $\xi \in P_{N-1}$. It follows that $v^{\prime}(1)=-\xi(1)$ and $v^{\prime \prime}(1)=$ $-2 \xi^{\prime}(1)$. By the trace inequality and Lemma 4.7,

$$
\left|v^{\prime}(1)\right|^{2}=|\xi(1)|^{2} \leq c(N-1)\|\xi\|^{2}=c(N-1) \int_{-1}^{1} \frac{|v|^{2} w}{(1-x)^{2}} \leq C N\left\|v^{\prime}\right\|^{2}
$$

Similarly, invoking the inverse estimate in addition to the other inequalities,

$$
\left|v^{\prime \prime}(1)\right|^{2}=4\left|\xi^{\prime}(1)\right|^{2} \leq c(N-1)\left\|\xi^{\prime}\right\|^{2} \leq C N^{5}\|\xi\|^{2} \leq C N^{5}\left\|v^{\prime}\right\|^{2}
$$

The boundary term can now be estimated directly:

$$
\left|v^{\prime \prime}(1) v^{\prime}(1)\right| \rho_{0} \leq C N^{5 / 2}\left\|v^{\prime}\right\| C N^{1 / 2}\left\|v^{\prime}\right\| \frac{\pi}{2 N}=c N^{2}\left\|v^{\prime}\right\|^{2} \leq C N^{6}\|v\|^{2}
$$

Finally, the magnitude of the eigenvalue can be estimated using Lemmas 4.6 and 4.7 and the fact that integration can be evaluated exactly for any integrand of degree $2 N-1$ or lower:

$$
\begin{aligned}
|\lambda| \sum_{j=0}^{N}\left|v\left(x_{j}\right)\right|^{2} \rho_{j} & \leq 3\left|\int_{-1}^{1} \frac{v^{\prime \prime} \bar{v} w}{1-x}\right|+\left|\int_{-1}^{1} v^{\prime \prime \prime} \bar{v} w\right|+C N^{6}\|v\|^{2} \\
& \leq 3\left\|v^{\prime \prime}\right\|\left(\int_{-1}^{1} \frac{|\bar{v}|^{2} w}{(1-x)^{2}}\right)^{1 / 2}+\left|\int_{-1}^{1} v^{\prime \prime}(\bar{v} w)^{\prime}\right|+C N^{6}\|v\|^{2} \\
& \leq c N^{2}\left\|v^{\prime}\right\|\left\|v^{\prime}\right\|+c\left\|v^{\prime \prime}\right\|\left\|\bar{v}^{\prime}\right\|+C N^{6}\|v\|^{2} \\
& \leq C N^{6}\|v\|^{2}
\end{aligned}
$$

In the last two lines, the inverse estimate has been invoked several times. It follows from the equivalence of the discrete and weighted $L^{2}$ norms that $|\lambda| \leq C N^{6}$.

Next a lower bound of $|\lambda|$ is estimated. Note that this has been an open problem for more than 20 years. [24] first reported numerical evidence that $\operatorname{Re} \lambda$ is positive and bounded away from zero. First, we state the following useful result due to [25].

Lemma 4.9. Let $N \geq 1$. If $F=\sum_{k=0}^{4 N-1} b_{k} T_{k}$ for some complex constants $b_{k}$, then

$$
\sum_{j=0}^{N} \rho_{j} F\left(t_{j}\right)=\int_{-1}^{1} F(t) w(t) d t+\pi b_{2 N}
$$

Proposition 4.10. Let $N \geq 2$ and $B_{1}$ be defined in (3.4). Suppose $\lambda$ is any eigenvalue of $B_{1}$. Then
$\operatorname{Re} \lambda \geq c$.
Proof. Let $\lambda$ be an eigenvalue of $B_{1}$. As in the previous proposition, we look for $v \in P_{N}$ vanishing at the end points $\pm 1$ so that $(1-x) v(x)$ satisfies the eigenvalue relation:

$$
\begin{equation*}
-3 v^{\prime \prime}(x)+(1-x) v^{\prime \prime \prime}(x)=\lambda(1-x) v(x)+\frac{\lambda}{N} \sum_{i=0}^{2} A_{i} x^{i} T_{N}^{\prime}(x) \tag{4.1}
\end{equation*}
$$

where the coefficients $A_{i}, i=0,1,2$, are chosen so that the right-hand side is a polynomial of degree at most $N-2$, matching that of the left-hand side. Let

$$
v=\sum_{k=0}^{N} a_{k} T_{k}
$$

Substitute this into the eigenvalue relation (4.1) and set the coefficients of $T_{N+1}, T_{N}, T_{N-1}$ of the expression on the righthand side to zero to obtain

$$
A_{0}=a_{N}, \quad A_{1}=\frac{1}{2} a_{N-1}-a_{N}, \quad A_{2}=\frac{1}{2}\left(\frac{1}{2} a_{N-2}-a_{N}-a_{N-1}\right)
$$

Evaluate (4.1) at the interior collocation points to get

$$
\begin{equation*}
-3 v^{\prime \prime}\left(x_{j}\right)+\left(1-x_{j}\right) v^{\prime \prime \prime}\left(x_{j}\right)=\lambda\left(1-x_{j}\right) v\left(x_{j}\right), \quad 1 \leq j \leq N-1 \tag{4.2}
\end{equation*}
$$

giving us the relation defining the eigenvalues of $B_{1}$. Let $\beta$ be a constant to be determined later. Evaluate (4.1) at $x_{j}$ for $0 \leq j \leq N$. Then multiply both sides of resultant equation by $\left(1+x_{j}\right)\left(1+\beta x_{j}\right) \overline{v\left(x_{j}\right)} \rho_{j}$, then add the result to the complex conjugate of (4.1) multiplied by $\left(1+x_{j}\right)\left(1+\beta x_{j}\right) v\left(x_{j}\right) \rho_{j}$, and finally sum $j$ from 0 to $N$ to get

$$
\begin{align*}
& -3 \sum_{j=0}^{N}\left(1+x_{j}\right)\left(1+\beta x_{j}\right)\left(v^{\prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}+\overline{v^{\prime \prime}\left(x_{j}\right)} v\left(x_{j}\right)\right) \rho_{j} \\
& \quad+\sum_{j=0}^{N}\left(1-x_{j}^{2}\right)\left(1+\beta x_{j}\right)\left(v^{\prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}+\overline{v^{\prime \prime \prime}\left(x_{j}\right)} v\left(x_{j}\right)\right) \rho_{j}  \tag{4.3}\\
& =2 \operatorname{Re} \lambda\left[\sum_{j=0}^{N}\left(1-x_{j}^{2}\right)\left(1+\beta x_{j}\right)\left|v\left(x_{j}\right)\right|^{2} \rho_{j}\right] .
\end{align*}
$$

Restricting $\beta \in(-1,1)$, the expression in square brackets is positive, say, equal to $C$. Use (2.3) to get

$$
C \leq 2 \int_{-1}^{1}\left(1-x^{2}\right)(1+\beta x)|v(x)|^{2} w(x) d x
$$

By the mean value theorem for integrals, there is some $z \in(-1,1)$ so that

$$
\begin{equation*}
C \leq 2\left(1-z^{2}\right)(1+\beta z)\|v\|^{2}=: C^{\prime}\|v\|^{2} . \tag{4.4}
\end{equation*}
$$

Define

$$
f(x)=-3(1+x)(1+\beta x)\left(v^{\prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime}(x)} v(x)\right):=\sum_{k=0}^{2 N} b_{k} T_{k}(x)
$$

and

$$
g(x)=\left(1-x^{2}\right)(1+\beta x)\left(v^{\prime \prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime \prime}(x)} v(x)\right):=\sum_{k=0}^{2 N} c_{k} T_{k}(x)
$$

Using Lemma 4.9, the left-hand side of (4.3) can be expressed as an integral plus some constants:

$$
\int_{-1}^{1} f(x) w(x) d x+\pi b_{2 N}+\int_{-1}^{1} g(x) w(x) d x+c_{2 N}
$$

where, after some calculations,

$$
b_{2 N}=-3 \beta N(N-1)\left|a_{N}\right|^{2}, \quad c_{2 N}=-\beta N(N-1)(N-2)\left|a_{N}\right|^{2}
$$

Observe that $b_{2 N}, c_{2 N} \geq 0$ if $\beta \leq 0$.
We now estimate the left-hand side of (4.3). Toward that end, write

$$
f(x)=\left(v^{\prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime}(x)} v(x)\right) p(x), \quad g(x)=\left(v^{\prime \prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime \prime}(x)} v(x)\right) q(x)
$$

where

$$
p(x)=-3(1+x)(1+\beta x) w(x), \quad q(x)=\left(1-x^{2}\right)(1+\beta x) w(x)
$$

The left-hand side of (4.3) becomes, after some integration by parts,

$$
\begin{align*}
I & :=\int_{-1}^{1}\left(v^{\prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime}(x)} v(x)\right) p(x) d x+\int_{-1}^{1}\left(v^{\prime \prime \prime}(x) \overline{v(x)}+\overline{v^{\prime \prime \prime}(x)} v(x)\right) q(x) d x+\pi\left(b_{2 N}+c_{2 N}\right) \\
& =\int_{-1}^{1}\left|v^{\prime}(x)\right|^{2}\left(-2 p(x)+3 q^{\prime}(x)\right) d x+\int_{-1}^{1}|v(x)|^{2}\left(p^{\prime \prime}(x)-q^{\prime \prime \prime}(x)\right) d x+\pi\left(b_{2 N}+c_{2 N}\right) \tag{4.5}
\end{align*}
$$

By a direct calculation,

$$
\begin{equation*}
-2 p+3 q^{\prime}=3(x(2 \beta+1)+\beta+2) w \geq 3(1-\beta) w \tag{4.6}
\end{equation*}
$$

because the linear function attains its minimum for $x \in[-1,1]$ at the left end point for $\beta \in[-1 / 2,0]$. Also,

$$
\begin{equation*}
p^{\prime \prime}-q^{\prime \prime \prime}=-3\left((\beta+2) x^{2}+(3 \beta+2) x+\beta+1\right) w^{5} \geq-3(1-\beta) w^{5} \tag{4.7}
\end{equation*}
$$

since the above quadratic is concave and attains its minimum for $x \in[-1,1]$ at the left end point if $\beta \in[-2 / 3,-1 / 2]$. Define $\beta=-1 / 2<0$. It follows that $b_{2 N}, c_{2 N} \geq 0$. Using the second inequality of Lemma 4.7 , (4.6) and (4.7), (4.5) can be estimated:

$$
\begin{aligned}
I & \geq \frac{9}{2} \int_{-1}^{1}\left|v^{\prime}\right|^{2} w d x-\frac{9}{2} \int_{-1}^{1}|v|^{2} w^{5} d x+\pi\left(b_{2 N}+c_{2 N}\right) \\
& \geq \frac{9}{2} \int_{-1}^{1}\left|v^{\prime}\right|^{2} w d x-\frac{9}{2} \frac{2}{3} \int_{-1}^{1}\left|v^{\prime}\right|^{2} w d x \\
& =\frac{3}{2} \int_{-1}^{1}\left|v^{\prime}\right|^{2} w d x
\end{aligned}
$$

Then (4.3) becomes

$$
2 C \operatorname{Re} \lambda \geq \frac{3}{2} \int_{-1}^{1}\left|v^{\prime}\right|^{2} w d x
$$

Using the weighted Poincaré inequality,

$$
2 C \operatorname{Re} \lambda \geq c \int_{-1}^{1}|v|^{2} w
$$

Finally, from (4.4), we conclude that

$$
\operatorname{Re} \lambda \geq c^{\prime}
$$

a positive constant independent of $N$.

Theorem 4.11. Let $N \geq 2$. Let $A_{a}$ be the Chebyshev spectral collocation matrix defined by (3.5) and $\lambda$ be any eigenvalue of $A_{a}$. Then

$$
c \leq|\lambda| \leq C N^{6}
$$

Consequently

$$
\kappa\left(A_{a}\right) \leq C N^{6}
$$

Proof. From (3.5), it follows that

$$
\lambda=\gamma+\mu
$$

where $\gamma=\gamma_{r}+i \gamma_{i}$ is an eigenvalue of [D] and $\mu=\mu_{r}+i \mu_{i}$ is an eigenvalue of $B_{1}$ defined in (3.4). By Lemma 4.1 and Proposition 4.10, it follows that the real part of $\lambda$ is bounded below by a positive constant. This yields $|\lambda| \geq c$ immediately. Using Lemma 4.2 and Proposition 4.8,

$$
|\lambda| \leq c N^{2}+C N^{6}
$$

Thus

$$
c \leq|\lambda| \leq C N^{6}, \quad \kappa\left(A_{a}\right) \leq C N^{6}
$$

### 4.4. Beam equation

First we state a lemma which will be needed in the estimate for the beam spectral operator. The lemma is actually a special case of a general result (Proposition III.1) from [27].

Lemma 4.12. Let $v \in H_{0, w}^{2}$. There is some positive constant $c$ independent of $v$ so that

$$
\operatorname{Re} \int_{-1}^{1} v^{\prime \prime}(\bar{v} w)^{\prime \prime} \geq c \int_{-1}^{1}\left|v^{\prime \prime}\right|^{2} w
$$

Next are estimates for the magnitude of eigenvalues of $B_{2}$ in the discrete beam operator. Note that [28] already showed that the real part of any eigenvalue is positive for the discrete fourth derivative for any Jacobi polynomials. Our result is specific to Chebyshev polynomials, and we show that the real part of every eigenvalue is bounded away from zero.

Proposition 4.13. Let $N \geq 2$ and $B_{2}$ be defined in (3.7). Suppose $\lambda$ is any eigenvalue of $B_{2}$. Then

$$
c \leq|\lambda| \leq C N^{8}
$$

Proof. Let $u_{h}$ be an eigenvector of $B_{2}$ corresponding to $\lambda$. Let $v \in P_{N}$ so that $v( \pm 1)=0$ and $v\left(x_{h}\right)=M^{-1} u_{h}$, where $M$ is diagonal with diagonal entries of the form $1-x_{j}^{2}$. Define $u(x)=\left(1-x^{2}\right) v(x) \in P_{N+2}$. Note that $u\left(x_{h}\right)=M v\left(x_{h}\right)=u_{h}$. Now

$$
\lambda u\left(x_{h}\right)=\lambda u_{h}=B_{2} u_{h}=\left(M \llbracket D^{4} \rrbracket-8 X \llbracket D^{3} \rrbracket-12 \llbracket D^{2} \rrbracket\right) v\left(x_{h}\right)=u^{\prime \prime \prime \prime}\left(x_{h}\right)
$$

by (3.6). Observe that $u( \pm 1)=0=u^{\prime}( \pm 1)$. Therefore

$$
\lambda \sum_{j=1}^{N-1} \frac{\left|u\left(x_{j}\right)\right|^{2}}{\left(1-x_{j}^{2}\right)^{2}} \rho_{j}=\sum_{j=1}^{N-1} \frac{u^{\prime \prime \prime \prime}\left(x_{j}\right) \overline{u\left(x_{j}\right)}}{\left(1-x_{j}^{2}\right)^{2}} \rho_{j} .
$$

Since $u^{\prime \prime \prime \prime}=-12 v^{\prime \prime}-8 x v^{\prime \prime \prime}+\left(1-x^{2}\right) v^{\prime \prime \prime \prime}$,

$$
\begin{aligned}
\lambda \sum_{j=1}^{N-1}\left|v\left(x_{j}\right)\right|^{2} \rho_{j}= & -12 \sum_{j=1}^{N-1} \frac{v^{\prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}^{2}} \rho_{j}-8 \sum_{j=1}^{N-1} \frac{x_{j} v^{\prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}^{2}} \rho_{j}+\sum_{j=1}^{N-1} v^{\prime \prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)} \rho_{j} \\
\lambda \sum_{j=0}^{N}\left|v\left(x_{j}\right)\right|^{2} \rho_{j}= & -12 \sum_{j=0}^{N} \frac{v^{\prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}^{2}} \rho_{j}-8 \sum_{j=0}^{N} \frac{x_{j} v^{\prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)}}{1-x_{j}^{2}} \rho_{j}+\sum_{j=0}^{N} v^{\prime \prime \prime \prime}\left(x_{j}\right) \overline{v\left(x_{j}\right)} \rho_{j} \\
& +\frac{3 \pi}{N}\left(v^{\prime \prime}(-1) \overline{v^{\prime}(-1)}-v^{\prime \prime}(1) \overline{v^{\prime}(1)}\right)+\frac{2 \pi}{N}\left(v^{\prime \prime \prime}(-1) \overline{v^{\prime}(-1)}-v^{\prime \prime \prime}(1) \overline{v^{\prime}(1)}\right) .
\end{aligned}
$$

Let $v(x)=\left(1-x^{2}\right) \xi(x)$ with $\xi \in P_{N-2}$. Using a similar technique as before, we estimate the boundary terms:

$$
\left|v^{\prime}( \pm 1)\right|^{2}=4|\xi( \pm 1)|^{2} \leq C N\left\|v^{\prime}\right\|^{2}, \quad\left|v^{\prime \prime}( \pm 1)\right|^{2}=\left|2 \xi( \pm 1)+4( \pm 1) \xi^{\prime}( \pm 1)\right|^{2} \leq C N^{5}\left\|v^{\prime}\right\|^{2}
$$

and

$$
\left|v^{\prime \prime \prime}( \pm 1)\right|^{2}=\left|6 \xi^{\prime}( \pm 1)+6( \pm 1) \xi^{\prime \prime}( \pm 1)\right|^{2} \leq C N^{9}\left\|v^{\prime}\right\|^{2}
$$

leading to a final upper bound of all boundary terms of $c N^{4}\left\|v^{\prime}\right\|^{2} \leq C N^{8}\|v\|^{2}$.
For the final estimate of the eigenvalue, again use Lemmas 4.6 and 4.7 and the fact that the integration can be evaluated exactly by summation since the integrand is of degree at most $2 N-1$ to get

$$
\begin{aligned}
|\lambda| \sum_{j=0}^{N}\left|v\left(x_{j}\right)\right|^{2} \rho_{j} & \leq 12\left|\int_{-1}^{1} \frac{v^{\prime \prime} \bar{v} w}{1-x^{2}}\right|+8\left|\int_{-1}^{1} \frac{x v^{\prime \prime \prime} \bar{v} w}{1-x^{2}}\right|+\left|\int_{-1}^{1} v^{\prime \prime \prime \prime} \bar{v} w\right|+C N^{8}\|v\|^{2} \\
& \leq 12\left\|v^{\prime \prime}\right\|\left(\int_{-1}^{1} \frac{|v|^{2} w}{\left(1-x^{2}\right)^{2}}\right)^{1 / 2}+8\left\|v^{\prime \prime \prime}\right\|\left(\int_{-1}^{1} \frac{|v|^{2} w}{\left(1-x^{2}\right)^{2}}\right)^{1 / 2}+\left|\int_{-1}^{1} v^{\prime \prime \prime}(\bar{v} w)^{\prime}\right|+C N^{8}\|v\|^{2} \\
& \leq C N^{2}\left\|v^{\prime}\right\|^{2}+C N^{4}\left\|v^{\prime}\right\|^{2}+\left\|v^{\prime \prime \prime}\right\|\left\|v^{\prime}\right\|+C N^{8}\|v\|^{2} \\
& \leq C N^{8}\|v\|^{2} .
\end{aligned}
$$

By the equivalence of the discrete and weighted $L^{2}$ norms, $|\lambda| \leq C N^{8}$
Now a lower bound of $|\lambda|$ is estimated. As in the first paragraph of this proof, let $u \in P_{N+2} \cap H_{0}^{2}(-1,1)$ so that $u_{h}=u\left(x_{h}\right)$ is an eigenvector of $B_{2}$ with corresponding eigenvalue $\lambda$. From the eigenvalue relation for $u$, it follows that

$$
\begin{equation*}
u^{\prime \prime \prime \prime}(x)=\lambda u(x)+\frac{\lambda}{N} \sum_{i=0}^{3} A_{i} x^{i} T_{N}^{\prime}(x) \tag{4.8}
\end{equation*}
$$

for some $A_{i} \in \mathbb{R}$ chosen so that the right-hand side is a polynomial of degree at most $N-2$. Multiply (4.8) evaluated at $x_{j}$ by $\overline{u\left(x_{j}\right)} \rho_{j}$ and then sum over $0 \leq j \leq N$ to obtain $S_{1}$. Next, multiply the complex conjugate of (4.8) evaluated at $x_{j}$ by $u\left(x_{j}\right) \rho_{j}$ and then sum over $0 \leq j \leq N$ to get $S_{2}$. Observe that for any $0 \leq j \leq N$, both $T_{N}^{\prime}\left(x_{j}\right) u\left(x_{j}\right)$ and its complex conjugate vanish. It follows that $\left(S_{1}+S_{2}\right) / 2$ equals

$$
\begin{equation*}
\operatorname{Re}\left(\sum_{j=0}^{N} u^{\prime \prime \prime \prime}\left(x_{j}\right) \overline{u\left(x_{j}\right)} \rho_{j}\right)=C_{1} \operatorname{Re} \lambda, \tag{4.9}
\end{equation*}
$$

where

$$
C_{1}=\sum_{j=0}^{N}\left|u\left(x_{j}\right)\right|^{2} \rho_{j} .
$$

Notice that $C_{1}>0$, otherwise $u \equiv 0$. By the equivalence of weighted $L^{2}$ and discrete norms,

$$
\begin{equation*}
C_{1} \leq C_{2}\|u\|^{2} \tag{4.10}
\end{equation*}
$$

Define

$$
f(x)=\operatorname{Re}\left(u^{\prime \prime \prime \prime}(x) \overline{u(x)}\right)
$$

Let

$$
f(x)=\sum_{k=0}^{2 N} b_{k} T_{k}(x), \quad u(x)=\sum_{k=0}^{N+2} a_{k} T_{k}(x)
$$

After some calculations, the coefficient of $T_{2 N}$ of $f$ is

$$
b_{2 N}=16\left(N^{2}-1\right)(N+2) N\left|a_{N+2}\right|^{2} \geq 0
$$

By Lemma 4.9, the left-hand side of (4.9) becomes

$$
\begin{aligned}
\operatorname{Re}\left(\int_{-1}^{1} u^{\prime \prime \prime \prime}(x) \overline{u(x)} w(x) d x+\pi b_{2 N}\right) & =\operatorname{Re}\left(\int_{-1}^{1} u^{\prime \prime}(x)(\overline{u(x)} w(x))^{\prime \prime} d x\right)+\pi b_{2 N} \\
& \geq C_{3} \int_{1}^{1}\left|u^{\prime \prime}(x)\right|^{2} w(x) d x \quad \text { (Lemma 4.12) } \\
& \geq \frac{3}{2} C_{3} \int_{-1}^{1}\left|u^{\prime}(x)\right|^{2} w^{5}(x) d x \quad \quad \text { (Hardy-type inequality) } \\
& \geq \frac{3}{2} C_{3} \int_{-1}^{1}\left|u^{\prime}(x)\right|^{2} w(x) d x \\
& \geq C_{4} \int_{-1}^{1}|u(x)|^{2} w(x) d x \quad \text { (weighted Poincaré inequality). }
\end{aligned}
$$

In summary, (4.9) and (4.10) together imply

$$
\operatorname{Re} \lambda \geq \frac{C_{4}\|u\|^{2}}{C_{2}\|u\|^{2}}:=c
$$

Numerically, every eigenvalue $\lambda$ of $B_{2}$ is found to be real ([29]). It remains an open problem to prove that $\operatorname{Im} \lambda=0$. By assuming this explicitly, it is possible to prove a condition number estimate of the space-time beam operator.

Theorem 4.14. Let $N \geq 2$. Let $A_{b}$ be the Chebyshev spectral collocation matrix defined by (3.8) and $\lambda$ be any eigenvalue of $A_{b}$. Assume every eigenvalue of $B_{2}$ is real. Then

$$
c \leq|\lambda| \leq C N^{8}
$$

Consequently

$$
\kappa\left(A_{b}\right) \leq C N^{8}
$$

Proof. From (3.8), it follows that

$$
\lambda=\gamma^{2}+\mu,
$$

where $\gamma=\gamma_{r}+i \gamma_{i}$ is an eigenvalue of [ $D$ ] and $\mu$ is an eigenvalue of $B_{2}$ defined in (3.7). By assumption, $\mu$ is real. The same calculation as before yields

$$
|\lambda|^{2}=\gamma_{r}^{4}+2 \mu \gamma_{r}^{2}+2 \gamma_{r}^{2} \gamma_{i}^{2}+\left(\mu-\gamma_{i}^{2}\right)^{2} \geq \gamma_{r}^{4}+2 \mu \gamma_{r}^{2} \geq c,
$$

by Lemma 4.1 and Proposition 4.13. Using Lemma 4.2 and Proposition 4.13, it follows that

$$
|\lambda|^{2} \leq c\left(N^{8}+N^{12}+N^{16}\right) .
$$

Thus

$$
c \leq|\lambda| \leq C N^{8}, \quad \kappa\left(A_{b}\right) \leq C N^{8} .
$$

## 5. Spectral convergence

In this section, we discuss space-time spectral convergence of our method for the Schrodinger and wave equations.
Theorem 5.1. Let $u$ be the solution of the Schrodinger equation. Assume $u$ is separately analytic in each variable. Let $N \geq 2$ and $\hat{u}_{h}$ be the solution of the space-time method with matrix defined by (3.1). Define the error vector $E_{h}$ as the difference of $u$ evaluated at the collocation points and $\hat{u}_{h}$. Then

$$
\left|W^{1 / 2} E_{h}\right| \leq c N^{3.5} e^{-C N}
$$

The proof of spectral convergence for the Schrodinger equation is almost identical to that of the heat equation in [2] and is omitted. What is perhaps surprising is that the method of proof is so similar despite the fact that this PDE is dispersive and has completely different properties from those of the heat equation which is diffusive.

Theorem 5.2. Let $u$ be the solution of the wave equation. Assume $u$ is separately analytic in each variable. Let $N \geq 2$ and $\hat{v}_{1 h}$ be the solution of the space-time method with matrix defined by (3.2). Define the error vector $E_{h}$ as the difference of $u$ evaluated at the collocation points and $\hat{v}_{1 h}$. Then

$$
\left|W^{1 / 2} E_{h}\right| \leq c N^{4.5} e^{-C N}
$$

Proof. Define

$$
u_{h}(t)=\left[\begin{array}{c}
u\left(x_{1}, t\right) \\
\vdots \\
u\left(x_{N-1}, t\right)
\end{array}\right], \quad f_{h}(t)=\left[\begin{array}{c}
f\left(x_{1}, t\right) \\
\vdots \\
f\left(x_{N-1}, t\right)
\end{array}\right] .
$$

A semi-discrete approximation of the wave equation is

$$
u_{h}^{\prime \prime}(t)=\sum_{j=0}^{N}\left(A u_{h}\left(t_{j}\right)+f_{h}\left(t_{j}\right)\right) \ell_{j}(t), \quad u_{h}(-1)=u_{0 h}, u_{h}^{\prime}(-1)=u_{1 h}
$$

where $A=\llbracket D^{2} \rrbracket$. Hence

$$
u_{h}^{\prime \prime}\left(t_{k}\right)=A u_{h}\left(t_{k}\right)+f_{h}\left(t_{k}\right), \quad 0 \leq k \leq N-1
$$

Define the error function $e_{h}(t)=u_{h}(t)-u\left(x_{h}, t\right)$ with components $e_{j}(t)=\left(e_{h}(t)\right)_{j}$. Using the above equation, it is easy to see that the error satisfies, for $0 \leq k \leq N-1$,

$$
\begin{equation*}
e_{h}^{\prime \prime}\left(t_{k}\right)=A e_{h}\left(t_{k}\right)+r\left(t_{k}\right), \quad r\left(t_{k}\right)=A u\left(x_{h}, t_{k}\right)-u_{x x}\left(x_{h}, t_{k}\right) \tag{5.1}
\end{equation*}
$$

For any analytic $z$ such that $z(-1)=0$, recall the definition of the interpolant

$$
\mathcal{I}_{N} z(t)=\sum_{j=0}^{N-1} z\left(t_{j}\right) \ell_{j}(t)
$$

For $0 \leq k \leq N-1$,

$$
\begin{equation*}
z^{\prime}\left(t_{k}\right)=\left(\mathcal{I}_{N} z\right)^{\prime}\left(t_{k}\right)+\tilde{\epsilon}_{k}=\left([D]\left(\mathcal{I}_{N} z\right)\left(t_{h}\right)\right)_{k}+\tilde{\epsilon}_{k}=\left([D] z\left(t_{h}\right)\right)_{k}+\tilde{\epsilon}_{k} \tag{5.2}
\end{equation*}
$$

where $\tilde{\epsilon}_{k}=\left(z-\mathcal{I}_{N} z\right)^{\prime}\left(t_{k}\right)$ satisfies

$$
\left|\tilde{\epsilon}_{k}\right| \leq c N^{2} e^{-C N}
$$

according to [30]. Take $z(t)=e_{j}(t)$ in (5.2), observing that $e_{j}(-1)=0$, then

$$
\begin{equation*}
e_{j}^{\prime}\left(t_{k}\right)=\left([D] e_{j}\left(t_{h}\right)\right)_{k}+\epsilon_{1 j k} \tag{5.3}
\end{equation*}
$$

where $\left|\epsilon_{1 j k}\right| \leq c N^{2} e^{-C N}$. Next take $z(t)=e_{j}^{\prime}(t)$ in (5.2), noting that $e_{j}^{\prime}(-1)=0$, then

$$
\begin{equation*}
e_{j}^{\prime \prime}\left(t_{k}\right)=\left([D] e_{j}^{\prime}\left(t_{h}\right)\right)_{k}+\epsilon_{2 j k} \tag{5.4}
\end{equation*}
$$

where $\left|\epsilon_{2 j k}\right| \leq c N^{2} e^{-C N}$. Considering (5.1) together with (5.3) and (5.4), we have

$$
\begin{align*}
& \left([D] e_{j}\left(t_{h}\right)\right)_{k}+\epsilon_{1 j k}=\left(e_{h}^{\prime}\left(t_{k}\right)\right)_{j}  \tag{5.5}\\
& \left([D] e_{j}^{\prime}\left(t_{h}\right)\right)_{k}+\epsilon_{2 j k}=\left(A e_{h}\left(t_{k}\right)\right)_{j}+r_{j}\left(t_{k}\right) \tag{5.6}
\end{align*}
$$

where residual vectors $r_{j}\left(t_{h}\right)=A u\left(x_{j}, t_{h}\right)-u_{x x}\left(x_{j}, t_{h}\right)$. Define the long vector

$$
\tilde{R}_{h}=\left[\begin{array}{c}
r_{1}\left(t_{h}\right) \\
\vdots \\
r_{N-1}\left(t_{h}\right)
\end{array}\right]
$$

and

$$
E_{h}=\left[\begin{array}{c}
e_{1}\left(t_{h}\right) \\
\vdots \\
e_{N-1}\left(t_{h}\right)
\end{array}\right]
$$

in vector notation, the equations (5.5) and (5.6) are

$$
\left(I_{N-1} \otimes[D]\right) E_{h}=E_{h}^{\prime}-\epsilon_{1}, \quad\left(I_{N_{1}} \otimes[D]\right) E_{h}^{\prime}=\left(A \otimes I_{N}\right) E_{h}+\tilde{R}_{h}-\epsilon_{2}
$$

where $\epsilon_{1}, \epsilon_{2}$ are long vectors formed by stacking together vectors $\left[\epsilon_{p j 0}, \ldots, \epsilon_{p j, N-1}\right]^{T}$ for $p=1,2$ and $1 \leq j \leq N-1$; and each component of $E_{h}^{\prime}$ has the form $e_{j}^{\prime}\left(t_{k}\right)$. Combine these two equations to obtain

$$
A_{w} E_{h}=R_{h}:=\tilde{R}_{h}-\epsilon_{2}-\left(I_{N-1} \otimes[D]\right) \epsilon_{1}
$$

Using Lemma 4.2 and the above estimates, it follows that $\left|R_{h}\right|_{\infty} \leq c N^{4} e^{-C N}$. Apply the result of Theorem 4.5 and proceed as in [2] to get the desired error estimate.

We expect similar spectral convergence for the Airy and beam equations. Numerical results certainly support this. We leave the analysis for future work.

## 6. Nonlinear PDEs

The purpose of this section is to show that it is simple, in a few lines of code in the spirit of [20], to adapt the above methodology to solve some of the most common nonlinear PDEs with spectral space-time convergence. We employ simple iterative schemes to solve the nonlinear system. While spectral convergence is observed numerically, we make no claims about theoretical convergence. In previous publications [1] and [2], we had considered the Allen-Cahn equation and Burgers' equation. We now look at some other nonlinear PDEs.

### 6.1. Nonlinear reaction diffusion equation

Consider

$$
u_{t}=u_{x x}+\lambda e^{u}+f(x, t), \text { on }(-1,1)^{2}
$$

with initial condition $u(x,-1)=u_{0}(x)$ and homogeneous Dirichlet boundary conditions. Here $\lambda$ is a positive constant. The spectral scheme is

$$
\left(I_{N+1} \otimes D\right) u_{h}=\left(D^{2} \otimes I_{N+1}\right) u_{h}+\lambda e^{u_{h}}+f_{h}
$$

where $f_{h}$ is $f$ evaluated at collocation points. Deleting the known boundary and initial values, the final scheme reads

$$
\left[\left(I_{N-1} \otimes[D]\right)-\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \hat{u}_{h}-\lambda e^{\hat{u}_{h}}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where $u_{0 h}$ is $u_{0}$ evaluated at the interior collocation points. This nonlinear system can be solved using the simple iteration ( $k \geq 0$ )

$$
\left[\left(I_{N-1} \otimes[D]\right)-\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \hat{u}_{h}^{(k+1)}=\lambda e^{\hat{u}_{h}^{(k)}}+\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

### 6.2. Nonlinear Schrodinger equation

Consider

$$
i u_{t}=-u_{x x}+|u|^{2} u+f(x, t), \text { on }(-1,1)^{2}
$$

with initial condition $u(x,-1)=u_{0}(x)$ and homogeneous Dirichlet boundary conditions. The spectral scheme is

$$
i\left(I_{N+1} \otimes D\right) u_{h}=-\left(D^{2} \otimes I_{N+1}\right) u_{h}+\left|u_{h}\right|^{2} u_{h}+f_{h}
$$

where $f_{h}$ is $f$ evaluated at collocation points. Deleting the known boundary and initial values, the final scheme reads

$$
i\left[\left(I_{N-1} \otimes[D]\right)+\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \hat{u}_{h}-\left|\hat{u}_{h}\right|^{2} \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where $u_{0 h}$ is $u_{0}$ evaluated at the interior collocation points. This nonlinear system can be solved using the simple iteration ( $k \geq 0$ ) with relaxation:

$$
i\left[\left(I_{N-1} \otimes[D]\right)+\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \tilde{u}_{h}^{(k+1)}-\left|\hat{u}_{h}^{(k)}\right|^{2} \tilde{u}_{h}^{(k+1)}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right), \quad \hat{u}_{h}^{(k+1)}=\frac{\tilde{u}_{h}^{(k+1)}+u_{h}^{(k)}}{2}
$$

### 6.3. Sine-Gordon equation

The Sine-Gordon equation is

$$
u_{t t}=u_{x x}+\sin u+f(x, t), \text { on }(-1,1)^{2}
$$

with initial conditions $u(x,-1)=u_{0}(x)$ and $u_{t}(x,-1)=u_{1}(x)$ and homogeneous Dirichlet boundary conditions. The spectral scheme is

$$
\left(I_{N+1} \otimes D^{2}\right) u_{h}=\left(D^{2} \otimes I_{N+1}\right) u_{h}+\sin u_{h}+f_{h}
$$

Deleting the known boundary and initial values, the final scheme reads

$$
\left[\left(I_{N-1} \otimes[D]^{2}\right)-\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \hat{u}_{h}-\sin \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes\left([D] d_{h}\right)\right)-\left(u_{1 h} \otimes d_{h}\right)
$$

This nonlinear system can be solved using the iteration ( $k \geq 0$ )

$$
\left[\left(I_{N-1} \otimes[D]^{2}\right)-\left(\llbracket D^{2} \rrbracket \otimes I_{N}\right)\right] \hat{u}_{h}^{(k+1)}=\sin \hat{u}_{h}^{(k)}+\hat{f}_{h}-\left(u_{0 h} \otimes\left([D] d_{h}\right)\right)-\left(u_{1 h} \otimes d_{h}\right)
$$

### 6.4. KdV equation

The KdV equation is

$$
u_{t}+u u_{x}+u_{x x x}=f(x, t), \text { on }(-1,1)^{2}
$$

with initial condition $u(x,-1)=u_{0}(x)$ and boundary conditions $u(-1, t)=0=u(1, t)=u_{x}(1, t)$. The spectral scheme is

$$
\left(I_{N+1} \otimes D\right) u_{h}+\operatorname{diag}\left(u_{h}\right)\left(D \otimes I_{N+1}\right) u_{h}+\left(D^{3} \otimes I_{N+1}\right) u_{h}=f_{h}
$$

Let $B_{1}$ be the spectral third derivative (3.4) defined for the Airy operator. The final system, removing the known boundary and initial values, becomes

$$
\left[\left(I_{N-1} \otimes[D]\right)+\left(B_{1} \otimes I_{N}\right)\right] \hat{u}_{h}+\operatorname{diag}\left(\left(\llbracket D \rrbracket \otimes I_{N}\right) \hat{u}_{h}\right) \hat{u}_{h}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

This can be solved using the iteration $(k \geq 0)$

$$
\left[\left(I_{N-1} \otimes[D]\right)+\left(B_{1} \otimes I_{N}\right)\right] \hat{u}_{h}^{(k+1)}+\operatorname{diag}\left(\left(\llbracket D \rrbracket \otimes I_{N}\right) \hat{u}_{h}^{(k)}\right) \hat{u}_{h}^{(k+1)}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

### 6.5. Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation reads

$$
u_{t}+u_{x x x x}+u_{x x}+u u_{x}=f(x, t), \text { on }(-1,1)^{2}
$$

with initial condition $u(x,-1)=u_{0}(x)$ and homogeneous Dirichlet boundary conditions. The scheme is then

$$
\left.\left(\left(I_{N-1} \otimes[D]\right)+\left(B_{2}+\llbracket D^{2} \rrbracket\right) \otimes I_{N}\right)\right) \hat{u}_{h}+\frac{1}{2}\left(\llbracket D \rrbracket \otimes I_{N}\right) \hat{u}_{h}^{2}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

where $B_{2}$ is defined in (3.7). This nonlinear system can be solved using the iteration ( $k \geq 0$ )

$$
\left.\left(\left(I_{N-1} \otimes[D]\right)+\left(B_{2}+\llbracket D^{2} \rrbracket\right) \otimes I_{N}\right)\right) \hat{u}_{h}^{(k+1)}+\operatorname{diag}\left(\left(\llbracket D \rrbracket \otimes I_{N}\right) \hat{u}_{h}^{(k)}\right) \hat{u}_{h}^{(k+1)}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

### 6.6. Cahn-Hilliard equation

The Cahn-Hilliard equation is

$$
u_{t}-\left(-u_{x x}+u^{3}-u\right)_{x x}=f(x, t)
$$

with initial condition $u(x,-1)=u_{0}(x)$ and boundary conditions

$$
u_{x}( \pm 1, t)=0=u_{x x x}( \pm 1, t)
$$

The full scheme, using Legendre space-time collocation, is

$$
\left(\left(I_{N+1} \otimes D\right)+\left(\left(D^{4}+D^{2}\right) \otimes I_{N+1}\right)\right) u_{h}-\left(D^{2} \otimes I_{N}\right) u_{h}^{3}=f_{h}
$$

Let $B=W^{-1} D^{T} W D$, where $W$ is the diagonal matrix whose diagonal entries are the weights of the collocation scheme. It is known ([23]) that $-B$ is a spectral approximation of the second derivative for functions whose derivative vanishes at the boundary. The spectral equations for $\hat{u}_{h}$, the entries of $u_{h}$ removing the initial values, become

$$
\left.\left(\left(I_{N+1} \otimes[D]\right)+\left(B^{2}-B\right) \otimes I_{N}\right)\right) \hat{u}_{h}+\left(B \otimes I_{N}\right) \hat{u}_{h}^{3}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
$$

This nonlinear system can be solved iteratively. Let $\tilde{D}$ be $D$ except that the first and last rows are replaced by a row of zeroes. This is a spectral approximation of the first derivative for functions whose derivative vanish at the boundary. There are several ways to discretize $\left(u^{3}\right)_{x x}=2 u u_{x}^{2}+u^{2} u_{x x}$. We attempted two, one of which worked, but not the other. The simple scheme

$$
\begin{aligned}
& \left.\left(\left(I_{N+1} \otimes[D]\right)+\left(B^{2}-B\right) \otimes I_{N}\right)\right) \hat{u}_{h}^{(k+1)}-6 \operatorname{diag}\left(\left(\tilde{D} \otimes I_{N}\right) \hat{u}_{h}^{(k)}\right)^{2} \hat{u}_{h}^{(k+1)} \\
& \quad+3 \operatorname{diag}\left(\hat{u}_{h}^{(k)}\right)^{2}\left(B \otimes I_{N}\right) \hat{u}_{h}^{(k+1)}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
\end{aligned}
$$

did not seem to converge. The following iteration with relaxation does seem to work ( $k \geq 0$ ):

$$
\begin{aligned}
\left.\left(\left(I_{N+1} \otimes[D]\right)+\left(B^{2}-B\right) \otimes I_{N}\right)\right) & \tilde{u}_{h}^{(k+1)}-6 \operatorname{diag}\left(\hat{u}_{h}^{(k)}\right) \operatorname{diag}\left(\left(\tilde{D} \otimes I_{N}\right) \hat{u}_{h}^{(k)}\right)\left(\tilde{D} \otimes I_{N}\right) \tilde{u}_{h}^{(k+1)} \\
& +3 \operatorname{diag}\left(\hat{u}_{h}^{(k)}\right)^{2}\left(B \otimes I_{N}\right) \tilde{u}_{h}^{(k+1)}=\hat{f}_{h}-\left(u_{0 h} \otimes d_{h}\right)
\end{aligned}
$$

$$
\hat{u}_{h}^{(k+1)}=\frac{\tilde{u}_{h}^{(k+1)}+u_{h}^{(k)}}{2}
$$

where $u_{0 h}$ is the initial data evaluated at all spatial collocation points. It is beyond the scope of this work to discuss convergence theories of the schemes in this section.

## 7. Numerical results

We implemented a very simple space-time Legendre and Chebyshev collocation method for each PDE discussed in this paper in MATLAB. Results for the Chebyshev space-time collocation in all but the Cahn-Hilliard equation are reported below. Almost identical results hold for the Legendre case and they are not given. Each linear system is solved by the Bartel-Stewart algorithm [31].

The convergence for the Schrodinger equation

$$
u_{t}=i u_{x x}+f
$$



Fig. 1. Convergence of Chebyshev collocation method for the Schrodinger (left) and wave (right) equations.


Fig. 2. Convergence of Chebyshev collocation method for the Airy (left) and beam (right) equations.
with boundary conditions $u( \pm 1, t)=0$ and initial condition $u(x,-1)=u_{0}(x)$. Take $f$ so that the exact solution is $u(x, t)=$ $e^{x+t} \sin (\pi t / 2) \sin \pi x$. Spectral convergence is clearly illustrated in the left figure of Fig. 1. The error is the largest error of the numerical solution at the Chebyshev nodes at the final time $t=1$. Note that the error is $O\left(10^{-14}\right)$ at $N=18$ which corresponds to a system with 306 unknowns.

The convergence of the Chebyshev collocation method for the wave equation

$$
u_{t t}=u_{x x}+f
$$

with boundary conditions $u( \pm 1, t)=0$ and initial conditions $u(x,-1)=u_{0}(x)$ and $u_{t}(x,-1)=u_{1}(x)$ can be found in the right figure of Fig. 1. Here we take $f$ so that the exact solution is the same as above.

For the Airy equation

$$
u_{t}+u_{x x x}=f
$$

with boundary conditions $u( \pm 1, t)=0=u_{x}(1, t)$ and initial condition $u(x, 0)=u_{0}(x)$, with the same exact solution as before, spectral convergence of the space-time Chebyshev collocation method is shown in the left figure of Fig. 2.

Next consider the beam equation

$$
u_{t t}+u_{x x x x}=f
$$

with clamped boundary conditions $u( \pm 1, t)=0=u_{x}( \pm 1, t)$ and initial conditions $u(x,-1)=u_{0}(x), u_{t}(x,-1)=u_{1}(x)$. Take $f$ so that the exact solution is $e^{x+t} \sin (\pi t / 2) \sin ^{2}(\pi x)$, the spectral convergence of the Chebyshev collocation method can be seen in the right figure of Fig. 2.

The spectrum of the various spectral Chebyshev operators for the case $N=60$ and plots of the spectral condition numbers as functions of $N$ are shown in Figs. 3, 4, 5, 6.


Fig. 3. Spectrum (left) and spectral condition number (right) for the Schrodinger operator $A_{s}$.


Fig. 4. Spectrum (left) and spectral condition number (right) for the wave operator $A_{w}$.


Fig. 5. Spectrum (left) and spectral condition number (right) for the Airy operator $A_{a}$.


Fig. 6. Spectrum (left) and spectral condition number (right) for the beam operator $A_{b}$.
Now we move onto nonlinear PDEs. For all nonlinear PDEs, we take as initial guess the zero function and use the iteration defined for each nonlinear PDE. The iteration is stopped whenever the infinity norm of the difference of two consecutive iterates is smaller than $\epsilon=10^{-13}$. Consider first the nonlinear reaction diffusion equation

$$
u_{t}=u_{x x}+\lambda e^{u}+f
$$

with homogeneous boundary conditions. Take $\lambda=0.5$ and $f$ so that the exact solution is $u(x, t)=e^{x+t} \cos (\pi x / 2)$. See the left figure of Fig. 7 for the convergence.

Next consider the nonlinear Schrodinger equation

$$
i u_{t}-u_{x x}+|u|^{2} u=f
$$

with homogeneous Dirichlet boundary conditions. Take $f$ so that the exact solution is $u(x, t)=e^{x+t} \sin (\pi x)$. See the right figure of Fig. 7 for the convergence.

Next consider the KdV equation

$$
u_{t}+u u_{x}+u_{x x x}=f
$$

with boundary conditions $u( \pm 1, t)=0=u_{x}(1, t)$. Take $f$ so that the exact solution is $u(x, t)=\cos (x-t)(x-1)^{2}(x+1)$. See the left figure of Fig. 8 for the convergence.

Next consider the Sine-Gordon equation

$$
u_{t t}=u_{x x}+\sin u+f
$$

with homogeneous Dirichlet boundary conditions. Take $f$ so that the exact solution is $u(x, t)=e^{x+t} \sin \pi x$. See the right figure of Fig. 8 for the convergence.

Next consider the nonlinear Kuramoto-Sivashinsky equation

$$
u_{t}+u_{x x x x}+u_{x x}+u u_{x}=f
$$

with clamped boundary conditions $u( \pm 1, t)=0=u_{x}( \pm 1, t)$ and initial conditions $u(x,-1)=u_{0}(x)$. Take $f$ so that the exact solution is $e^{x+t} \sin ^{2}(\pi x)$, the spectral convergence of the Chebyshev collocation method can be seen in the left figure of Fig. 9.

Finally, consider the Cahn-Hilliard equation

$$
u_{t}+u_{x x x x}+u_{x x}-\left(u^{2}\right)_{x x}=f
$$

with boundary conditions $u_{x}( \pm 1, t)=0=u_{x x x}( \pm 1, t)$ and initial conditions $u(x,-1)=u_{0}(x)$. Take $f$ so that the exact solution is $\cos (t) \cos (\pi x)$, the spectral convergence of the Legendre space-time collocation method can be seen in the right figure of Fig. 9. This PDE was the most difficult to solve. The stopping criterion was reduced to $\epsilon=10^{-9}$.

## 8. Discussion and conclusion

In this paper, we have proposed very simple space-time spectral collocation methods for some linear time dependent PDEs: Schrodinger, wave, Airy and beam equations. The entire MATLAB code of each PDE (including calculation of Legendre


Fig. 7. Convergence of Chebyshev collocation method for the nonlinear reaction diffusion equation (left) and nonlinear Schrodinger equation (right).


Fig. 8. Convergence of Chebyshev collocation method for the KdV (left) and Sine-Gordon (right) equations.


Fig. 9. Convergence of Chebyshev collocation method for the Kuramoto-Sivashinsky (left) and Cahn-Hilliard (right) equations.
and Chebyshev Lobatto points and plotting the errors) consists of no more than 50 lines. As far as we know, the proof of spectral convergence for the Schrodinger and wave equations is new. The condition number estimates of the global Chebyshev space-time operators are also new. The proofs require a new estimate of eigenvalues of the Chebyshev third derivative matrix, which was conjectured more than 20 years ago, and is proved for the first time here. The proof of the fourth-order case is still incomplete because we still need to prove that the spectrum of the Chebyshev fourth derivative matrix is real.

We did manage to show that the spectrum lies in the right half-plane and bounded away from the imaginary axis. Numerical results confirm nicely the theory of the paper. Some simple experiments for common nonlinear PDEs (nonlinear reaction diffusion equation from combustion, nonlinear Schrodinger, Sine-Gordon, KdV, Kuramoto-Sivashinsky and CahnHilliard equations) demonstrate numerically full space-time convergence. It is remarkable that space-time spectral methods work so well for these different classical PDEs with different features: diffusion, dispersion, nonlinear advection, etc.

Although we have only considered one spatial dimension, the general case of a spatial domain $(-1,1)^{d}$ follows immediately. Also, the implementation of the collocation method for general linear variable coefficient PDEs with standard linear boundary conditions is quite straightforward.

Space-time methods are extremely robust methods which converge spectrally for most standard linear PDEs with standard boundary conditions. They deserve more investigations, especially more sophisticated algorithms to speed up the linear algebra.

## CRediT authorship contribution statement

S.H. Lui: Conceptualization, Formal analysis, Funding acquisition, Software, Supervision, Writing S. Nataj: Formal analysis, Software, Writing

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

[1] S.H. Lui, Legendre spectral collocation in space and time for PDEs, Numer. Math. 136 (2017) 75-99.
[2] S.H. Lui, S. Nataj, Chebyshev spectral collocation in space and time for the heat equation, Electron. Trans. Numer. Anal. 52 (2020) $295-319$.
[3] T. Tang, X. Xu, Accuracy enhancement using spectral postprocessing for differential equations and integral equations, Commun. Comput. Phys. 5 (2) (2009) 779-792.
[4] B.Y. Guo, Z.Q. Wang, Legendre-Gauss collocation methods for ordinary differential equations, Adv. Comput. Math. 30 (3) (2009) $249-280$.
[5] H. Tal-Ezer, Spectral methods in time for parabolic problems, SIAM J. Numer. Anal. 26 (1) (1989) 1-11.
[6] H. Tal-Ezer, Spectral methods in time for hyperbolic equations, SIAM J. Numer. Anal. 23 (1) (1986) 11-26.
[7] G. Ierley, B. Spencer, R. Worthing, Spectral methods in time for a class of parabolic partial differential equations, J. Comput. Phys. 102 (1) (1992) $88-97$.
[8] J. Shen, L.-L. Wang, Fourierization of the Legendre-Galerkin method and a new space-time spectral method, Appl. Numer. Math. 57 (5) (2007) 710-720.
[9] W. Liu, B. Wu, J. Sun, Space-time spectral collocation method for the one-dimensional Sine-Gordon equation, Numer. Methods Partial Differ. Equ. 31 (3) (2015) 670-690.
[10] L. Yi, Z. Wang, Legendre spectral collocation method for second-order nonlinear ordinary partial differential equations, Discrete Continuous Dyn. Syst., Ser. B 19 (2014) 299-322.
[11] L. Yi, Z. Wang, Legendre Gauss type spectral collocation algorithms for nonlinear ordinary partial differential equations, Int. J. Comput. Math. 91 (7) (2014) 1434-1460.
[12] J.G. Tang, H.P. Ma, Single and multi-interval Legendre $\tau$-methods in time for parabolic equations, Adv. Comput. Math. 17 (4) (2002) $349-367$.
[13] J.G. Tang, H.P. Ma, A Legendre spectral method in time for first-order hyperbolic equations, Appl. Numer. Math. 57 (1) (2007) 1-11.
[14] A.Y. Suhov, A spectral method for the time evolution in parabolic problems, J. Sci. Comput. 29 (2) (2006) 201-217.
[15] A. Townsend, S. Olver, The automatic solution of partial differential equations using a global spectral method, J. Comput. Phys. 299 (2015) 106-123.
[16] Y. Qin, H.P. Ma, Legendre-tau-Galerkin and spectral collocation method for nonlinear evolution equations, Appl. Numer. Math. 153 (2020) $52-65$.
[17] M.J. Gander, 50 years of time parallel time integration: multiple shooting and time domain decomposition methods, Contrib. Math. Comput. Sci. 9 (2015) 69-113.
[18] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods-Fundamentals in Single Domains, Springer-Verlag, New York, 2006.
[19] C. Bernardi, Y. Maday, Spectral methods, in: Techniques of Scientific Computing, Part 2, in: Handbook of Numerical Analysis, vol. V, Elsevier, Amsterdam, 1997.
[20] L.N. Trefethen, Spectral Methods in Matlab, Society for Industrial and Applied Mathematics, Philadelphia, 2000.
[21] J. Shen, T. Tang, L.L. Wang, Spectral Methods, Springer-Verlag, Berlin, Heidelberg, 2011.
[22] H.P. Ma, W. Sun, A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations, SIAM J. Numer. Anal. 38 (5) (2000) 1425-1438.
[23] S.H. Lui, Numerical Analysis of Partial Differential Equations, John Wiley \& Sons, Inc., Hoboken, NJ, 2011.
[24] W. Heinrichs, Spectral approximation of third-order problems, J. Sci. Comput. 14 (3) (1999) 275-289.
[25] A. Solomonoff, E. Turkel, Global properties of pseudospectral methods, J. Comput. Phys. 81 (2) (1989) 239-276.
[26] L.N. Trefethen, J.A.C. Weideman, The eigenvalues of second-order spectral differentiation matrices, SIAM J. Numer. Anal. 25 (6) (1988) $1279-1298$.
[27] C. Bernardi, Y. Maday, Some spectral approximations of one-dimensional fourth-order problems, Progress in Approximation Theory, Academic Press, Boston, MA, 1991.
[28] D. Funaro, Polynomial Approximation of Differential Equations, Springer-Verlag, Berlin, 1992.
[29] D. Funaro, W. Heinrichs, Some results about the pseudospectral approximation of one-dimensional fourth-order problems, Numer. Math. 58 (4) (1990) 399-418.
[30] S.C. Reddy, J.A.C. Weideman, The accuracy of the Chebyshev differencing method for analytic functions, SIAM J. Numer. Anal. 42 (5) (2005) $2176-2187$.
[31] R.H. Bartels, G.W. Stewart, Solution of the matrix equation $A X+X B=C$, Commun. ACM 15 (9) (1972) 820-826.


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