Applied Numerical Mathematics 187 (2023) 206–234

Contents lists available at ScienceDirect

Applied Numerical Mathematics

journal homepage: www.elsevier.com/locate/apnum

Space-time spectral method for the Stokes problem

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ARTICLE INFO

Article history:
Received 17 October 2022
Received in revised form 31 January 2023
Accepted 17 February 2023
Available online 24 February 2023

Keywords:
Condition number
Recombined Legendre polynomials
Space-time
Spectral collocation
Spectral Galerkin
Stokes problem

ABSTRACT

The Stokes equations are a linearized version of the Navier-Stokes equations and model incompressible viscous fluid flow with low Reynolds numbers. Several spectral methods, exhibiting exponential decay in error when the solution is analytic, are known to solve the steady-state Stokes problem numerically. A common strategy to solve such a problem in the time-dependent case involves extending the spectral scheme in spatial derivatives by implementing a low-order finite difference scheme for the time derivatives. Instead, we implement and analyze a space-time spectral method for the Stokes problem, which converges exponentially in both space and time. This numerical scheme imposes spectral collocation in time and $P_N - P_{N-2}$ spectral Galerkin scheme in space by using a recombined Legendre polynomial basis, resulting in a global spectral operator that is a saddle point matrix. The main objectives of the research are estimating the condition number of the global spectral operators and proving the spectral convergence of this scheme in space and time. The analysis is not quite complete because two of the estimates are based on numerical evidence. However, throughout the project, some intermediate results-such as the 2-norm of the pseudospectral derivative matrix for Chebyshev-Gauss-Lobatto nodes as well as the condition number of the mass matrix and discrete Laplacian for a recombined Legendre basis-were proved to obtain the aforementioned findings. Numerical experiments of this scheme verify the theoretical results. Also, the numerical result of applying this scheme to the Navier-Stokes equations is presented.

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1. Introduction

One of the topics investigated extensively in fluid dynamics is devising numerical schemes to solve the Stokes problem. The type of flow for which the Reynolds number is low, say $R_e \ll 1$, i.e., the fluid velocity is extremely small, or the viscosity is very large, or an infinitesimal length scale is considered, is called the Stokes flow (or creeping flow). This type of flow is evident in many cases, such as swimming of a microorganism, flow of lava, flow of polymers, etc. The equations of motion for Stokes flow are called the Stokes equations, which along with suitable boundary and initial conditions are termed the Stokes problem:

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[1] This work was funded by the University of Manitoba Graduate Fellowship (UMGF) from the University of Manitoba, and parts of the research were enabled by the Digital Research Alliance of Canada.
[2] This work was funded by the Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant number RGPIN-2022-03665.

https://doi.org/10.1016/j.apnum.2023.02.013
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$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= f \text{ in } \Omega := (-1,1)^2,\\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega,\\
\mathbf{u} &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{1}
$$

where velocity field and pressure are denoted by $\mathbf{u} = [u; v] \in V := (H_0^1(\Omega))^2$ and $p \in L_0^2(\Omega) := \left\{ q \in L^2(\Omega) \,\middle|\, \int_\Omega q = 0 \right\}$, respectively. In an unsteady state,

$$
\begin{aligned}
\mathbf{u}_t - \Delta \mathbf{u} + \nabla p &= f \text{ in } \Omega_t := \Omega \times (-1,1),\\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega_t,\\
\mathbf{u} &= 0 \text{ on } \partial\Omega \times (-1,1),\\
\mathbf{u}(x,y,-1) &= \mathbf{u}_0(x,y) \text{ in } \Omega,
\end{aligned}
\tag{2}
$$

where $\mathbf{u}(\cdot,t) \in (H_0^1(\Omega))^2$, $p(\cdot,t) \in L_0^2(\Omega)$, and $\mathbf{u}_0 \in (H_0^1(\Omega))^2$.

Spectral methods are numerical methods that solve differential and integral equations. These methods have been used extensively due to their fast convergence rates, i.e., the error decreases exponentially when the solution is analytic, also termed spectral convergence. In general, the error depends super-algebraically on the smoothness of the solution. For instance, Theorem 4.1 and (4.77) on [34, p. 166] imply that the convergence rate in H^1-norm for solution $u \in H_0^1(-1,1)$, $\partial_x u \in B_{0,0}^{m-1}(-1,1)$ of a second order differential equation by Legendre-Galerkin method is at least $\mathcal{O}(n^{1-m})$, where n is the discretization parameter. A considerable body of literature on spectral methods exist, including [23,28,34,44,43,11].

In [3], the authors describe three spectral methods for solving the Stokes problem. The first method is called the single grid scheme, a collocation type using the same degree of polynomials for velocity and pressure; however, it generates spurious modes for pressure and, hence, is not used. The second method, the $P_N - P_{N-2}$ scheme, is a spectral Galerkin scheme that uses polynomials of degree N for velocity and $N-2$ for pressure. The third one, the staggered grid scheme, is a spectral collocation method that uses staggered grids for velocity and pressure. In this research work, we focus on the second scheme. For all three methods, the inf-sup condition is not bounded independently of the discretization parameter of the scheme that decreases the accuracy of the error for pressure, which has been improved in [4] by proposing smaller discrete spaces for pressure.

In the past few years, space-time spectral methods, exhibiting spectral convergence in both space and time, are being used to solve time-dependent PDEs. A set practice was to implement a low-order finite difference approximation of the time derivative, which does not give spectral convergence for the whole scheme due to the dominance of the time discretization error. See [22,15] for such schemes for linear PDEs, [27,42,7,2,6] and the references therein for problems related to the Stokes problem. The numerical schemes involving spectral collocation in space and finite difference schemes in time possess the fundamental and crucial theoretical difficulty of controlling the aliasing error. For incompressible Navier-Stokes equations, long-time stability analysis for some semi-implicit numerical schemes in time and Fourier pseudospectral schemes in space was presented in [13,8], and that for Burgers' equation was conducted in [12]. Growing appeals for faster convergence in time generated the class of space-time spectral methods, some references for which are [17,29,35,14,33,38,37,39,40,21,45,46]. A space-time spectral collocation method given in [41] was analyzed in [24] and [25] for Legendre and Chebyshev polynomials, respectively, based on which schemes for some linear PDEs were analyzed in [26], which serves as the motivation for this paper.

The aim of this work is to perform a condition number estimate for a spectral method for the steady Stokes equations, and propose and analyze a space-time spectral method for the Stokes problem based upon an $P_N - P_{N-2}$ scheme. The Stokes equations are more difficult to handle than heat or wave equations because they are a system of PDEs possessing different spaces for velocity and pressure. Thus, it requires an analysis of various terms appearing in the discrete problem, which includes proving condition number estimates for the stiffness matrix, mass matrix and discrete Laplacian for a recombined Legendre basis derived in [32].

We also prove an estimate for the maximum singular value or the 2-norm for the Chebyshev-Gauss-Lobatto pseudospectral derivative matrix. This matrix is non-symmetric with an indefinite symmetric part, which makes the analysis more challenging. We believe our analysis of spectral convergence of the unsteady Stokes equations is new. We have also laid the groundwork for a condition number estimate of the global space-time operator. Moreover, a scheme is designed for the unsteady Navier-Stokes problem, by considering Chebyshev collocation in time and the $P_N - P_{N-2}$ scheme in space.

A shortcoming of using such spectral in time schemes is that they do not allow time stepping, the unknowns for all time need to be solved simultaneously. However, far fewer unknowns are required in comparison to finite difference discretizations in time. The results of the numerical experiments found clear support for the spectral convergence for space-time spectral schemes for less than 20 spectral modes in each dimension, see the numerical results provided in [24].

We begin by summarizing some of the notations. Throughout this paper, the discretization parameter is denoted by N, besides, c and C denote some positive constants independent of N. A column vector x with n components is represented by $x = [x_1; x_2; \ldots; x_n] \in \mathbb{R}^n$, and its 2-norm is denoted by $|\cdot|$, i.e., $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, whereas ∞-norm is given as $|x|_\infty = \max\limits_{1\leqslant i\leqslant n} |x_i|$.

For an $n \times n$ matrix M, let $M]$ denote the $n \times (n-1)$ matrix obtained from M by deleting the first column, $[M$ denote the $(n-1) \times n$ matrix obtained from M by deleting the first row, $\mathbf{[}M\mathbf{]}$ denote the $(n-1) \times (n-1)$ matrix obtained from M by deleting the first column and row, and $[\![M]\!]$ denote the $(n-2) \times (n-2)$ matrix obtained from M by deleting the first and last columns and rows. The spectrum of M is denoted by $\Lambda(M)$. Let the eigenvalues of M be represented by $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \ldots \geqslant \lambda_n(M)$. Also, $\lambda_{\max}(M) := \lambda_1(M)$ and $\lambda_{\min}(M) := \lambda_n(M)$ denote the maximum and minimum eigenvalues of M, respectively.

For any matrix, $M \in \mathbb{R}^{m\times n}$, $\sigma(M)$ denotes the set of singular values of M, which are represented by $\sigma_1(M) \geqslant \sigma_2(M) \geqslant \ldots \geqslant \sigma_{\min(m,n)}(M) \geqslant 0$. The maximum and minimum singular values of M are represented by $\sigma_{\max}$ and $\sigma_{\min}$, respectively. The 2-norm, 1-norm, and ∞-norm of a matrix $M \in \mathbb{R}^{m\times n}$ are denoted by $\|M\|$, $\|M\|_1$, and $\|M\|_\infty$, respectively. Also, the term SPD is used to refer to a symmetric positive definite matrix.

The Kronecker product of the matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p\times q}$ is denoted by $K = A \otimes B \in \mathbb{R}^{mp\times nq}$ and is defined to be the block matrix with blocks $K_{ij} = a_{ij}B$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. Also, their direct sum is defined as $A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathbb{R}^{(m+p)\times(n+q)}$. They satisfy the properties $(A \otimes B)^T = A^T \otimes B^T$, and $(A \otimes B)(C \otimes D) = AC \otimes BD$, where $C \in \mathbb{R}^{n\times r}$, $D \in \mathbb{R}^{q\times s}$. Also, $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \cdot \operatorname{rank}(B)$, $\operatorname{rank}(A \oplus B) = \operatorname{rank}(A) + \operatorname{rank}(B)$, $\sigma(A \otimes B) = \sigma(A) \times \sigma(B)$, $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, thus, $\|A \otimes B\| = \|A\| \cdot \|B\|$. For $m = n$ and $p = q$, $\Lambda(A \otimes B) = \Lambda(A) \times \Lambda(B)$, $\Lambda(A \oplus B) = \Lambda(A) \cup \Lambda(B)$, including algebraic multiplicities in all cases.

Let P_N be the space of polynomials of degree less than or equal to N, and P_N^0 denote the polynomials in P_N that vanish at the endpoints $x = \pm 1$. Let $\mathbb{P}_{n_1,n_2}$ be the space of polynomials of degree at most n_1, n_2 in x, y, respectively and $\mathbb{P}^0_{n_1,n_2} = \mathbb{P}_{n_1,n_2} \cap V$, i.e., they vanish on $\partial\Omega$. Let $\mathbb{P}_{n_1,n_2,m}$ be the space of polynomials of degree at most n_1, n_2 in x, y, and degree at most m in time. Moreover, define $\mathbb{P}^0_{n_1,n_2,m}$ as the polynomials in $\mathbb{P}_{n_1,n_2,m}$ that vanish on the boundary of the spatial domain Ω. Let L_j and T_j denote the Legendre and Chebyshev polynomial of the first kind of degree j, respectively. The norm of p is given as $\|p\|_0 = \left[\int_\Omega |p|^2\right]^{\frac{1}{2}}$, and the inner product on the space $L^2_0(\Omega)$ is defined to be the same as that for $L^2(\Omega)$, which is defined as $(f, g) = \int_\Omega fg dx$, for $f, g \in L^2(\Omega)$.

This paper is structured as follows. In section 2, we list some important existing results and derive Proposition 1, which will be used in the next sections. In section 3, we implement the $P_N - P_{N-2}$ scheme by using a recombined Legendre polynomial basis for the steady Stokes problem and prove the condition number estimates for the scheme in sections 3.1 and 3.2, respectively. We extend the $P_N - P_{N-2}$ scheme to the unsteady Stokes problem by using Chebyshev Gauss-Lobatto collocation in time in section 4.1. Moreover, in sections 4.2 and 4.3, we respectively prove the condition number estimates and spectral convergence in space and time. In section 5, we adapt the $P_N - P_{N-2}$ scheme for the unsteady Navier-Stokes equations. Section 6 describes the numerical experiments verifying the spectral convergence of all the schemes derived in the previous sections. Finally, we conclude the findings of this paper and discuss some future work in section 7.

2. Fundamentals

We begin by presenting some basic definitions, the first three of which are given in [34, p. 145–146]. For $x \in [-1, 1]$ and $j \in \mathbb{N} \cup \{0\}$, define the polynomial $\phi_j(x) := L_j(x) - L_{j+2}(x)$, so that $\phi_j(\pm 1) = 0$, thus $\phi_j \in P^0_{j+2}$. The stiffness matrix, denoted by S, is defined as $s_{jk} := -\int_{-1}^1 \phi_k''(x)\phi_j(x)dx$, is a diagonal matrix with entries given as follows,

$$s_{kk} = (4k + 6). \tag{3}$$

The mass matrix, denoted by M, is defined as $m_{jk} = \int_{-1}^1 \phi_j(x)\phi_k(x)dx$. It is a symmetric penta-diagonal matrix whose non-zero elements are given as follows,

$$m_{jk} = m_{kj} = \begin{cases} \frac{2}{2k+1} + \frac{2}{2k+5}, & j = k, \\ -\frac{2}{2k+5}, & j = k+2. \end{cases} \tag{4}$$

For $N \geqslant 4$, let x_j be the Chebyshev Gauss-Lobatto quadrature nodes, defined as $x_j = -\cos(\frac{\pi j}{N})$ for $0 \leqslant j \leqslant N$. Let $\tilde{c}_0 = \tilde{c}_N = 2$ and $\tilde{c}_j = 1$ for $1 \leqslant j \leqslant N_1$. The Chebyshev Gauss-Lobatto pseudospectral derivative matrix is defined as $D := [d_{kj}]_{0\leqslant k,j\leqslant N+1}$, where $d_{kj} = \ell_j'(x_k)$ given as follows in [34, p. 110].

$$d_{kj} = \begin{cases} -\frac{2N^2+1}{6}, & j = k = 0, \\ \frac{\tilde{c}_k(-1)^{k+j}}{\tilde{c}_j(x_k - x_j)}, & 0 \leqslant k \neq j \leqslant N, \\ -\frac{x_k}{2(1-x_k^2)}, & 1 \leqslant k = j \leqslant N-1, \\ \frac{2N^2+1}{6}, & k = j = N. \end{cases} \tag{5}$$

Some results from matrix analysis play a vital role for proving the results in sections 3 and 4, such as Weyl's inequalities in [16, p. 239] for symmetric matrices $A, E \in \mathbb{R}^{n\times n}$:

$$\lambda_k(A) + \lambda_n(E) \leqslant \lambda_k(A+E) \leqslant \lambda_k(A) + \lambda_1(E), \quad 1 \leqslant k \leqslant n, \tag{6}$$

and the spectrum of a symmetric saddle point matrix, $\mathcal{X} = \begin{bmatrix} A & B^T \\ B & O \end{bmatrix}$ such that $B \in \mathbb{R}^{m\times n}$ is full rank and its Schur complement $BA^{-1}B^T$ is SPD, then [1] proves that

$$\Lambda(\mathcal{X}) \subseteq \left[\frac{-\lambda_1}{\frac{1}{2}\left(1+\sqrt{1+4\frac{\lambda_1}{\mu_1}}\right)}, \frac{-\lambda_m}{\frac{1}{2}\left(1+\sqrt{1+4\frac{\lambda_m}{\mu_n}}\right)} \right] \cup \left[\mu_n, \mu_1 \frac{1+\sqrt{1+4\frac{\lambda_1}{\mu_1}}}{2} \right], \tag{7}$$

where $0 < \mu_n \leqslant \ldots \leqslant \mu_1$ denote the eigenvalues of A and $0 < \lambda_m \leqslant \ldots \leqslant \lambda_1$ are the eigenvalues of $BA^{-1}B^T$. In general, for a non-singular $\mathcal{X} = \begin{bmatrix} A & B^T \\ B & O \end{bmatrix}$, so that A and B are full rank, [18, Chap. 5] and [20] give the following estimate

$$\sigma_{\min}(\mathcal{X}) \geqslant \sqrt{1-\cos\theta} \cdot \min\{\sigma_{\min}(A), \sigma_{\min}(B)\}, \tag{8}$$

where θ is the minimum principal angle[3] between the range space $\mathcal{R}\left(\begin{bmatrix} A & B^T \end{bmatrix}^T\right)$ and $\mathcal{R}\left(\begin{bmatrix} B & O \end{bmatrix}^T\right)$. Finally, we derive the following result for assistance in the analysis performed in the next sections.

Proposition 1. For $0 \leqslant j, k \leqslant N-1$, the matrices R and Q defined by $r_{jk} := \int_{-1}^{1} L_k(x)\phi_j'(x)dx$ and $q_{jk} := \int_{-1}^{1} L_k(x)\phi_j(x)dx$, respectively, satisfy

$$r_{j,j+1} = -2, \qquad q_{jk} = \begin{cases} \gamma_j, & k = j, \\ -\gamma_{j+2}, & k = j+2, \end{cases} \tag{9}$$

where $\gamma_j = \|L_j\|_0^2 = \frac{2}{2j+1}$.

Proof. Since $\phi_j = L_j - L_{j+2}$,

$$r_{jk} = \int_{-1}^{1} L_k(x)\phi_j'(x)dx = \int_{-1}^{1} L_k(x)(L_j'(x) - L_{j+2}'(x))dx.$$

Using the recurrence relation, $(2j+1)L_j(x) = L_{j+1}'(x) - L_{j-1}'(x)$, for $j \in \mathbb{N}$,

$$r_{jk} = -(2j+3)\int_{-1}^{1} L_k(x)L_{j+1}(x)dx = -(2j+3)\frac{2\delta_{k,j+1}}{2j+3}.$$

Similarly,

$$q_{jk} = \int_{-1}^{1} L_k(x)(L_j(x) - L_{j+2}(x))dx = \delta_{k,j}\gamma_j - \delta_{k,j+2}\gamma_{j+2}. \quad \square$$

3. Steady state

The Stokes problem in the steady state is given by (1), which in component form can be expressed as:

$$-\Delta u + p_x = f_1 \text{ in } \Omega, \tag{10a}$$
$$-\Delta v + p_y = f_2 \text{ in } \Omega, \tag{10b}$$
$$u_x + v_y = 0 \text{ in } \Omega, \tag{10c}$$
$$u = 0,\ v = 0 \text{ on } \partial\Omega, \tag{10d}$$

We first describe a spectral discretization, the analysis for which will come into play in the later sections.

[3] $\cos\theta = \sigma_{\min}(Q_1^T Q_2)$, where $Q_1 \in \mathbb{R}^{m\times p}$ and $Q_2 \in \mathbb{R}^{m\times q}$ represent a orthogonal bases for $\mathcal{R}\left(\begin{bmatrix} A & B^T \end{bmatrix}^T\right)$ and $\mathcal{R}\left(\begin{bmatrix} B & O \end{bmatrix}^T\right)$, see [18, p. 123] or [20] for details.

3.1. Discretization

We implement the $P_N - P_{N-2}$ scheme described in [3], by defining the variables as follows:

$$u_N(x,y) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} u_{ij}\phi_i(x)\phi_j(y) \in \mathbb{P}^0_{N,N},$$

$$v_N(x,y) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} v_{ij}\phi_i(x)\phi_j(y) \in \mathbb{P}^0_{N,N},$$

$$p_{N-2}(x,y) = \sum_{i=0}^{N-2}\sum_{\substack{j=0\\ i+j>0}}^{N-2} p_{ij}L_i(x)L_j(y) \in \mathbb{P}_{N-2,N-2}\cap L^2_0(\Omega),$$

so that $\int_\Omega p_{N-2} = 0$, i.e., it has zero average. Define $\vartheta = (N-1)^2$, the number of unknowns for u_N and v_N each, and $\wp = (N-1)^2 - 1$, the number of unknowns for p_{N-2}. The total number of unknowns in the discrete Stokes equations are $2\vartheta + \wp = 3(N-1)^2 - 1$.

Define the discrete unknowns as $u_h = [u_{00}; u_{10}; \ldots; u_{N-2,0}; u_{01}; \ldots; u_{N-2,N-2}] \in \mathbb{R}^\vartheta$, and similarly define v_h, $p_h = [p_{10}; p_{20}; \ldots; p_{N-2,0}; p_{01}; \ldots; p_{N-2,N-2}] \in \mathbb{R}^\wp$, and $F_k = [f^k_{00}; f^k_{10}; \ldots; f^k_{N-2,0}; f^k_{01}; \ldots; f^k_{N-2,N-2}] \in \mathbb{R}^\vartheta$, where

$$f_k = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} f^k_{ij}L_i(x)L_j(y),$$

for $k = 1, 2$. Then, the discrete Stokes problem for the weak form of (10) is

$$\begin{aligned}
&(M\otimes S + S\otimes M)\,u_h - (Q\otimes R)]p_h = (Q\otimes Q)\,F_1,\\
&(M\otimes S + S\otimes M)\,v_h - (R\otimes Q)]p_h = (Q\otimes Q)\,F_2,\\
&-[\left(Q^T\otimes R^T\right)u_h - [\left(R^T\otimes Q^T\right)v_h = O_{\wp,1},
\end{aligned}$$

where $S = [s_{ij}]$, $M = [m_{ij}]$, $Q = [q_{ij}]$, and $R = [r_{ij}]$, for $0 \leqslant i, j \leqslant N-2$. Define the discrete Laplacian $A = M\otimes S + S\otimes M \in \mathbb{R}^{\vartheta\times\vartheta}$, $B_1 = -(Q\otimes R)] \in \mathbb{R}^{\vartheta\times\wp}$, and $B_2 = -(R\otimes Q)] \in \mathbb{R}^{\vartheta\times\wp}$, recall that] means the first column is deleted. The spectral convergence of this scheme is suggested by Fig. 1. Furthermore, the global spectral operator of the discrete Stokes problem is defined as,

$$G = \begin{bmatrix} \mathcal{A} & B \\ B^T & O_{\wp,\wp}\end{bmatrix} \in \mathbb{R}^{(2\vartheta+\wp)\times(2\vartheta+\wp)}, \text{ where } B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}^{2\vartheta\times\wp}, \tag{11}$$

and $\mathcal{A} = A\oplus A \in \mathbb{R}^{2\vartheta\times 2\vartheta}$.

Although for (1) the velocity $\mathbf{u}$ is divergence-free or $\nabla\cdot\mathbf{u} = 0$, that is not the case for the approximate solution obtained from the $P_N - P_{N-2}$ scheme. Observe that this method implements the weak form of the continuity equation, that is (10c), which is given as follows,

$$(q_{N-2}, u_{N_x} + v_{N_y}) = 0, \tag{12}$$

for all $q_{N-2} \in \mathbb{P}_{N-2,N-2}\cap L^2_0(\Omega)$. Therefore, $\mathbf{u}_N = [u_N; v_N]$ are not divergence-free, however, all divergence-free polynomials satisfy the above equation. This is not a drawback as it is overpowered by the property that this scheme eliminates the presence of any spurious modes on pressure. For more details, see [3, p. 416].

3.2. Analysis

In this section, we estimate the condition number of the global matrix G for the discretized steady Stokes problem given by (11). Since G is a symmetric saddle point matrix with an SPD leading block and full rank matrix B, finding the bounds for the spectrum of G is facilitated by the theory of the spectrum of a symmetric saddle point matrix with the desired properties. This analysis requires bounds on the spectrum of the sub-blocks of G, thus we proceed as follows.

Lemma 1. *For $N \geqslant 2$, let $S \in \mathbb{R}^{(N-1)\times(N-1)}$ be the Stiffness matrix defined by (3), then it is SPD with $\lambda_{\min}(S) = 6$ and $\lambda_{\max}(S) = 4N-2$, thus $\kappa(S) = \frac{2N-1}{3}$.*

Proof. By (3), S is a diagonal matrix with entries $s_{kk} = 4k+6$ for $0 \leqslant k \leqslant N-2$, thus $\lambda_{\min}(S) = 6$ and $\lambda_{\max}(S) = 4N-2$. Since the stiffness matrix S is an SPD, its condition number is $\kappa(S) = \frac{\sigma_{\max}(S)}{\sigma_{\min}(S)} = \frac{\lambda_{\max}(S)}{\lambda_{\min}(S)} = \frac{4N-2}{6}$. $\square$

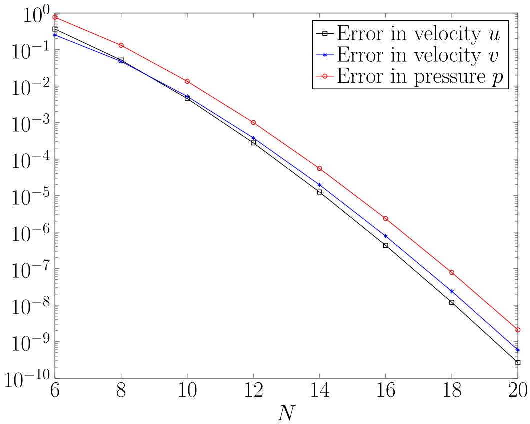


Fig. 1. Convergence for the $P_N - P_{N-2}$ scheme for the Stokes problem in a steady state.

The following results give optimal condition number estimates for the mass matrix and discrete Laplacian matrix in two dimensions for the recombined Legendre basis considered in this scheme, derived in [34] for Dirichlet boundary conditions. These results appear to be new.

Lemma 2. *For $N \geqslant 4$, let $M \in \mathbb{R}^{(N-1)\times(N-1)}$ be the mass matrix defined by (4). Then, M is SPD and $\dfrac{c}{N^3} \leqslant \Lambda(M) \leqslant C$, thus $\kappa(M) \leqslant cN^3$.*

Proof. Let $u(x) = \sum_{k=0}^{N-2} u_k \phi_k(x) \in \mathbb{P}_N^0$, where ϕ_k represent recombined Legendre basis functions and define $u_h := [u_0; u_1; \ldots; u_{N-2}] \in \mathbb{R}^{N-1}$, then

$$\|u\|_0^2 = \int_{-1}^{1} u(x)^2 dx = \sum_{j=0}^{N-2}\sum_{k=0}^{N-2} u_j u_k m_{jk} = u_h^T M u_h.$$

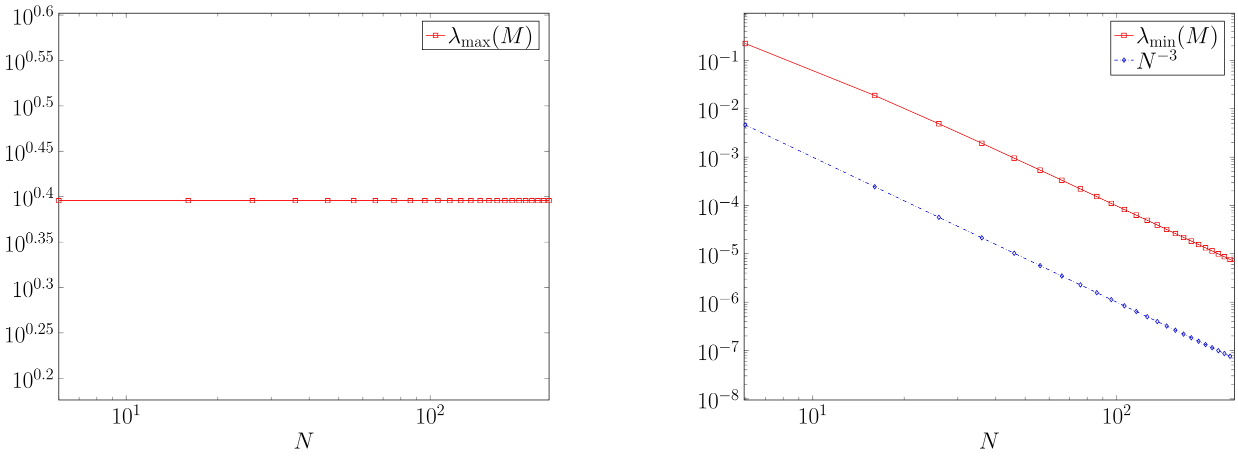
Hence M is SPD, for any $x \in \mathbb{R}^{N-1} \setminus \{0\}$, the bounds on the eigenvalues of M by estimating $x^T Mx$ are derived as follows

$$\begin{aligned} x^T Mx &= \sum_{k=0}^{N-2} x_k^2 m_{kk} + 2\sum_{k=0}^{N-4} x_k x_{k+2} m_{k,k+2} \\ &= 2\sum_{k=0}^{N-2} x_k^2 \left(\frac{1}{2k+1} + \frac{1}{2k+5}\right) - 4\sum_{k=0}^{N-4} \frac{x_k x_{k+2}}{2k+5}, \qquad (13) \\ &\leqslant 2\sum_{k=0}^{N-2} x_k^2 \left(1 + \frac{1}{5}\right) + \frac{4}{5}\sum_{k=0}^{N-4} |x_k||x_{k+2}| \\ &\leqslant \frac{12}{5}\sum_{k=0}^{N-2} x_k^2 + \frac{4}{5}\sum_{k=0}^{N-2} x_k^2 = \frac{16}{5}\sum_{k=0}^{N-2} x_k^2. \end{aligned}$$

Hence, $x^T Mx \leqslant C|x|^2$, therefore $\lambda_{\max}(M) \leqslant C$.

Note that

$$\begin{aligned} 4\sum_{k=0}^{N-4} \frac{x_k x_{k+2}}{2k+5} &\leqslant 4\sum_{k=0}^{N-4} \frac{\sqrt{2k+9}\cdot|x_k|}{2k+5} \cdot \frac{|x_{k+2}|}{\sqrt{2k+9}} \\ &\leqslant 2\sum_{k=0}^{N-4} \left(\frac{(2k+9)|x_k|^2}{(2k+5)^2} + \frac{|x_{k+2}|^2}{(2k+9)}\right) \\ &= 2\sum_{k=0}^{N-4} \frac{(2k+9)x_k^2}{(2k+5)^2} + 2\sum_{k=2}^{N-2} \frac{x_k^2}{(2k+5)}. \qquad (14) \end{aligned}$$



(a) Maximum eigenvalue of M. (b) Minimum eigenvalue of M

Fig. 2. Numerical results for $\Lambda(M)$.

Thus, the above result in (13) leads to

$$\begin{aligned} x^T M x &\geqslant 2\sum_{k=0}^{N-2} x_k^2\left(\frac{1}{2k+1}+\frac{1}{2k+5}\right) - 2\sum_{k=0}^{N-4}\frac{(2k+9)x_k^2}{(2k+5)^2} - 2\sum_{k=2}^{N-2}\frac{x_k^2}{(2k+5)} \\ &\geqslant 2\sum_{k=0}^{N-2} x_k^2\left(\frac{1}{2k+1}+\frac{1}{2k+5}-\frac{(2k+9)}{(2k+5)^2}-\frac{1}{(2k+5)}\right) \\ &= 32\sum_{k=0}^{N-2}\frac{x_k^2}{(2k+1)(2k+5)^2} \geqslant \frac{c}{N^3}|x|^2. \end{aligned}$$

Therefore, $\lambda_{\min}(M) \geqslant \dfrac{c}{N^3}$ and $\kappa(M) = \dfrac{\lambda_{\max}(M)}{\lambda_{\min}(M)} \leqslant cN^3$. $\square$

Note that the optimality of bounds derived in the above theorem is suggested by the results of the numerical experiments presented in Fig. 2. Since the discrete Laplacian matrix A, given in section 3.1, is defined in terms of the stiffness and mass matrices, thus we are ready to analyze the spectrum of A.

Theorem 1. *For $N \geqslant 4$, let $A \in \mathbb{R}^{\vartheta\times\vartheta}$ be the discrete Laplacian matrix defined by (11). Then, it is SPD and $\dfrac{c}{N^2} \leqslant \Lambda(A) \leqslant CN$, thus $\kappa(A) \leqslant cN^3$.*

Proof. Since $A \in \mathbb{R}^{\vartheta\times\vartheta}$ and is defined as $A = M \otimes S + S \otimes M$, it is SPD, as both M and S are SPD. Hence, (6) yields

$$\begin{aligned} \lambda_{\max}(A) &\leqslant \lambda_{\max}(M\otimes S) + \lambda_{\max}(S\otimes M) \\ &= \lambda_{\max}(M)\lambda_{\max}(S) + \lambda_{\max}(S)\lambda_{\max}(M) = 2\lambda_{\max}(M)\lambda_{\max}(S), \end{aligned}$$

where the last equality results from a property of the spectrum of a Kronecker product. Thus, Lemmas 1 and 2 give $\lambda_{\max}(A) \leqslant C(4N-2) \leqslant CN$.

The definition of S and M, given by the equations (3) and (4) respectively, implies that $A \in \mathbb{R}^{\vartheta\times\vartheta}$ is a symmetric block matrix with non-zero 0, 2 and -2 block diagonals with blocks defined as

$$A_{jk} = A_{kj} = \begin{cases} s_{jj}M + m_{jj}S, & j = k, \\ m_{jk}S, & j = k+2, \end{cases} \tag{15}$$

for $0 \leqslant j, k \leqslant N-2$. Let $x = [x_0; x_1; \ldots; x_{N-2}] \in \mathbb{R}^{\vartheta} \setminus \{0\}$, $x_k = \left[x_k^0; x_k^1; \ldots; x_k^{N-2}\right] \in \mathbb{R}^{N-1} \setminus \{0\}$ for each $0 \leqslant k \leqslant N-2$. Then

$$x^T A x = \sum_{k=0}^{N-2} x_k^T\left(s_{kk}M + m_{kk}S\right)x_k + 2\sum_{k=0}^{N-4} x_k^T m_{k,k+2} S x_{k+2}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N-2}(4k+6)\,x_k^T M x_k + \sum_{k=0}^{N-2}\left(\frac{2}{2k+1}+\frac{2}{2k+5}\right)x_k^T S x_k + 2\sum_{k=0}^{N-4}\frac{-2x_k^T S x_{k+2}}{2k+5} && \text{(by eqs. (3) and (4))}\\
&= \sum_{k=0}^{N-2}(4k+6)\left(\sum_{j=0}^{N-2}(x_k^j)^2\left(\frac{2}{2j+1}+\frac{2}{2j+5}\right)-4\sum_{j=0}^{N-4}\frac{x_k^j x_k^{j+2}}{2j+5}\right) && \text{(by eqs. (3) and (13))}\\
&\quad + \sum_{j=0}^{N-2}(4j+6)\left(\sum_{k=0}^{N-2}(x_k^j)^2\left(\frac{2}{2k+1}+\frac{2}{2k+5}\right)-4\sum_{k=0}^{N-4}\frac{x_k^j x_{k+2}^j}{2k+5}\right)\\
&\geqslant 2\sum_{k=0}^{N-2}(4k+6)\left(\sum_{j=0}^{N-2}(x_k^j)^2\left(\frac{1}{2j+1}+\frac{1}{2j+5}\right)-\sum_{j=0}^{N-4}\frac{(2j+9)(x_k^j)^2}{(2j+5)^2}\right. && \text{(by eq. (14))}\\
&\quad \left. -\sum_{j=2}^{N-2}\frac{(x_k^j)^2}{(2j+5)}\right) + 2\sum_{j=0}^{N-2}(4j+6)\left(\sum_{k=0}^{N-2}(x_k^j)^2\left(\frac{1}{2k+1}+\frac{1}{2k+5}\right)\right.\\
&\quad \left. -\sum_{k=0}^{N-4}\frac{(2k+9)(x_k^j)^2}{(2k+5)^2}-\sum_{k=2}^{N-2}\frac{(x_k^j)^2}{(2k+5)}\right)\\
&= 2\sum_{k=0}^{N-2}(4k+6)\left(\sum_{j=0}^{1}\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right)(x_k^j)^2\right.\\
&\quad \left. + \sum_{j=2}^{N-4}\frac{16(x_k^j)^2}{(2j+1)(2j+5)^2}+\sum_{j=N-3}^{N-2}\frac{(x_k^j)^2}{2j+1}\right)\\
&\quad + 2\sum_{j=0}^{N-2}(4j+6)\left(\sum_{k=0}^{1}\left(\frac{1}{2k+1}+\frac{1}{2k+5}-\frac{2k+9}{(2k+5)^2}\right)(x_k^j)^2\right.\\
&\quad \left. + \sum_{k=2}^{N-4}\frac{16(x_k^j)^2}{(2k+1)(2k+5)^2}+\sum_{k=N-3}^{N-2}\frac{(x_k^j)^2}{2k+1}\right).
\end{aligned}
$$

The above expression can be expressed in terms of nine double summations as

$$
x^T A x \geqslant 2\sum_{m=1}^{9}\mathcal{S}_m, \tag{16}
$$

which along with eqs. (A.1) to (A.7) imply $x^T A x \geqslant \frac{c}{N^2}x^T x$, hence the desired result. □

Since $\Lambda(A\oplus A)=\Lambda(A)$, the above theorem gives the following estimate.

Corollary 1. *For $N\geqslant 4$, $\mathcal{A}\in\mathbb{R}^{2\vartheta\times 2\vartheta}$ defined by (11) is SPD and satisfies $\frac{c}{N^2}\leqslant\Lambda(\mathcal{A})\leqslant cN$, thus $\kappa(\mathcal{A})\leqslant cN^3$.*

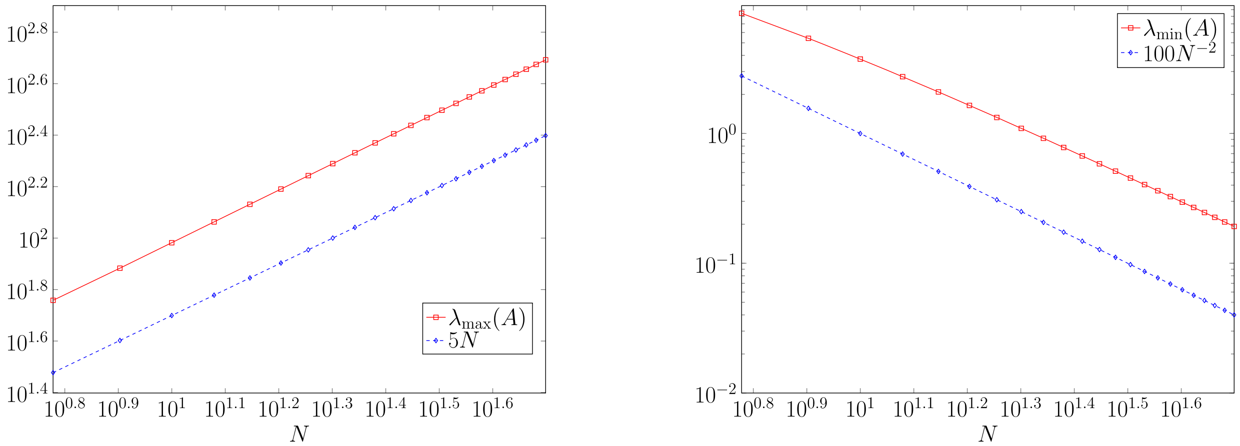
Remark 1. Note that the best upper bound for $\lambda_{\max}(A)$, as seen in Fig. 3a, is obtained by (6), which can also be applied for estimating the $\lambda_{\min}(A)$. However, (6) does not provide an optimal lower bound as by reiterating the process used for estimating $\lambda_{\max}(A)$,

$$
\lambda_{\min}(A)\geqslant\lambda_{\min}(M\otimes S)+\lambda_{\min}(S\otimes M)=2\lambda_{\min}(S)\lambda_{\min}(M)\geqslant 12\frac{c}{N^3}.
$$

While on the contrary, Fig. 3b suggests a lower bound of $\frac{c}{N^2}$, thus implying the need for more careful analysis.

We now proceed to analyze the matrix $B\in\mathbb{R}^{2\vartheta\times\wp}$, a sub-block of G defined by (11), which is a rectangular matrix defined in terms of matrices R and Q given by Proposition 1. Singular value estimates of G require sharp bounds on singular values of B as well, thus we proceed to derive them as follows.

Lemma 3. *Let $N\geqslant 4$, R, $Q\in\mathbb{R}^{(N-1)\times(N-1)}$ be defined by Proposition 1, then $\sigma_{\min}(R)=0$, $\sigma_{\max}(R)=2$, and $\frac{c}{N^2}\leqslant\sigma(Q)\leqslant C$.*



(a) Maximum eigenvalue of A.

(b) Minimum eigenvalue of A.

Fig. 3. Numerical results for $\Lambda(A)$.

Proof. The definition of R and (9) implies that $R^T R$ is a diagonal matrix with entries $(R^T R)_{00} = 0$, and $(R^T R)_{jj} = 4$, for $1 \leqslant j \leqslant N-2$. Therefore, $\sigma_{\min}(R) = 0$ and $\sigma_{\max}(R) = \sqrt{\lambda_{\max}(R^T R)} = 2$.

Recall (9), since the 1-norm of a matrix is its maximum absolute column sum, $\|Q\|_1 = \max\{\gamma_0, \gamma_1, 2\gamma_2, 2\gamma_3, \ldots, 2\gamma_{N-2}\} = \gamma_0 = 2$. Also, the maximum absolute row sum, $\|Q\|_\infty = \max\{\gamma_0 + \gamma_2, \gamma_1 + \gamma_3, \ldots, \gamma_{N-4} + \gamma_{N-2}, \gamma_{N-3}, \gamma_{N-2}\} = \gamma_0 + \gamma_2 = 2 + \frac{2}{5} = \frac{12}{5}$, hence $\sigma_{\max}(Q) = \|Q\| \leqslant \sqrt{\|Q\|_1 \|Q\|_\infty} = \sqrt{4.8}$.

We now estimate $\sigma_{\min}(Q) = \|Q^{-1}\|^{-1}$. It is easily verified that Q^{-1} is upper triangular and is non-zero along every other diagonal:

$$Q^{-1} = \begin{bmatrix} \gamma_0^{-1} & 0 & \gamma_0^{-1} & 0 & \gamma_0^{-1} & 0 & \cdots \\ & \gamma_1^{-1} & 0 & \gamma_1^{-1} & 0 & \gamma_1^{-1} & \cdots \\ & & \gamma_2^{-1} & 0 & \gamma_2^{-1} & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ & & & & \gamma_{N-4}^{-1} & 0 & \gamma_{N-4}^{-1} \\ & & & & & \gamma_{N-3}^{-1} & 0 \\ & & & & & & \gamma_{N-2}^{-1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$

Label the columns of Q^{-1} as $C_0, C_1, \ldots, C_{N-2}$. Note that the maximum absolute column sum of Q^{-1} is attained at either C_{N-3} or C_{N-2}, denoted by $S_{C_{N-3}}$ or $S_{C_{N-2}}$, respectively, and are given as follows,

$$S_{C_{N-3}} = \begin{cases} \sum_{k=0}^{\lfloor \frac{N-2}{2} \rfloor} \gamma_{2k}^{-1}, & N \text{ is odd}, \\ \sum_{k=0}^{\lfloor \frac{N-3}{2} \rfloor} \gamma_{2k+1}^{-1}, & N \text{ is even}, \end{cases} \qquad S_{C_{N-2}} = \begin{cases} \sum_{k=0}^{\lfloor \frac{N-3}{2} \rfloor} \gamma_{2k+1}^{-1}, & N \text{ is odd}, \\ \sum_{k=0}^{\lfloor \frac{N-2}{2} \rfloor} \gamma_{2k}^{-1}, & N \text{ is even}. \end{cases}$$

Since

$$\sum_{k=0}^{\lfloor \frac{N-2}{2} \rfloor} \frac{1}{\gamma_{2k}} = \sum_{k=0}^{\lfloor \frac{N-2}{2} \rfloor} \frac{2(2k)+1}{2} = \frac{1}{2} \sum_{k=0}^{\lfloor \frac{N-2}{2} \rfloor} (4k+1) \leqslant cN^2,$$

and similarly, $\sum_{k=0}^{\lfloor \frac{N-3}{2} \rfloor} \frac{1}{\gamma_{2k+1}} \leqslant cN^2$, it follows that $\|Q^{-1}\|_1 \leqslant cN^2$.

For $0 \leqslant k \leqslant N-2$, the absolute sum of the kth row of Q^{-1} is $\frac{1}{\gamma_k} \left\lfloor \frac{N-k}{2} \right\rfloor$, thus

$$\begin{aligned} \|Q^{-1}\|_\infty &= \max_{0 \leqslant k \leqslant N-2} \frac{1}{\gamma_k} \left\lfloor \frac{N-k}{2} \right\rfloor \leqslant \max_{0 \leqslant k \leqslant N-2} \frac{1}{\gamma_k} \max_{0 \leqslant k \leqslant N-2} \left\lfloor \frac{N-k}{2} \right\rfloor \\ &= \frac{1}{\gamma_{N-2}} \left\lfloor \frac{N}{2} \right\rfloor = \frac{2(N-2)+1}{2} \left\lfloor \frac{N}{2} \right\rfloor \leqslant cN^2. \end{aligned}$$

Therefore, $\|Q^{-1}\| \leqslant \sqrt{\|Q^{-1}\|_1 \|Q^{-1}\|_\infty} \leqslant \sqrt{cN^2 \cdot cN^2} = cN^2$, hence the result. $\square$

Lemma 4. *For $N \geqslant 4$, the matrix $B \in \mathbb{R}^{2\vartheta \times \wp}$ defined by (11) is full rank, that is, rank$(B) = \wp$ and $\frac{c}{N^2} \leqslant \sigma(B) \leqslant C$.*

Proof. Let R_i be the rows of B for $1 \leqslant i \leqslant 2\vartheta$. On exchanging $R_{k(N-1)}$ with $R_{\vartheta+1+(k-1)(N-1)}$, for all $1 \leqslant k \leqslant N-2$, the first $\wp$ rows of B form an upper triangular matrix of size $\wp \times \wp$ with non-zero diagonal entries, hence rank$(B) = \wp$.

We now estimate the singular values of B, which are the square-root of the eigenvalues of $B^T B \in \mathbb{R}^{\wp\times\wp}$. Note that rank$(B^T B) =$ rank$(B) = \wp$, and $B^T B = B_1^T B_1 + B_2^T B_2$. So we consider the blocks B_1 and B_2.

Since $B_1 = -Q \otimes R]$ and $B_2 = -R \otimes Q]$, that is, their first column is deleted, which only contains zero, therefore rank$(B_i) =$ rank(Q)rank$(R) = (N-1)(N-2) < \wp$, for $i = 1, 2$. Thus, B_i are rank deficient, so that $\sigma_{\min}(B_i) = 0$ for $i = 1, 2$. Furthermore, $\sigma(B_i) = \sigma(Q) \times \sigma(R)$, so Lemma 3 implies that $\sigma_{\max}(B_i) = \sigma_{\max}(R)\sigma_{\max}(Q) \leqslant 2C$, for $i = 1, 2$. Therefore, (6) gives

$$\lambda_{\max}(B^T B) \leqslant \lambda_{\max}(B_1^T B_1) + \lambda_{\max}(B_2^T B_2) = \sigma^2_{\max}(B_1) + \sigma^2_{\max}(B_2) \leqslant 4C^2 + 4C^2,$$

thus, $\sigma_{\max}(B) \leqslant C$. However, (6) gives a trivial bound for the minimum singular value of B, but we need a positive value as $B^T B$ is full rank. To this end, we perform the following analysis.

Let $\alpha_{jk} := (Q^T Q)_{jk}$ and $\beta_{jk} := (R^T R)_{jk}$, where $0 \leqslant j, k \leqslant N-2$. Note that $Q^T Q$ is a symmetric matrix, so that for $0 \leqslant j, k \leqslant N-2$

$$\alpha_{jk} = (Q^T Q)_{jk} = \begin{cases} \gamma_j^2, & j = k = 0, 1, \\ 2\gamma_j^2, & 2 \leqslant j = k \leqslant N-2, \\ -\gamma_j \gamma_{j+2}, & k = j+2. \end{cases}$$

Since Lemma 3 implies that $\sigma_{\min}(Q) \geqslant \frac{c}{N^2}$, for $y \in \mathbb{R}^{N-1} \setminus \{0\}$

$$\frac{c}{N^4} y^T y \leqslant y^T Q^T Q y = \sum_{j=0}^{N-2}\sum_{k=0}^{N-2} y_j \alpha_{jk} y_k = \sum_{j=0}^{N-2} \alpha_{jj}(y_j)^2 + 2\sum_{j=0}^{N-4} \alpha_{j,j+2} y_j y_{j+2}. \tag{17}$$

Recall that $R^T R$ is a diagonal matrix, with $\beta_{00} = 0$, and $\beta_{jj} = 4$ for $1 \leqslant j \leqslant N-2$. By a direct calculation, $B^T B = \textbf{[}Q^T Q \otimes R^T R + R^T R \otimes Q^T Q\textbf{]}$, that is, delete the first row and first column, then the (j,k)-th block of $B^T B$ is given as

$$(B^T B)_{jk} = \begin{cases} \alpha_{00}\textbf{[}R^T R\textbf{]}, & j = k = 0, \\ \alpha_{jj} R^T R + 4 Q^T Q, & 1 \leqslant j = k \leqslant N-2, \\ \alpha_{02}[R^T R, & j = 0, k = 2, \\ \alpha_{j,j+2} R^T R, & k = j+2, 1 \leqslant j \leqslant N-4. \end{cases}$$

Let $x = [x_0; x_1; \ldots; x_{N-2}] \in \mathbb{R}^{\wp} \setminus \{0\}$, where $x_0 = [x_0^1; x_0^2; \ldots; x_0^{N-2}] \in \mathbb{R}^{N-2} \setminus \{0\}$ and $x_j = [x_j^0; x_j^1; \ldots; x_j^{N-2}] \in \mathbb{R}^{N-1} \setminus \{0\}$ for $1 \leqslant j \leqslant N-2$, then

$$\begin{aligned} x^T B^T B x &= \sum_{j=0}^{N-2} x_j^T (B^T B)_{jj} x_j + 2\sum_{j=0}^{N-4} x_j^T (B^T B)_{j,j+2} x_{j+2} \\ &= x_0^T \alpha_{00}\textbf{[}R^T R\textbf{]} x_0 + \sum_{j=1}^{N-2} x_j^T (\alpha_{jj} R^T R + \beta_{jj} Q^T Q) x_j + 2 x_0^T \alpha_{02}[R^T R \cdot x_2 \\ &\quad + 2\sum_{j=1}^{N-4} x_j^T \alpha_{j,j+2} R^T R x_{j+2} \\ &= \alpha_{00}\sum_{k=1}^{N-2} 4(x_0^k)^2 + \sum_{j=1}^{N-2}\alpha_{jj}\sum_{k=1}^{N-2} 4(x_j^k)^2 + \sum_{j=1}^{N-2}\beta_{jj} x_j^T Q^T Q x_j \\ &\quad + 2\alpha_{02}\sum_{\ell=1}^{N-2}\sum_{k=0}^{N-2} x_0^\ell ([R^T R)_{\ell k} x_2^k + 2\sum_{j=1}^{N-4}\alpha_{j,j+2}\sum_{k=1}^{N-2} 4 x_j^k x_{j+2}^k \\ &= \alpha_{00}\sum_{k=1}^{N-2} 4(x_0^k)^2 + \sum_{j=1}^{N-2}\alpha_{jj}\sum_{k=1}^{N-2} 4(x_j^k)^2 + \sum_{j=1}^{N-2} 4 x_j^T Q^T Q x_j \end{aligned}$$

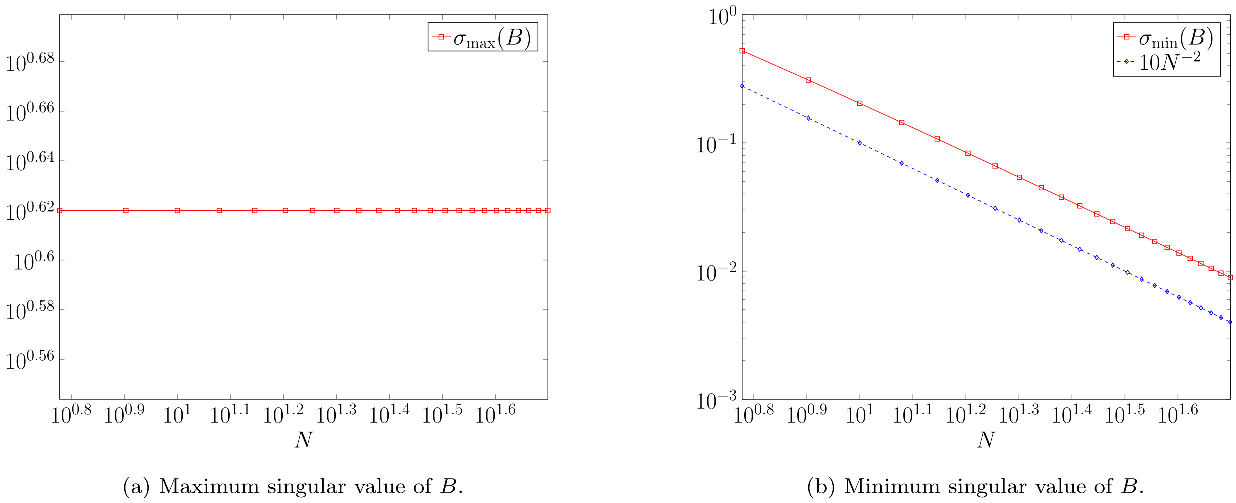


Fig. 4. Numerical results for $\sigma(B)$.

$$
\begin{aligned}
&+2\alpha_{02}\sum_{k=1}^{N-2}4x_0^kx_2^k+2\sum_{j=1}^{N-4}\alpha_{j,j+2}\sum_{k=1}^{N-2}4x_j^kx_{j+2}^k\\
&=4\sum_{k=1}^{N-2}\left(\sum_{j=0}^{N-2}\alpha_{jj}(x_j^k)^2+2\sum_{j=0}^{N-4}\alpha_{j,j+2}x_j^kx_{j+2}^k\right)+\sum_{j=1}^{N-2}4x_j^TQ^TQx_j.
\end{aligned}
$$

Define $\xi_k=[x_0^k;x_1^k;x_2^k;\dots;x_{N-2}^k]\in\mathbb{R}^{N-1}\setminus\{0\}$, then

$$
\begin{aligned}
x^TB^TBx&=4\sum_{k=1}^{N-2}\xi_k^TQ^TQ\xi_k+\sum_{j=1}^{N-2}4x_j^TQ^TQx_j &&(18)\\
&\geqslant 4\frac{c}{N^4}\sum_{k=1}^{N-2}\xi_k^T\xi_k+4\frac{c}{N^4}\sum_{j=1}^{N-2}x_j^Tx_j &&(\text{by eq. (17)})\\
&=4\frac{c}{N^4}\sum_{k=1}^{N-2}\sum_{j=0}^{N-2}(x_j^k)^2+4\frac{c}{N^4}\sum_{j=1}^{N-2}\sum_{k=0}^{N-2}(x_j^k)^2\\
&=4\frac{c}{N^4}\sum_{k=1}^{N-2}(x_0^k)^2+4\frac{c}{N^4}\sum_{j=1}^{N-2}\left((x_j^0)^2+\sum_{k=1}^{N-2}2(x_j^k)^2\right)\\
&\geqslant 4\frac{c}{N^4}\sum_{k=1}^{N-2}(x_0^k)^2+4\frac{c}{N^4}\sum_{j=1}^{N-2}\sum_{k=0}^{N-2}(x_j^k)^2=\frac{4c}{N^4}x^Tx.
\end{aligned}
$$

Thus, $\sigma^2_{\min}(B)=\lambda_{\min}(B^TB)\geqslant\frac{c}{N^4}$, which gives us the desired result. □

Fig. 4 suggests that the estimates on the singular values of B derived above are sharp. Note that the Schur complement matrix of the global spectral operator G, defined by (11), is defined as $\Upsilon_h:=B^T\mathcal{A}^{-1}B\in\mathbb{R}^{\wp\times\wp}$. This matrix is essential because we need bounds on the spectrum of Υ_h in order to use (7) to estimate the spectrum of G, which leads us to the following results.

Lemma 5. *For given $N\geqslant 4$, let $\Upsilon_h=B^T\mathcal{A}^{-1}B\in\mathbb{R}^{\wp\times\wp}$, where $\mathcal{A}$ and B are defined by (11), then $\frac{c}{N^3}\leqslant\lambda_{\min}(\Upsilon_h)\leqslant c\lambda_{\min}(\mathcal{A})$.*

Proof. Consider $\mathfrak{M}\in\mathbb{R}^{\wp\times\wp}$ defined by (B.1). Since Υ_h is symmetric,

$$
\lambda_{\min}(\Upsilon_h)=\min_{p\in\mathbb{R}^{\wp}\setminus 0}\frac{p^T\Upsilon_hp}{p^T\mathfrak{M}p}\cdot\frac{p^T\mathfrak{M}p}{p^Tp}
$$

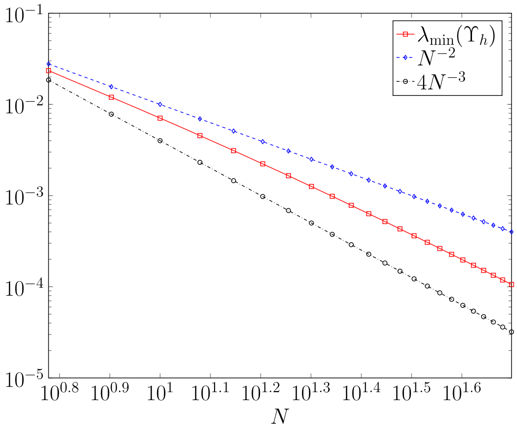


Fig. 5. Minimum eigenvalue of Υ_h.

$$\geqslant \min_{p\in\mathbb{R}^{\wp}\backslash 0} \frac{p^T\Upsilon_h p}{p^T\mathfrak{M}p}\cdot \min_{p\in\mathbb{R}^{\wp}\backslash 0}\frac{p^T\mathfrak{M}p}{p^Tp} \geqslant \frac{c}{N}\cdot\frac{c}{N^2} = \frac{c}{N^3}. \qquad \text{(by eqs. (B.1) and (B.2))}$$

Since $B\in\mathbb{R}^{\vartheta\times\wp}$ is full rank, therefore Corollary 1 and Lemma 7 imply

$$\begin{aligned}
\lambda_{\min}(\Upsilon_h) &= \min_{x\in\mathbb{R}^{\wp}\backslash\{0\}} \frac{x^TB^T\mathcal{A}^{-1}Bx}{x^TB^TBx}\cdot\frac{x^TB^TBx}{x^Tx}\\
&\leqslant \max_{x\in\mathbb{R}^{\wp}\backslash\{0\}} \frac{x^TB^T\mathcal{A}^{-1}Bx}{x^TB^TBx}\cdot \min_{x\in\mathbb{R}^{\wp}\backslash\{0\}}\frac{x^TB^TBx}{x^Tx}\\
&= \max_{y\in\mathbb{R}^{\vartheta}\backslash\{0\}} \frac{y^T\mathcal{A}^{-1}y}{y^Ty}\cdot \min_{x\in\mathbb{R}^{\wp}\backslash\{0\}}\frac{x^TB^TBx}{x^Tx}\\
&= \lambda_{\min}^{-1}(\mathcal{A})\cdot\sigma_{\min}^2(B)\\
&\leqslant cN^2\cdot\frac{c}{N^4}\\
&\leqslant \frac{c}{N^2}\leqslant c\lambda_{\min}(\mathcal{A}). \quad \square
\end{aligned}$$

Fig. 5 suggests that the bounds proved above are strong. Hence, the above results aid us to prove our main goal of this section, that is, an optimal bound for the condition number of the global spectral operator G for the steady Stokes problem as depicted by Fig. 6.

Theorem 2. *For $N\geqslant 4$, let $G\in\mathbb{R}^{(2\vartheta+\wp)\times(2\vartheta+\wp)}$ be defined by (11), then $\frac{c}{N^2}\leqslant\sigma(G)\leqslant cN^2$ and $\kappa(G)\leqslant cN^4$.*

Proof. Note that $G=\begin{bmatrix}\mathcal{A} & O\\ O & O_{\wp,\wp}\end{bmatrix}+\begin{bmatrix}O & B\\ B^T & O_{\wp,\wp}\end{bmatrix}=: G_1+G_2$, thus it is a sum of two symmetric matrices. Hence, by (6)

$$\lambda_{\max}(G)\leqslant\lambda_{\max}(G_1)+\lambda_{\max}(G_2). \tag{19}$$

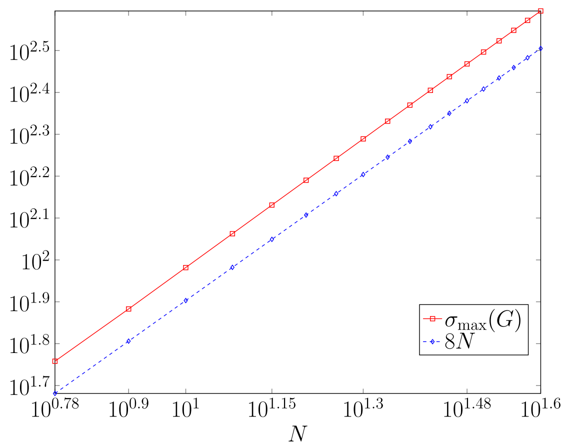
Note that $\lambda_{\max}(G_1)=\lambda_{\max}(\mathcal{A})$, thus Corollary 1 implies $\lambda_{\max}(G_1)\leqslant cN^2$. Also, a simple triangle inequality[4] for 2-norm implies that $\lambda_{\max}(G_2)=\sigma_{\max}(B)\leqslant C$ by Lemma 4. Hence, these results along with (19) yield $\lambda_{\max}(G)\leqslant cN$.

It remains to estimate the absolute minimum value of the eigenvalues of G, denoted by $|\lambda|_{\min}(G)$, for which (7) gives,

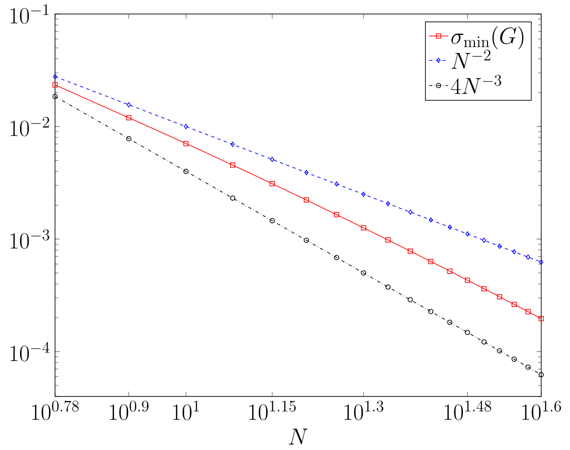
$$|\lambda|_{\min}(G)\geqslant\min\left\{\lambda_{\min}(\mathcal{A}),\ \frac{\lambda_{\min}(\Upsilon_h)}{\frac{1}{2}\left(1+\sqrt{1+\frac{4\lambda_{\min}(\Upsilon_h)}{\lambda_{\min}(\mathcal{A})}}\right)}\right\},$$

and by Lemma 5, $\frac{\lambda_{\min}(\Upsilon_h)}{\lambda_{\min}(\mathcal{A})}\leqslant c$, leading to $\frac{1}{2}\left(1+\sqrt{1+\frac{4\lambda_{\min}(\Upsilon_h)}{\lambda_{\min}(\mathcal{A})}}\right)\leqslant c$, thus

[4] Moreover, [5] gives $\Lambda(G_2)=\sigma(B)\cup-\sigma(B)\cup\mathbf{0}_{|2\vartheta-\wp|}$.



(a) Maximum eigenvalue of G.



(b) Minimum eigenvalue of G.

Fig. 6. Numerical results for $\Lambda(G)$.

$$\frac{\lambda_{\min}(\Upsilon_h)}{\frac{1}{2}\left(1+\sqrt{1+\frac{4\lambda_{\min}(\Upsilon_h)}{\lambda_{\min}(\mathcal{A})}}\right)} \geqslant c\lambda_{\min}(\Upsilon_h).$$

Hence, the minimum absolute value of eigenvalues of G satisfies,

$$|\lambda|_{\min}(G) \geqslant \min\{\lambda_{\min}(\mathcal{A}), c\lambda_{\min}(\Upsilon_h)\} \geqslant \min\left\{\frac{c}{N^2}, \frac{c}{N^3}\right\} = \frac{c}{N^3}.$$

Since $\kappa(G) = \dfrac{|\lambda|_{\max}(G)}{|\lambda|_{\min}(G)}$, therefore $\kappa(G) \leqslant cN \cdot N^3 = cN^4$. $\square$

4. Unsteady state

Consider the unsteady Stokes problem, given by equation (2), which can also be written as:

$$u_t - \Delta u + p_x = f_1 \text{ in } \Omega_t, \tag{20a}$$

$$v_t - \Delta v + p_y = f_2 \text{ in } \Omega_t, \tag{20b}$$

$$u_x + v_y = 0 \text{ in } \Omega \times (-1,1), \tag{20c}$$

$$u = 0,\ v = 0 \text{ on } \partial\Omega \times (-1,1), \tag{20d}$$

$$u(x,y,-1) = u_0(x,y),\ v(x,y,-1) = v_0(x,y) \text{ in } \Omega. \tag{20e}$$

We extend the $P_N - P_{N-2}$ scheme of the last section to the unsteady case by applying Chebyshev Gauss-Lobatto spectral collocation in time. These particular polynomial bases are chosen for simplicity of analysis of this scheme. In practice, Chebyshev recombined basis given in [34, p. 149] or Jacobi collocation can be chosen in place of Legendre recombined basis or Chebyshev collocation, respectively, without any difficulties. The goal is to show spectral convergence of a space-time spectral method and a condition number estimate for the scheme. The analysis of the latter is incomplete because two of the estimates are based on numerical evidence.

4.1. Discretization

For given $N \geqslant 4$, consider the Chebyshev Gauss-Lobatto nodes t_k for $0 \leqslant k \leqslant N$, so that $t_0 = -1$ and $t_N = 1$. Let ℓ_k denote the Lagrange basis polynomials for t_k, therefore $\ell_k(t_j) = \delta_{kj}$ for $0 \leqslant k, j \leqslant N$. Let D denote the Chebyshev Gauss-Lobatto pseudospectral derivative matrix of size $(N+1)\times(N+1)$, defined by (5). Additionally, we denote the first column of $[D]$ by $\mathbf{d} = [d_{10}; d_{20}; \ldots; d_{N0}] \in \mathbb{R}^{N\times 1}$. For this scheme, we define the velocity u, v and the pressure p as follows,

$$\begin{aligned}
u_N(x,y,t) &= \sum_{i=0}^{N-2}\sum_{j=0}^{N-2}\sum_{k=0}^{N} u_{ijk}\phi_i(x)\phi_j(y)\ell_k(t) \in \mathbb{P}^0_{N,N,N},\\
v_N(x,y,t) &= \sum_{i=0}^{N-2}\sum_{j=0}^{N-2}\sum_{k=0}^{N} v_{ijk}\phi_i(x)\phi_j(y)\ell_k(t) \in \mathbb{P}^0_{N,N,N},\\
p_N(x,y,t) &= \sum_{i=0}^{N-2}\sum_{\substack{j=0\\ i+j>0}}^{N-2}\sum_{k=0}^{N} p_{ijk}L_i(x)L_j(y)\ell_k(t) \in \mathbb{P}_{N-2,N-2,N}.
\end{aligned} \tag{21}$$

The number of unknowns for u and v each are $N\vartheta$, and the number of unknowns for p are $N\wp$. The total number of unknowns in the discrete Stokes equations is $2N\vartheta + N\wp = 3N(N-1)^2 - N$. Define the discrete unknowns as $u_h = [u_h^1; u_h^2; \dots; u_h^N]$, where $u_h^\ell = [u_{0,0,\ell}; u_{1,0,\ell}; \dots u_{N-2,0,\ell}; u_{0,1,\ell}; \dots u_{N-2,N-2,\ell}]$, similarly define v_h, p_h. Also, $\mathbf{F}_k = [F_k^1; F_k^2; \dots; F_k^N]$, where

$$F_k^\ell = [f_{00}^{k,\ell}; f_{10}^{k,\ell}; \dots f_{N-2,0}^{k,\ell}; f_{01}^{k,\ell}; \dots; f_{N-2,N-2}^{k,\ell}],$$

so that $f_k(x,y,t_\ell) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} f_{ij}^{k,\ell} L_i(x)L_j(y)$ for $k = 1, 2$, $1 \leqslant \ell \leqslant N$.

Note that the initial condition $u(x,y,-1) = u_0(x,y)$ gives $u_{ij0} = u_{ij}^0$, where the truncated Legendre series gives

$$u_0(x,y) \approx \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} u_{ij}\phi_i(x)\phi_j(y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \mathrm{u}_{ij}^0 L_i(x)L_j(y), \tag{22}$$

which implies $(\mathcal{L}\otimes\mathcal{L})u_{0h} = \mathrm{u}_{0h}$, where $\mathcal{L}$ is a Toeplitz matrix of size $(N+1)\times(N-1)$ with 1 on the main diagonal and -1 on the -2-diagonal. The coefficient vectors are defined as $u_{0h} = \left[u_{00}^0; u_{10}^0; \dots u_{N-20}^0; u_{01}^0; \dots u_{N-2N-2}^0\right] \in \mathbb{R}^\vartheta$, and $\mathrm{u}_{0h} = \left[\mathrm{u}_{00}^0; \mathrm{u}_{10}^0; \dots \mathrm{u}_{N0}^0; \mathrm{u}_{01}^0; \dots \mathrm{u}_{NN}^0\right] \in \mathbb{R}^{N^2}$. Similarly, v_{0h} is obtained. Consequently, for given $N \geqslant 4$, the discrete weak formulation of the unsteady Stokes problem becomes

$$\begin{aligned}
\left([D]\otimes\mathcal{M} + I_N\otimes A\right)u_h + \left(I_N\otimes B_1\right)p_h &= \left(I_N\otimes\mathcal{Q}\right)\mathbf{F}_1 - \mathbf{d}\otimes\left(\mathcal{M}u_{0h}\right)\\
\left([D]\otimes\mathcal{M} + I_N\otimes A\right)v_h + \left(I_N\otimes B_2\right)p_h &= \left(I_N\otimes\mathcal{Q}\right)\mathbf{F}_2 - \mathbf{d}\otimes\left(\mathcal{M}v_{0h}\right)\\
\left(I_N\otimes B_1^T\right)u_h + \left(I_N\otimes B_2^T\right)v_h &= \mathbf{0}_{N\wp},
\end{aligned} \tag{23}$$

where $\mathcal{M} = M\otimes M$, $\mathcal{Q} = Q\otimes Q \in \mathbb{R}^{\vartheta\times\vartheta}$. The space-time spectral convergence of the above scheme is observed in Fig. 7. Thus, the coefficient matrix of the discrete Stokes problem or the global space-time spectral operator becomes,

$$G_t = \begin{bmatrix}\mathcal{A}_t & \mathcal{B}\\ \mathcal{B}^T & O\end{bmatrix}, \text{where } \mathcal{B} = \begin{bmatrix} I_N\otimes B_1\\ I_N\otimes B_2\end{bmatrix} \in \mathbb{R}^{2N\vartheta\times N\wp}, \tag{24}$$

and $\mathcal{A}_t = A_t \oplus A_t \in \mathbb{R}^{2N\vartheta\times 2N\vartheta}$ with $A_t = [D]\otimes\mathcal{M} + I_N\otimes A \in \mathbb{R}^{N\vartheta}$. Analogous to the steady case, the main features of this scheme for the unsteady Stokes problem are: the velocity is not exactly divergence-free, this method is a spectral-Galerkin scheme in space and collocation in time, and there are no spurious modes for pressure.

4.2. Analysis

In this section, we undertake an analysis of the proposed scheme for the unsteady Stokes problem, with the objective of formulating a condition number estimate for the global space-time spectral operator. We begin our analysis by proving a well-known conjecture about the norm of the Chebyshev derivative matrix, stated in [9, p. 499] and depicted by Fig. 8.

Lemma 6. *For $N \geqslant 2$, let $D \in \mathbb{R}^{(N+1)\times(N+1)}$ be the Chebyshev Gauss-Lobatto pseudospectral derivate matrix, then $\|[D]\| \leqslant cN^2$.*

Proof. Since $\|[D]\| \leqslant \sqrt{\|[D]\|_1\|[D]\|_\infty}$, we evaluate the maximum absolute row and column sum of $[D]$ by using (5). Let C_k and R_k denote the absolute sum of k-th column and k-th row respectively, for $1 \leqslant k \leqslant N$, then

$$C_k = |d_{kk}| + \sum_{\substack{k=1\\ k\neq j}}^{N} |d_{kj}|. \tag{25}$$

On simplification of the diagonal terms,

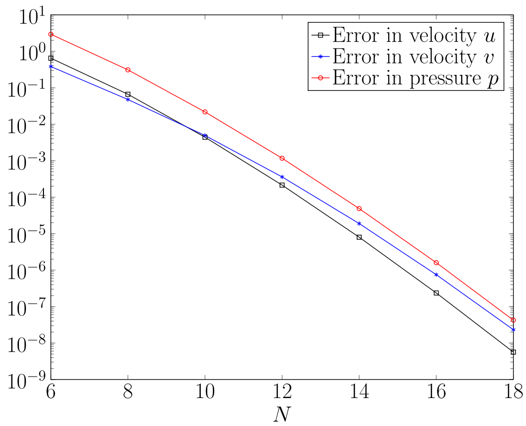


Fig. 7. Convergence for the $P_N - P_{N-2}$ scheme for the unsteady Stokes problem.

$$|d_{kk}| = \left| \frac{-x_k}{2(1-x_k^2)} \right| = \frac{\cos \frac{\pi k}{N}}{2\left(1-\cos^2 \frac{\pi k}{N}\right)} \leqslant \frac{1}{2\sin^2 \frac{\pi k}{N}}.$$

Note that $\frac{\pi k}{N} \leqslant \frac{\pi}{2}$ for $1 \leqslant k \leqslant \frac{N}{2}$, and since for $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\frac{2x}{\pi} \leqslant \sin x \leqslant x, \tag{26}$$

thus for $1 \leqslant k \leqslant \frac{N}{2}$, $|d_{kk}| \leqslant \frac{1}{2\left(\frac{2}{N}\right)^2} \leqslant \frac{N^2}{8}$, and for $\frac{N}{2} < k \leqslant N-1$, $\frac{\pi}{N} \leqslant \frac{\pi(N-k)}{N} < \frac{\pi}{2}$, (26) yields

$$|d_{kk}| \leqslant \frac{1}{2\sin^2 \frac{\pi k}{N}} = \frac{1}{2\sin^2\left(\frac{\pi}{N}(N-k)\right)} \leqslant \frac{1}{2\left(\frac{2}{N}\right)^2} = \frac{N^2}{8}.$$

Also, for $k = N$, $|d_{NN}| = \frac{2N^2+1}{6} \leqslant cN^2$, thus for all $1 \leqslant k \leqslant N$,

$$|d_{kk}| \leqslant cN^2. \tag{27}$$

For a fixed $1 \leqslant j \leqslant N$,

$$\sum_{\substack{k=1\\k\neq j}}^{N} |d_{kj}| = \sum_{\substack{k=1\\k\neq j}}^{N} \left| \frac{\tilde{c}_k(-1)^{k+j}}{\tilde{c}_j(x_k - x_j)} \right| \leqslant 2\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{|x_k - x_j|} = \sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{\left|\sin \frac{(j+k)\pi}{2N} \sin \frac{(j-k)\pi}{2N}\right|}.$$

Using (26), for all $k \neq j$ and $1 \leqslant k \leqslant N$, $\frac{1}{\left|\sin\left(\frac{(j-k)\pi}{2N}\right)\right|} \leqslant \frac{N}{|j-k|}$, thus

$$\sum_{\substack{k=1\\k\neq j}}^{N} |d_{kj}| \leqslant N \sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{\left|\sin\left(\frac{(j+k)\pi}{2N}\right)\right| |j-k|}.$$

Since $(j+k)\frac{\pi}{2N} \leqslant \frac{\pi}{2}$ implies $k \leqslant N-j$, split the above sum as follows

$$\begin{aligned}\sum_{\substack{k=1\\k\neq j}}^{N} |d_{kj}| &\leqslant N \sum_{\substack{k=1\\k\neq j}}^{N-j} \frac{1}{\left|\sin\left(\frac{(j+k)\pi}{2N}\right)\right| |j-k|} + N \sum_{\substack{k=N-j+1\\k\neq j}}^{N} \frac{1}{\left|\sin\left(\frac{(j+k)\pi}{2N}\right)\right| |j-k|} \\ &=: NS_1 + NS_2. \end{aligned} \tag{28}$$

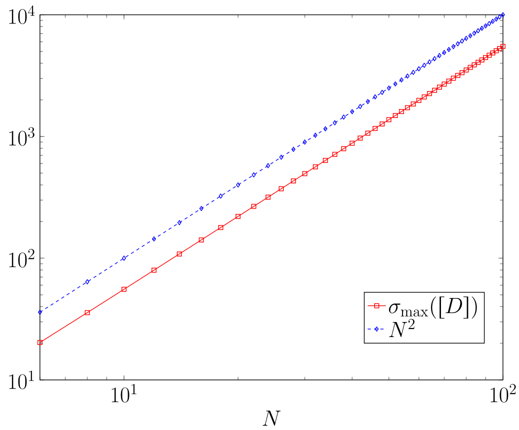


Fig. 8. Maximum singular value of $[D]$.

Note that $1 \leqslant k \leqslant N-j$ gives $\dfrac{\pi}{N} \leqslant \dfrac{(k+j)\pi}{2N} \leqslant \dfrac{\pi}{2}$, by applying (26)

$$\sin\left(\frac{(k+j)\pi}{2N}\right) \geqslant \frac{2}{\pi}\frac{(k+j)\pi}{2N} = \frac{(k+j)}{N},$$

which implies

$$S_1 \leqslant N \sum_{\substack{k=1\\k\neq j}}^{N-j} \frac{1}{(k+j)|j-k|} \leqslant N \sum_{\substack{k=1\\k\neq j}}^{N-j} \frac{1}{|j-k|^2} \leqslant N\frac{\pi^2}{6},$$

as $|j-k| \leqslant k+j$ and $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. A similar analysis gives $S_2 \leqslant \frac{\pi^2}{6}N$, which along with (28) gives, $\sum_{\substack{k=1\\k\neq j}}^{N} |d_{kj}| \leqslant cN^2$. Therefore, (27) in (25) yields $C_k \leqslant cN^2$, for all $1 \leqslant k \leqslant N$. Hence, $\|[D]\|_1 = \max_{1\leqslant k\leqslant N} C_k \leqslant cN^2$, and similarly $\|[D]\|_\infty = \max_{1\leqslant k\leqslant N} R_k \leqslant cN^2$, which gives the desired result. □

Remark 2. The above proof is easily extended to prove that $\sigma_{\max}(D) \leqslant cN^2$, since we only need to add the contribution of $|d_{0k}| \leqslant cN^2$ to each C_k. For details on the proof see [18, p. 70–74].

The analysis of the unsteady Stokes problem is much harder than in the steady state because of the presence of the Chebyshev derivative matrix D, which is a non-symmetric matrix with an indefinite symmetric part. These properties are inherited by the leading block $\mathcal{A}_t$ of the global space-time spectral operator G_t. We could not find any results in the literature for approximating the spectrum of a saddle point matrix with the leading block of the form $\mathcal{A}_t$. Several results exist for estimating the spectrum of a symmetric saddle point matrix, thus creating scope for approximating the singular values of G_t, as they are the square-root of the eigenvalues of $G_t^T G_t$. In the following, we provide a condition number estimate for G_t by using computational and theoretical techniques. Note that (8) gives

$$\sigma_{\min}(G_t) \geqslant \sqrt{1-\cos\theta} \cdot \min\{\sigma_{\min}(\mathcal{A}_t), \sigma_{\min}(\mathcal{B})\}. \tag{29}$$

We could not estimate the term $\sqrt{1-\cos\theta}$, for which a numerical evidence Fig. 9a suggests

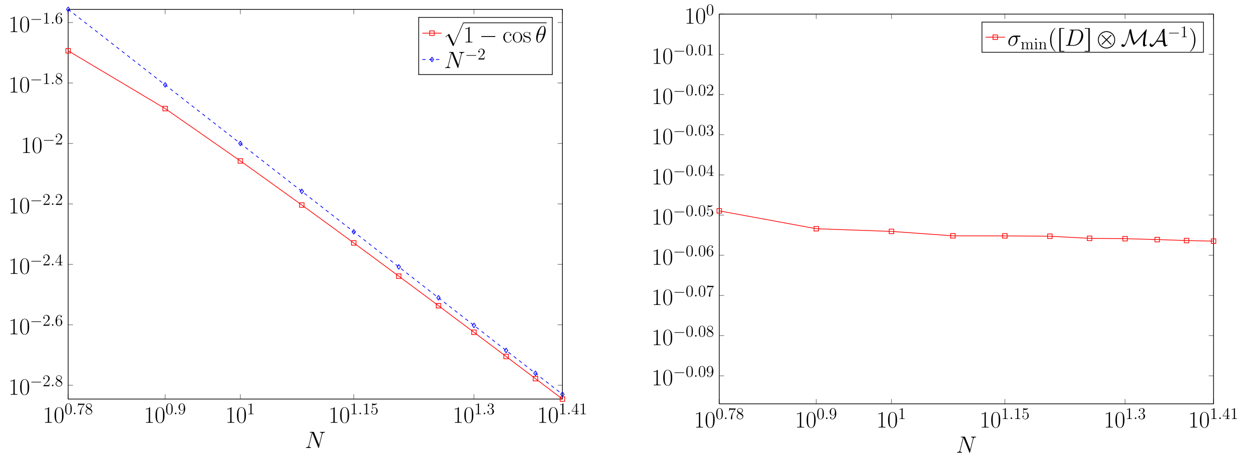
$$\sqrt{1-\cos\theta} \geqslant \frac{c}{N^2}. \tag{30}$$

Another estimate that has been difficult to show is

$$\sigma_{\min}\left([D] \otimes \mathcal{M}A^{-1} + I_{N\vartheta}\right) \geqslant c_1, \tag{31}$$

where $0 < c_1 < 1$ is a constant, as portrayed by numerical evidence Fig. 9b.

Theorem 3. *For $N \geqslant 4$, let G_t be defined by (24). Assume (30) and (31) hold, then $\kappa(G_t) \leqslant cN^6$.*



(a) Value of $\sqrt{1-\cos\theta}$. (b) $\sigma_{\min}([D]\otimes\mathcal{M}\mathcal{A}^{-1}+I)$

Fig. 9. Numerical results for (30) and (31).

Proof. We begin by estimating the maximum singular value of A_t,

$$\begin{aligned}
\sigma_{\max}(A_t) &= \|A_t\| = \|[D]\otimes\mathcal{M}+I_N\otimes A\| \\
&\leqslant \|[D]\otimes M\otimes M\| + \|I_N\otimes A\| \qquad\qquad (\text{since } \mathcal{M}=M\otimes M) \\
&= \|[D]\|\|M\|^2 + \|A\| \\
&\leqslant cN^2\cdot c + cN \leqslant cN^2,
\end{aligned}$$

which is obtained by using Lemmas 2 and 6, and Theorem 1. It remains to estimate the minimum singular value of A_t.

$$\begin{aligned}
\sigma_{\min}(A_t) &= \sigma_{\min}\left(\left([D]\otimes\mathcal{M}A^{-1}+I_{N\vartheta}\right)(I_N\otimes A)\right) \\
&\geqslant \sigma_{\min}\left([D]\otimes\mathcal{M}A^{-1}+I_{N\vartheta}\right)\sigma_{\min}\left(I_N\otimes A\right) \\
&\geqslant c_1\sigma_{\min}(A),
\end{aligned}$$

is obtained by using (31), thus applying Theorem 1 leads to the desired result $\sigma_{\min}(A_t)\geqslant\dfrac{c}{N^2}$, and $\sigma(A_t\oplus A_t)=\sigma(A_t)$ implies $\sigma_{\min}(\mathcal{A}_t)\geqslant\dfrac{c}{N^2}$.

Next, we estimate the singular values of $\mathcal{B}$. Since $\mathcal{B}^T\mathcal{B}=I_N\otimes B^TB$, Lemma 4 gives $\operatorname{rank}(\mathcal{B}^T\mathcal{B})=\operatorname{rank}(I_N)\cdot\operatorname{rank}(B^TB)=N\cdot\operatorname{rank}(B)=N\wp$. Hence, $\mathcal{B}$ is full rank. Also, $\Lambda(\mathcal{B}^T\mathcal{B})=\Lambda(I_N)\Lambda(B^TB)=\Lambda(B^TB)$, hence $\sigma(\mathcal{B})=\sigma(B)$, thus Lemma 4 implies $\sigma_{\max}(\mathcal{B})\leqslant c$ and $\sigma_{\min}(\mathcal{B})\geqslant\dfrac{c}{N^2}$.

Finally, for G_t, by following the proof of (19), $\sigma_{\max}(G_t)=\|G_t\|\leqslant\sigma_{\max}(\mathcal{A}_t)+\sigma_{\max}(\mathcal{B})\leqslant cN^2+c\leqslant cN^2$. For the minimum singular value of G_t, (29) and (30) imply $\sigma_{\min}(G_t)\geqslant\dfrac{c}{N^2}\min\left(\dfrac{c}{N^2},\dfrac{c}{N^2}\right)\geqslant\dfrac{c}{N^4}$, thus $\kappa(G_t)=\dfrac{\sigma_{\max}(G_t)}{\sigma_{\min}(G_t)}\leqslant cN^6$. $\square$

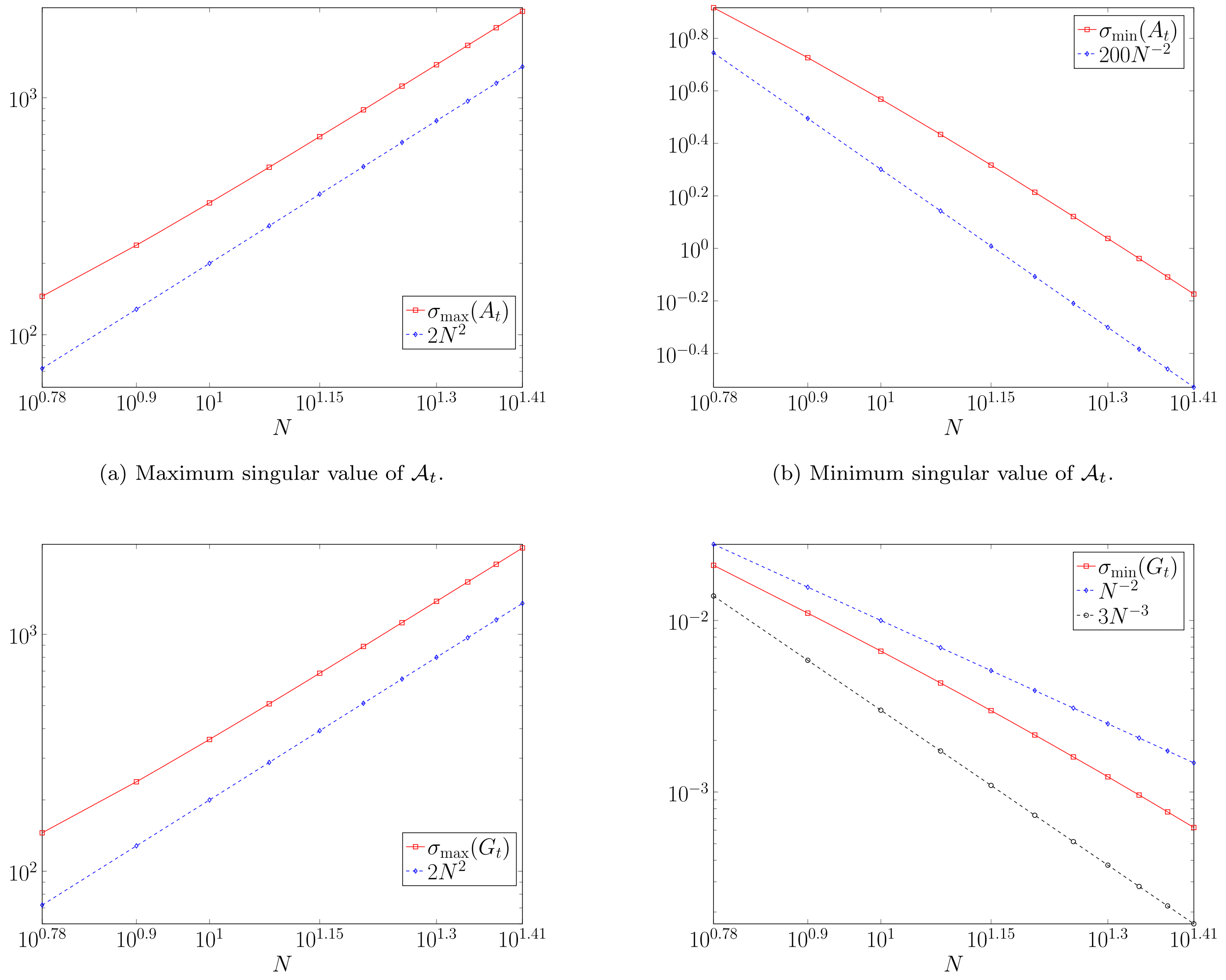
The above estimate is not sharp, as numerically Fig. 10d hints that $\sigma_{\min}(G_t)$ behaves at least like $\mathcal{O}\left(N^{-2.5}\right)$, suggesting $\kappa(G_t)$ is at least $\mathcal{O}(N^{4.5})$. The main objective of the above result is its application in deriving proof of convergence for the scheme devised and analyzed in this section.

4.3. Convergence

In this section, we discuss the space-time spectral convergence of our method for the unsteady Stokes problem, as spectral convergence of the P_N-P_{N-2} scheme for the Stokes problem in steady state was proved in [3].

Let $\|\cdot\|_{0,\omega}$ denote a weighted L^2-norm defined as

$$\|f\|_{0,\omega}^2=\int\limits_{\Omega_t} f(x,y,t)\frac{1}{\sqrt{1-t^2}}dxdydt.$$



(a) Maximum singular value of $\mathcal{A}_t$. (b) Minimum singular value of $\mathcal{A}_t$.

(c) Maximum singular value of G_t. (d) Minimum singular value of G_t.

Fig. 10. Numerical results for Theorem 3.

The above norm is designed to incorporate the weight functions for the Legendre polynomials in space and Chebyshev polynomials in time. Recall that the velocity obtained by the scheme devised in this section for the unsteady Stokes problem is not exactly divergence-free, as implied by (12). Moreover, the uniqueness of the solution for this scheme is a direct consequence of Theorem 3, thus we prove the following result by infusing the conditions of the aforementioned result.

Theorem 4. *Let u, v, and p be the solution of (2). Assume u, v, and p are separately analytic in each variable. Let $N \geqslant 4$ and u_N, v_N, and p_N be the solution of the space-time method of the form (21), with matrix defined by (24). Assume (30) and (31) hold, then the following holds*

$$\|u - u_N\|_{0,\omega} + \|v - v_N\|_{0,\omega} + \|p - p_{N-2}\|_{0,\omega} \leqslant cN^8 e^{-CN}.$$

Proof. Consider the exact solution and its truncation as follows,

$$u(x, y, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{u}_{ij}(t)\phi_i(x)\phi_j(y), \qquad \Pi_N u(x, y, t) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \hat{u}_{ij}(t)\phi_i(x)\phi_j(y),$$

$$v(x, y, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{v}_{ij}(t)\phi_i(x)\phi_j(y), \qquad \Pi_N v(x, y, t) = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \hat{v}_{ij}(t)\phi_i(x)\phi_j(y),$$

$$p(x,y,t)=\sum_{i=0}^{\infty}\sum_{\substack{j=0\\i+j>0}}^{\infty}\hat{p}_{ij}(t)L_i(x)L_j(y),\qquad \Pi_{N-2}p(x,y,t)=\sum_{i=0}^{N-2}\sum_{\substack{j=0\\i+j>0}}^{N-2}\hat{p}_{ij}(t)L_i(x)L_j(y).$$

Let $\mathscr{T}_N u$ denote the truncation error in velocity u, that is, $\mathscr{T}_N u(x,y,t)=(u-\Pi_N u)(x,y,t)$, similarly, let $\mathscr{T}_N v$ and $\mathscr{T}_{N-2}p$ denote the truncation error of velocity v and pressure p, defined as $v-\Pi_N v$ and $p-\Pi_{N-2}p$, respectively.

Define semi-discrete solutions for (2) representing (21) as follows

$$u_N(x,y,t)=\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}u_{ij}(t)\phi_i(x)\phi_j(y),$$
$$v_N(x,y,t)=\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}v_{ij}(t)\phi_i(x)\phi_j(y),$$
$$p_{N-2}(x,y,t)=\sum_{i=0}^{N-2}\sum_{\substack{j=0\\i+j>0}}^{N-2}p_{ij}(t)L_i(x)L_j(y),$$

where $u_{ij}(t)=\sum_{k=0}^{N}u_{ijk}\ell_k(t)$, and similarly $v_{ij}(t)$ and $p_{ij}(t)$ are defined, which imply $u_{ij}(t_k)=u_{ijk}$, $v_{ij}(t_k)=v_{ijk}$, and $p_{ij}(t_k)=p_{ijk}$, for t_k are Chebyshev Gauss-Lobatto nodes, $1\leqslant k\leqslant N$. Also, define $t_h=[t_1;t_2;\dots;t_N]\in\mathbb{R}^N$.

Define the error in truncated and approximated solutions as

$$e^u(x,y,t)=(\Pi_N u-u_N)(x,y,t),$$
$$e^v(x,y,t)=(\Pi_N v-v_N)(x,y,t),$$
$$e^p(x,y,t)=(\Pi_{N-2}p-p_{N-2})(x,y,t).$$

Also, define the error vectors as $E^u=[E^u_1;E^u_2;\dots;E^u_N]$, $E^v=[E^v_1;E^v_2;\dots;E^v_N]$, $E^p=[E^p_1;E^p_2;\dots;E^p_N]$, where for $0\leqslant i,j\leqslant N-2$ and $1\leqslant k\leqslant N$, $E^u_k=[\hat{u}_{ij}(t_k)-u_{ijk}]$, $E^v_k=[\hat{v}_{ij}(t_k)-v_{ijk}]$, and only $E^p_k=[\hat{p}_{ij}(t_k)-p_{ijk}]$ is considered along with the condition $i+j>0$.

Recall that for given f_r in (20a) and (20b), for $r=1,2$, so that at time $t=t_k$, for $1\leqslant k\leqslant N$, $f_r(x,y,t_k)$ is analytic in Ω, then it can be expressed as

$$f_r(x,y,t_k)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}f^{r,k}_{ij}\phi_i(x)\phi_j(y),\qquad \Pi_N f_r(x,y,t_k)=\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}f^{r,k}_{ij}\phi_i(x)\phi_j(y),$$

where $\Pi_N f_r$ is the truncation for f_r and the truncation error is defined as $\mathscr{T}_N f^k_r=(f_r-\Pi_N f_r)(x,y,t_k)$, for $r=1,2$ and $1\leqslant k\leqslant N$.

For $w\in V$, the first equation of the Stokes problem implies that the exact solution u,p satisfy the following weak form, for all $t\in(-1,1)$, thus at time $t=t_k$, where $1\leqslant k\leqslant N$

$$\big((u_t-\Delta u)(x,y,t_k),w\big)-\big(p(x,y,t_k),w_x\big)=\big(f_1(x,y,t_k),w\big) \tag{32}$$

and the approximated solution u_N,p_{N-2} satisfy

$$\big(((u_N)_t-\Delta u_N)(x,y,t_k),w_N\big)-\big(p_{N-2}(x,y,t_k),(w_N)_x\big)=\big(\Pi_N f_1(x,y,t_k),w_N\big), \tag{33}$$

for all $w_N\in\mathbb{P}_{N,N}\cap V$. Subtracting (32) and (33) for all $0\leqslant m,n\leqslant N-2$ gives

$$\begin{aligned}&\big(((u-u_N)_t-\Delta(u-u_N))(x,y,t_k),\phi_m(x)\phi_n(y)\big)\\&\quad-\big((p-p_{N-2})(x,y,t_k),\phi'_m(x)\phi_n(y)\big)=\big((f_1-\Pi_N f_1)(x,y,t_k),\phi_m(x)\phi_n(y)\big)\end{aligned}$$

which gives

$$\begin{aligned}&\big((e^u_t-\Delta e^u)(x,y,t_k),\phi_m(x)\phi_n(y)\big)-\big(e^p(x,y,t_k),\phi'_m(x)\phi_n(y)\big)\\&\quad=\big(\mathscr{T}_N f^k_1-((\mathscr{T}_N u)_t+\Delta(\mathscr{T}_N u))(x,y,t_k),\phi_m(x)\phi_n(y)\big)\\&\qquad+\big((\mathscr{T}_{N-2}p)(x,y,t_k),\phi'_m(x)\phi_n(y)\big).\end{aligned} \tag{34}$$

Define $g(t)$ as follows,

$$g(t) = (e^u(x, y, t), \phi_m(x)\phi_n(y))$$
$$= \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \left(\hat{u}_{ij}(t) - u_{ij}(t)\right)(\phi_i(x)\phi_j(y), \phi_m(x)\phi_n(y)), \tag{35}$$

then (34) becomes

$$\begin{aligned} g'(t_k) + (-\Delta e^u(x, y, t_k), \phi_m(x)\phi_n(y)) - (e^p(x, y, t_k), \phi_m'(x)\phi_n(y)) \\ = (\mathscr{T}_N f_1^k - ((\mathscr{T}_N u)_t + \Delta(\mathscr{T}_N u))(x, y, t_k), \phi_m(x)\phi_n(y)) \\ + ((\mathscr{T}_{N-2} p)(x, y, t_k), \phi_m'(x)\phi_n(y)). \end{aligned} \tag{36}$$

For any analytic z such that $z(-1) = 0$, recall the definition of the interpolant $\mathcal{I}_N z(t) = \sum_{i=1}^{N} z(t_i)\ell_i(t)$. For $0 \leqslant k \leqslant N-1$,

$$\begin{aligned} z'(t_k) &= (\mathcal{I}_N z)'(t_k) + \tilde{\epsilon}_k \\ &= ([D](\mathcal{I}_N(z)(t_h)))_k + \tilde{\epsilon}_k \\ &= ([D](z(t_h)))_k + \tilde{\epsilon}_k, \end{aligned}$$

where $\tilde{\epsilon}_k = (z - \mathcal{I}_N z)'(t_k)$, according to [30],[5] satisfies

$$|\tilde{\epsilon}_k| \leqslant cN^2 e^{-CN}. \tag{37}$$

Since the initial condition is $u(x, y-1) = u_0(x, y)$, recall that $\hat{u}_{ij}(-1) = u_{ij0} = u_{ij}^0$, therefore $g(-1) = \hat{u}_{ij}(-1) - u_{ij}(-1) = u_{ij}^0 - u_{ij}^0 = 0$. Hence, the above expression and (35) imply

$$\begin{aligned} g'(t_k) &= ([D]g(t_h))_k + \epsilon_k^1 \\ &= \left([D] \cdot \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} \left(\hat{u}_{ij}(t_h) - u_{ij}(t_h)\right)(\phi_i(x)\phi_j(y), \phi_m(x)\phi_n(y))\right)_k + \epsilon_k^1, \end{aligned}$$

thus (36) gives the first $(N-1)^2$ equations for each time step t_k for $1 \leqslant k \leqslant N$ and $0 \leqslant m, n \leqslant N-2$ as follows,

$$\begin{aligned} \left([D] \cdot (e^u(x, y, t_h), \phi_m(x)\phi_n(y))\right)_k + (-\Delta e^u(x, y, t_k), \phi_m(x)\phi_n(y)) \\ - (e^p(x, y, t_k), \phi_m'(x)\phi_n(y)) = -\epsilon_k^1 \\ + (\mathscr{T}_N f_1^k - ((\mathscr{T}_N u)_t - \Delta(\mathscr{T}_N u))(x, y, t_k), \phi_m(x)\phi_n(y)) \\ + ((\mathscr{T}_{N-2} p)(x, y, t_k), \phi_m'(x)\phi_n(y)). \end{aligned} \tag{38}$$

Thus, the $(N-1)^2$ equations together for all time steps $1 \leqslant k \leqslant N$ give

$$([D] \otimes \mathcal{M} + I_N \otimes \mathcal{A})\, E^u + (I_N \otimes B_1)\, E^p = -\epsilon_1 + R_1^u + R_2^u, \tag{39}$$

where we define $\epsilon_1 = \left[\epsilon_1^1; \epsilon_2^1; \ldots; \epsilon_N^1\right]$, and for $1 \leqslant i \leqslant 2$

$$R_i^u = \left[r_i^u(t_1); r_i^u(t_2); \ldots; r_i^u(t_N)\right],$$

with the following two vectors of length ϑ,

$$\begin{aligned} r_1^u(t_k) &= [(\mathscr{T}_N f_1^k - ((\mathscr{T}_N u)_t - \Delta(\mathscr{T}_N u))(x, y, t_k), \phi_m(x)\phi_n(y))], \\ r_2^u(t_k) &= \left[((\mathscr{T}_{N-2} p)(x, y, t_k), \phi_m'(x)\phi_n(y))\right], \end{aligned}$$

for $0 \leqslant m, n \leqslant N-2$ and $1 \leqslant k \leqslant N$.

Similarly, the error equation for the velocity v in matrix form is given as

$$([D] \otimes \mathcal{M} + I_N \otimes \mathcal{A})\, E^v + (I_N \otimes B_2)\, E^p = -\epsilon_2 + R_1^v + R_2^v, \tag{40}$$

[5] which was an improvement of a factor of $\mathcal{O}(N^{1.5})$ over the one derived in [36], however, the latter also provides results for more general Sobolev spaces.

where $r_2^v(t_k) = \left[(\mathscr{T}_{N-2}p)(x, y, t_k), \phi_m(x)\phi_n'(y))\right] \in \mathbb{R}^{\vartheta}$, as $0 \leqslant m, n \leqslant N-2$. The exact solution u, v satisfy the weak form of the third equation of the Stokes problem, for all $q \in L_0^2(\Omega)$ and time $t = t_k$,

$$(q, (u_x + v_y)(x, y, t_k)) = 0, \tag{41}$$

also, the approximate solutions satisfy the following for all $q_{N-2} \in \mathbb{P}_{N-2,N-2} \cap L_0^2(\Omega)$,

$$(q_{N-2}, ((u_N)_x + (v_N)_y)(x, y, t_k)) = 0, \tag{42}$$

for all $q_{N-2} \in \mathbb{P}_{N-2,N-2} \cap L_0^2(\Omega)$. Thus, subtracting (41) and (42) for all $0 \leqslant m, n \leqslant N-2$ with $m+n > 0$ and incorporating the truncated solution yields

$$-(L_m(x)L_n(y), (e_x^u - e_y^v)(x, y, t_k)) = (L_m(x)L_n(y), ((\mathscr{T}_N u)_x + (\mathscr{T}_N v)_y)(x, y, t_k)).$$

Hence, the following linear system is obtained.

$$(I_N \otimes B_1^T)E^u + (I_N \otimes B_2^T)E^v = R_2^p, \tag{43}$$

where $R_2^p = \left[r_2^p(t_1); r_2^p(t_2); \ldots; r_2^p(t_N)\right]$, and for $1 \leqslant k \leqslant N$,

$$r_2^p(t_k) = [(L_m(x)L_n(y), \left((\mathscr{T}_N u)_x + (\mathscr{T}_N v)_y\right)(x, y, t_k))] \in \mathbb{R}^{\wp},$$

as $0 \leqslant m, n \leqslant N-2$ and $m+n > 0$. Thus, (36), (40), and (43) imply

$$\begin{bmatrix} \mathcal{A}_t & O_{N\vartheta,N\vartheta} & I_N \otimes B_1 \\ O_{N\vartheta,N\vartheta} & \mathcal{A}_t & I_N \otimes B_2 \\ I_N \otimes B_1^T & I_N \otimes B_2^T & O_{\wp,\wp} \end{bmatrix} \begin{bmatrix} E^u \\ E^v \\ E^p \end{bmatrix} = -\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ 0 \end{bmatrix} + \begin{bmatrix} R_1^u \\ R_1^v \\ 0 \end{bmatrix} + \begin{bmatrix} R_2^u \\ R_2^v \\ R_2^p \end{bmatrix}$$

which is expressed as the following linear system

$$G_t E = -\epsilon + \sum_{i=1}^{2} R_i, \tag{44}$$

by (C.3), $|G_t E|_\infty \leqslant cN^3 e^{-CN}$.

The next stage is to estimate the norm of error between the truncated and approximated solution defined e^u, e^v and e^p in the beginning of this proof. Since $\phi_i = L_i - L_{i+2}$,

$$e^u(x, y, t_k) = \sum_{i=0}^{N}\sum_{j=0}^{N} \mathfrak{c}_{ij}^k L_i(x)L_j(y),$$

where $\mathfrak{c}_{ij}^k = (\mathcal{L} \otimes \mathcal{L})\, E_k^u$, $1 \leqslant k \leqslant N$ as defined from (22). Moreover, it is easy to prove that

$$\|\mathcal{L}\| \leqslant \sqrt{\|\mathcal{L}\|_1 \|\mathcal{L}\|_\infty} \leqslant \sqrt{2 \cdot 2} = 2. \tag{45}$$

Let the Chebyshev Gauss-Lobatto quadrature weights be denoted by $\omega_i = \dfrac{\pi}{Nd_i}$, where $d_0 = 2 = d_N$ and $d_i = 1$ for $1 \leqslant i \leqslant N-1$, and W denote the diagonal matrix containing the weights, $W_{ii} = \omega_i$, for $0 \leqslant i \leqslant N$, thus $\|[W]\| \leqslant \frac{c}{N}$. The weighted norm of e^u is given as

$$\begin{aligned} \|e^u\|_{0,\omega}^2 &= \int_\Omega |e^u|^2 \frac{1}{\sqrt{1-t^2}} dxdydt \\ &\leqslant c \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=1}^{N} |\mathfrak{c}_{ij}^k|^2 \omega_k && (\text{since } \|L_j(x)\|_0 \leqslant c, \text{ for } j \geqslant 0) \\ &= c\left|\left([W]^{\frac{1}{2}} \otimes \mathcal{L} \otimes \mathcal{L}\right) E^u\right|^2. \end{aligned}$$

Similarly, the other two error estimates can be derived to get the following

$$\|e^v\|_{0,\omega}^2 \leqslant c\left|\left([W]^{\frac{1}{2}} \otimes \mathcal{L} \otimes \mathcal{L}\right) E^v\right|^2$$
$$\|e^p\|_{0,\omega}^2 \leqslant c\left|\left([W]^{\frac{1}{2}} \otimes I_{N_\wp}\right) E^p\right|^2$$

Define $W_h = \left([W]^{\frac{1}{2}} \otimes \mathcal{L} \otimes \mathcal{L}\right) \oplus \left([W]^{\frac{1}{2}} \otimes \mathcal{L} \otimes \mathcal{L}\right) \oplus \left([W]^{\frac{1}{2}} \otimes I_{N_\wp}\right)$, then

$$\|W_h\| \leqslant \max\left\{\|[W]^{\frac{1}{2}} \otimes \mathcal{L} \otimes \mathcal{L}\|, \|[W]^{\frac{1}{2}} \otimes I_{N_\wp}\|\right\} \leqslant \frac{c}{\sqrt{N}}.$$

Define $\|e\|_2 = \sqrt{\|e^u\|_{0,\omega}^2 + \|e^v\|_{0,\omega}^2 + \|e^p\|_{0,\omega}^2}$, then addition of the three estimates for weighted norms of e^u, e^v, and e^p yields,

$$\begin{aligned}
\|e\|_2 &\leqslant c\,|W_h E| \\
&\leqslant c\|W_h\||E_h| \\
&\leqslant \frac{c}{\sqrt{N}}\|G_t^{-1}\||G_t E| \\
&\leqslant \frac{c}{\sqrt{N}}\|G_t^{-1}\|\sqrt{N(2(N-1)^2-1)}|G_t E|_\infty && (\text{as } |x| \leqslant \sqrt{m}|x|_\infty, \text{ for any } x \in \mathbb{R}^m) \\
&\leqslant cN^8 e^{-CN},
\end{aligned}$$

the last inequality results from Theorem 3 and (C.3). Thus, $\|e\|_2 \leqslant cN^8e^{-CN}$ and Theorem 5.12 in [23, p. 248] for the Legendre truncation error estimate yields the following estimate on the error in exact and approximate solution

$$\begin{aligned}
\|u-u_N\|_{0,\omega} + \|v-v_N\|_{0,\omega} + \|p-p_{N-2}\|_{0,\omega} &\leqslant \|\mathscr{T}_N u\|_{0,\omega} + \|e^u\|_{0,\omega} + \|\mathscr{T}_N v\|_{0,\omega} \\
&\quad + \|e^v\|_{0,\omega} + \|\mathscr{T}_{N-2} p\|_{0,\omega} + \|e^p\|_{0,\omega} \\
&\leqslant ce^{-CN} + c\|e\|_2 \\
&\leqslant cN^8 e^{-CN}. \quad \square
\end{aligned}$$

This concludes the proof of the spectral convergence in both space and time of the $P_N - P_{N-2}$ scheme in space and Chebyshev Gauss-Lobatto collocation in time. Thus, completing the analysis for a space-time spectral method for the Stokes problem.

Remark 3. In [18, Chap. 4], a space-time spectral scheme with spectral collocation in time and staggered-grid spectral collocation in space for (20) is presented. It may be difficult to perform the theoretical analysis presented in this paper to the aforementioned scheme. The main challenge of performing such an analysis is estimating the singular values of its global space-time spectral operator in terms of the sub-blocks which consist of dense interpolation matrices.

5. The Navier-Stokes equations

One of the most significant problems in fluid dynamics is the Navier-Stokes equations, which model the conservation of momentum and conservation of mass for Newtonian fluids. Its applications include modeling water flow in a pipe, ocean currents, airflow around a wing, weather, etc. Therefore, they help in the design process of vehicles and airplanes, the study of blood flow, the area of magneto-hydrodynamics, and the analysis of pollution, among others. We extend the space-time spectral scheme discussed in section 4 to the unsteady Navier-Stokes equations, which are given as:

$$\begin{aligned}
u_t + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{1}{R_e}\Delta u + p_x &= f_1 \text{ in } \Omega_t, \\
v_t + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{1}{R_e}\Delta v + p_y &= f_2 \text{ in } \Omega_t, \\
u_x + v_y &= 0 \text{ in } \Omega_t, \\
u = 0,\ v &= 0 \text{ on } \partial\Omega, \\
u(x,y,-1) = u_0(x,y),\ v(x,y,-1) &= v_0(x,y) \text{ in } \Omega.
\end{aligned} \tag{46}$$

Since (23) is a linearized version of the above problem, we define square matrices $\mathfrak{P}^{(\ell)}$ and $\mathfrak{T}^{(\ell)}$ of size $N-1$ for formulating the non-linear terms. For a given index $0 \leqslant \ell \leqslant N-2$, (i,j) entries of $\mathfrak{P}^{(\ell)}$ and $\mathfrak{T}^{(\ell)}$ are defined as $\mathfrak{P}_{ij}^{(\ell)} = \int_{-1}^{1} \phi_i(x)\phi_j'(x)\phi_\ell(x)dx$ and $\mathfrak{T}_{ij}^{(\ell)} = \int_{-1}^{1} \phi_i(x)\phi_j(x)\phi_\ell(x)dx$, respectively, for $0 \leqslant i,j \leqslant N-2$. A simple fixed point scheme yields the following linear system,

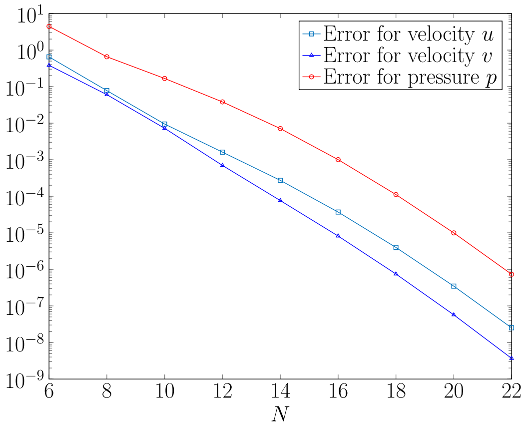


Fig. 11. Convergence for the $P_N - P_{N-2}$ scheme for the unsteady Navier-Stokes problem with $R_e = 1$.

$$\begin{aligned}
&\left(W^{(k-1)} + [D] \otimes \mathcal{M} + \frac{1}{R_e} I_N \otimes A\right) u_h^{(k)} + (I_N \otimes B_1)\, p_h^{(k)} = (I_N \otimes \mathcal{Q})\, \mathbf{F}_1 - \mathbf{d} \otimes (\mathcal{M} u_{0h}),\\
&\left(W^{(k-1)} + [D] \otimes \mathcal{M} + \frac{1}{R_e} I_N \otimes A\right) v_h^{(k)} + (I_N \otimes B_2)\, p_h^{(k)} = (I_N \otimes \mathcal{Q})\, \mathbf{F}_2 - \mathbf{d} \otimes (\mathcal{M} v_{0h}),\\
&\qquad\qquad \left(I_N \otimes B_1^T\right) u_h^{(k)} + \left(I_N \otimes B_2^T\right) v_h^{(k)} = O,
\end{aligned} \tag{47}$$

where the non-linear term $W^{(k-1)}$ is a block diagonal matrix with N blocks and is defined as $W^{(k-1)} = \oplus_{j=1}^{N} \left((I_\vartheta \otimes (u_h^{j,(k-1)})^T) W_1 + (I_\vartheta \otimes (v_h^{j,(k-1)})^T) W_2\right)$ and $W_1, W_2 \in \mathbb{R}^{\vartheta^2 \times \vartheta}$ are defined as the block column matrices with $(m,n) \times 1$ block as $\mathfrak{T}^{(n)} \otimes \mathfrak{P}^{(m)}, \mathfrak{P}^{(n)} \otimes \mathfrak{T}^{(m)} \in \mathbb{R}^{\vartheta \times \vartheta}$, respectively, for $0 \leqslant m, n \leqslant N-2$. Also, $u_h^{j,(k-1)}$ and $v_h^{j,(k-1)}$, vectors of length ϑ, represent the components of u_h and v_h, vectors of length $N\vartheta$, for time $t = t_j$ at $(k-1)$st iteration, for $1 \leqslant j \leqslant N$. The space-time spectral convergence of the scheme given by (47) is observed in Fig. 11. For more details, see [18, Chap. 4]. We expect that the result of Theorem 4 can be extended to (46) provided that R_e is kept sufficiently small.

6. Numerical results

For the Stokes problem in the steady state, given by (1), we implemented the proposed $P_N - P_{N-2}$ scheme in space by using recombined Legendre basis functions on MATLAB®. Take f_1, f_2 so that the exact solutions are $u(x, y) = (\cos(\pi x) + 1)\sin(2\pi y)$, $v(x, y) = 0.5\sin(\pi x)(1 - \cos(2\pi y))$, and $p(x, y) = \sin(\pi x)\cos(\pi y)$. The spectral convergence of the $P_N - P_{N-2}$ scheme analyzed in section 3 for the Stokes problem in the steady state is depicted by Fig. 1.

For the unsteady state, we implemented the scheme derived in section 4 for the Stokes problem defined by (2). Based on our interest in the analysis, we selected Chebyshev Gauss-Lobatto collocation in time, which can easily be replaced by other polynomials. For our implementation on MATLAB®, we take f_1, f_2, so that the exact solutions are $u(x, y, t) = (\cos(\pi x) + 1)\sin(2\pi y)\sin(0.5\pi t)$, $v(x, y, t) = 0.5\sin(\pi x)(1 - \cos(2\pi y))\sin(0.5\pi t)$, and $p(x, y, t) = \sin(\pi x)\cos(\pi y)\sin(0.5\pi t)$. The space-time spectral convergence of the unsteady Stokes scheme is depicted by Fig. 7. The same set of exact solutions is used for the MATLAB® implementation of (46), the Navier-Stokes problem in an unsteady state. The spectral convergence of the unsteady Navier-Stokes problem, using the same scheme as before, in section 5 is evident from Fig. 11. The iteration is stopped whenever the infinity norm of the difference between two consecutive iterates is smaller than $\epsilon = 10^{-12}$. All our MATLAB® implementations are provided in [19].

7. Conclusion and future work

In this paper, we proposed a space-time spectral method for the Stokes problem, which implements the $P_N - P_{N-2}$ scheme in space and spectral collocation in time. For simplicity of analysis, we considered a recombined Legendre basis in space and implemented Chebyshev collocation in time. Note that this scheme can easily be adapted to other orthogonal polynomial bases. The optimal condition number estimates were derived for the sub-block appearing in the global spectral operator for the Stokes problem in the steady state. Analysis of the scheme in the unsteady state required a new estimate of the maximum singular value (or 2-norm) of the Chebyshev Gauss-Lobatto pseudospectral derivative matrix, as per our knowledge, which appeared in existing literature solely through numerical experiments. The condition number estimate of the global space-time operator is still incomplete because we relied on numerical results for two estimates. We proved the spectral convergence of this scheme in space and time. Furthermore, space-time spectral convergence is evident for the scheme presented for the unsteady Navier-Stokes problem.

The global space-time spectral operator of the scheme analyzed in this paper is a non-symmetric saddle point matrix so that the symmetric part of its leading block is indefinite. This problem highlights a potential linear algebra problem. We desire some estimates on the spectrum of such type of a saddle point matrix, which will be significant for deriving spectral condition numbers for such schemes as seen in [26]. Since the linear systems arising from space-time spectral methods are coupled at all times, another future question to be studied is whether these schemes can be formulated as parallel in time. A problem of great interest is to study and estimate the high limit of the Reynolds number for the $P_N - P_{N-2}$ scheme derived in section 5 for the unsteady Navier-Stokes problem.

In [31], pseudo-spectral collocation methods on finite domains for initial value problems were reformulated in terms of the summation-by-parts and simultaneous-approximation-terms (SBP–SAT). Thus, another step towards the advancement of high-order time-dependent schemes is the application of SBP approximations, for which the presence of space-time stability is a merit.

CRediT authorship contribution statement

Avleen Kaur: Conceptualization, Data curation, Formal analysis, Methodology, Software, Visualization, Writing – original draft. **S.H. Lui:** Conceptualization, Formal analysis, Funding acquisition, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We sincerely thank the anonymous reviewers for taking the time and effort necessary to review this manuscript. We are grateful to the reviewers for their valuable comments and suggestions, which helped us in improving the quality of the manuscript.

Appendix A. Double summations in Theorem 1

The nine double summations denoted by $\mathcal{S}_k$, for $1 \leqslant k \leqslant 9$, in (16) are defined as follows

$$\mathcal{S}_1 = \sum_{k=0}^{1}\sum_{j=0}^{1}(x_k^j)^2\Bigg((4k+6)\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right) + (4j+6)\left(\frac{1}{2k+1}+\frac{1}{2k+5}-\frac{2k+9}{(2k+5)^2}\right)\Bigg),$$

$$\mathcal{S}_2 = \sum_{k=2}^{N-4}\sum_{j=0}^{1}(x_k^j)^2\left((4k+6)\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right)+\frac{16(4j+6)}{(2k+1)(2k+5)^2}\right),$$

$$\mathcal{S}_3 = \sum_{k=N-3}^{N-2}\sum_{j=0}^{1}(x_k^j)^2\left((4k+6)\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right)+\frac{4j+6}{2k+1}\right),$$

$$\mathcal{S}_4 = \sum_{k=0}^{1}\sum_{j=2}^{N-4}(x_k^j)^2\left(\frac{16(4k+6)}{(2j+1)(2j+5)^2}+(4j+6)\left(\frac{1}{2k+1}+\frac{1}{2k+5}-\frac{2k+9}{(2k+5)^2}\right)\right),$$

$$\mathcal{S}_5 = \sum_{k=2}^{N-4}\sum_{j=2}^{N-4}(x_k^j)^2\left(\frac{16(4k+6)}{(2j+1)(2j+5)^2}+\frac{16(4j+6)}{(2k+1)(2k+5)^2}\right),$$

$$\mathcal{S}_6 = \sum_{k=N-3}^{N-2}\sum_{j=2}^{N-4}(x_k^j)^2\left(\frac{16(4k+6)}{(2j+1)(2j+5)^2}+\frac{4j+6}{2k+1}\right),$$

$$\mathcal{S}_7 = \sum_{k=0}^{1}\sum_{j=N-3}^{N-2}(x_k^j)^2\left(\frac{4k+6}{2j+1}+(4j+6)\left(\frac{1}{2k+1}+\frac{1}{2k+5}-\frac{2k+9}{(2k+5)^2}\right)\right),$$

$$\mathcal{S}_8 = \sum_{k=2}^{N-4}\sum_{j=N-3}^{N-2}(x_k^j)^2\left(\frac{4k+6}{2j+1}+\frac{16(4j+6)}{(2k+1)(2k+5)^2}\right),$$

and, finally $\mathcal{S}_9 = \sum_{k=N-3}^{N-2}\sum_{j=N-3}^{N-2}(x_k^j)^2\left(\frac{4k+6}{2j+1}+\frac{4j+6}{2k+1}\right)$. We claim that $S_m \geqslant \frac{c}{N^2}$, for all $1 \leqslant m \leqslant 9$. First of all, the sum $\mathcal{S}_1$ contains only constants independent of N, therefore

$$\mathcal{S}_1 \geqslant \frac{c}{N^2}\sum_{k=0}^{1}\sum_{j=0}^{1}(x_k^j)^2. \tag{A.1}$$

For the second one, note that

$$\begin{aligned}\mathcal{S}_2 &\geqslant \sum_{k=2}^{N-4}\sum_{j=0}^{1}(x_k^j)^2\,(4(2)+6)\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right)\\ &\geqslant \frac{c}{N^2}\sum_{k=2}^{N-4}\sum_{j=0}^{1}(x_k^j)^2.\end{aligned} \tag{A.2}$$

Similarly,

$$\begin{aligned}\mathcal{S}_3 &\geqslant \sum_{k=N-3}^{N-2}\sum_{j=0}^{1}(x_k^j)^2\,(4(N-3)+6)\left(\frac{1}{2j+1}+\frac{1}{2j+5}-\frac{2j+9}{(2j+5)^2}\right)\\ &\geqslant cN\sum_{k=N-3}^{N-2}\sum_{j=0}^{1}(x_k^j)^2 \geqslant \frac{c}{N^2}\sum_{k=N-3}^{N-2}\sum_{j=0}^{1}(x_k^j)^2.\end{aligned} \tag{A.3}$$

Since $2 \leqslant k, j \leqslant N-4$ implies $(2j+5) \leqslant 6j$, $\frac{16(4k+6)}{(2j+1)(2j+5)^2} \geqslant \frac{16\cdot 4k}{(6j)^3} \geqslant c\frac{k}{j^3}$, thus $\mathcal{S}_5$ gives

$$\mathcal{S}_5 \geqslant c\sum_{k=2}^{N-4}\sum_{j=2}^{N-4}(x_k^j)^2\left(\frac{k}{j^3}+\frac{j}{k^3}\right) \geqslant \frac{c}{N^2}\sum_{k=2}^{N-4}\sum_{j=2}^{N-4}(x_k^j)^2, \tag{A.4}$$

as it is easily proved by using calculus that for $2 \leqslant k, j \leqslant N-4$, $\frac{k}{j^3}+\frac{j}{k^3} \geqslant \frac{c}{N^2}$, for details see [18, p. 49]. Moving forward to the term $\mathcal{S}_6$, which gives

$$\begin{aligned}\mathcal{S}_6 &\geqslant \sum_{k=N-3}^{N-2}\sum_{j=2}^{N-4}(x_k^j)^2\frac{4j+6}{2k+1} \geqslant \sum_{k=N-3}^{N-2}\sum_{j=2}^{N-4}(x_k^j)^2\frac{4(2)+6}{2k+1}\\ &\geqslant \sum_{k=N-3}^{N-2}\sum_{j=2}^{N-4}(x_k^j)^2\frac{14}{2(N-2)+1} \geqslant \frac{c}{N^2}\sum_{k=N-3}^{N-2}\sum_{j=2}^{N-4}(x_k^j)^2.\end{aligned} \tag{A.5}$$

Note that the terms $\mathcal{S}_4, \mathcal{S}_7$, and $\mathcal{S}_8$ are similar to the term $\mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_6$, respectively, hence (A.2), (A.3), and (A.5) yield

$$\mathcal{S}_4 \geqslant \frac{c}{N^2}\sum_{k=0}^{1}\sum_{j=2}^{N-4}(x_k^j)^2, \mathcal{S}_7 \geqslant \frac{c}{N^2}\sum_{k=0}^{1}\sum_{j=N-3}^{N-2}(x_k^j)^2, \mathcal{S}_8 \geqslant \frac{c}{N^2}\sum_{k=2}^{N-4}\sum_{j=N-3}^{N-2}(x_k^j)^2. \tag{A.6}$$

Finally, since for any $k,j \in \mathbb{N}$, $\frac{4k+6}{2j+1} \geqslant \frac{4(k+1)}{2(j+1)} \geqslant \frac{k}{j}$, thus using it in $\mathcal{S}_9$ gives

$$\mathcal{S}_9 \geqslant \sum_{k=N-3}^{N-2}\sum_{j=N-3}^{N-2}(x_k^j)^2\left(\frac{k}{j}+\frac{j}{k}\right) \geqslant \frac{c}{N^2}\sum_{k=N-3}^{N-2}\sum_{j=N-3}^{N-2}(x_k^j)^2. \tag{A.7}$$

Appendix B. Schur complement for the steady Stokes problem

In this section, we prove some results that are required for proving estimates for the Schur complement for the Stokes problem in steady state. The Uzawa pressure operator, denoted by $\Upsilon : L_0^2(\Omega) \to L_0^2(\Omega)$, is defined as $\Upsilon := \nabla\cdot\Delta^{-1}\nabla$. It is a self-adjoint, bounded, coercive and hence a bijective operator with $\lambda_{\max}(\Upsilon) = 1$. Also, $\Delta^{-1} : (H^{-1}(\Omega))^2 \to V$ denotes the inverse Laplacian. Let $u \in (H^{-1}(\Omega))^2$, we say $\Delta^{-1}u = v \in V$ if

$$\Delta v = u \text{ in } \Omega,$$
$$v = 0 \text{ on } \partial\Omega.$$

Note that Δ is the vector Laplacian as $v \in V$ is a vector having two components. From [3, p. 422], the following inf-sup condition holds for the $P_N - P_{N-2}$ scheme

$$\inf_{q_N \in \mathbb{P}_{N-2,N-2} \cap L_0^2(\Omega)} \sup_{v_N \in \mathbb{P}^0_{N,N}} \frac{(\nabla \cdot v_N, q_N)}{\|v_N\|_1 \|q_N\|_0} \geqslant \frac{c}{\sqrt{N}},$$

which, as stated in [10, p. 173], is equivalent to

$$\min_{q \in \mathbb{R}^{\wp} \setminus 0} \sqrt{\frac{q^T B^T \mathcal{A}^{-1} B q}{q^T \mathfrak{M} q}} \geqslant \frac{c}{\sqrt{N}}, \quad \text{or} \quad \min_{q \in \mathbb{R}^{\wp} \setminus 0} \frac{q^T \Upsilon_h q}{q^T \mathfrak{M} q} \geqslant \frac{c}{N}, \tag{B.1}$$

which is also observed numerically in Fig. 12a. Here q is the vector of coefficients of $q_N = \sum_{i=0}^{N-2} \sum_{\substack{j=0 \\ i+j>0}}^{N-2} q_{ij} L_i(x) L_j(y)$, and $\mathfrak{M}$ is the mass matrix, so that

$$q^T \mathfrak{M} q = \|q_N\|_0^2 = \sum_{i=0}^{N-2} \sum_{\substack{j=0 \\ i+j>0}}^{N-2} q_{ij}^2 \gamma_i \gamma_j = q^T [\Gamma \otimes \Gamma] q,$$

where $\Gamma := \operatorname{diag}(\gamma_0, \gamma_1, \ldots, \gamma_{N-2}) \in \mathbb{R}^{(N-1)\times(N-1)}$ with $\gamma_j = \frac{2}{2j+1}$ for $0 \leqslant j \leqslant N-2$. Recall that $[\,\cdot\,]$ signifies that its first row and first column are deleted, thus $\mathfrak{M} := [\Gamma \otimes \Gamma] \in \mathbb{R}^{\wp \times \wp}$ is a diagonal matrix, and

$$\lambda_{\min}(\mathfrak{M}) = \gamma_{N-2}^2 \geqslant \frac{c}{N^2}. \tag{B.2}$$

Lastly, the following result is consequential for proving an upper bound for $\lambda_{\min}(\Upsilon_h)$ given by Lemma 5.

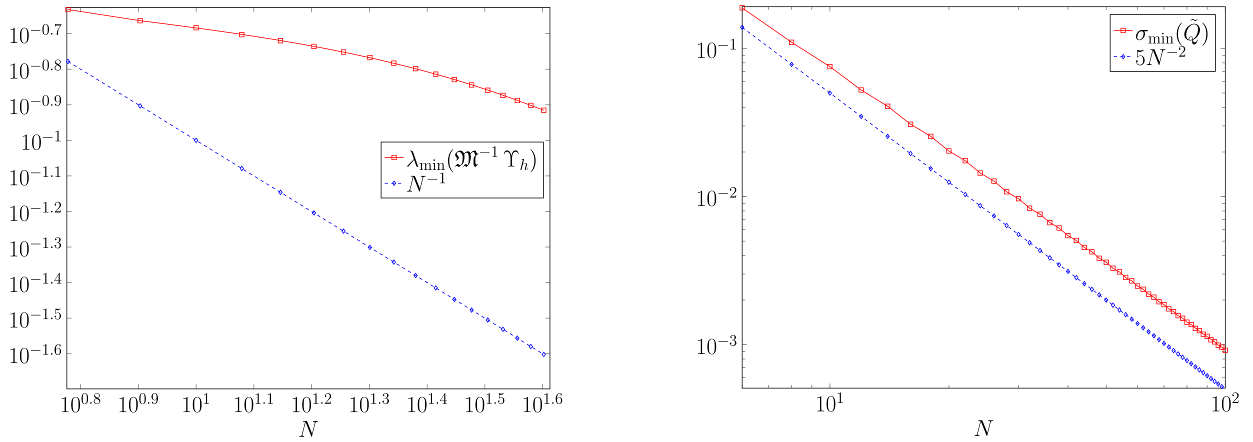
Lemma 7. *For given $N \geqslant 4$, the matrix $B \in \mathbb{R}^{2\vartheta \times \wp}$ defined by (11), then $\sigma_{\min}(B) \leqslant \dfrac{c}{N^2}$.*

Proof. As seen in proof of Lemma 4, $B^T B = [Q^T Q \otimes R^T R + R^T R \otimes Q^T Q]$, where $R, Q \in \mathbb{R}^{(N-1)\times(N-1)}$ are defined by Proposition 1. First, we claim that $\sigma_{\min}(Q) \leqslant \frac{c}{N^2}$. To this end, recall that $Q_{kk} = \gamma_k$, for all $0 \leqslant k \leqslant N-2$, and $Q_{k,k+2} = -\gamma_{k+2}$, for all $0 \leqslant k \leqslant N-4$.

$$\begin{aligned}
\sigma_{\min}(Q) &= \min_{\substack{x \in \mathbb{R}^{N-1} \\ |x|=1}} |Qx| \\
&= \min_{\substack{x \in \mathbb{R}^{N-1} \\ |x|=1}} \Big| \big[(\gamma_0 x_0 - \gamma_2 x_2); (\gamma_1 x_1 - \gamma_3 x_3); \ldots; (\gamma_k x_k - \gamma_{k+2} x_{k+2}); \ldots; \\
&\qquad \ldots; \gamma_{N-3} x_{N-3}; \gamma_{N-2} x_{N-2}\big] \Big|
\end{aligned}$$

Define $m = \left\lfloor \frac{N-2}{2} - \frac{1}{2} \left\lfloor \frac{N-2}{2} \right\rfloor \right\rfloor + 1$, $n = \left\lfloor \frac{N-2}{2} \right\rfloor$, and $y \in \mathbb{R}^{N-1}$ so that $y_{n+2k} = \frac{\sqrt{2(m-k)-1}}{m}$, for $0 \leqslant k \leqslant m-1$ and is zero otherwise, thus $|y| = 1$. Note that $cN \leqslant n \leqslant n+2k \leqslant N-2$, for all $0 \leqslant k \leqslant m-1$, and there exist some positive constants c_1, c_2 such that $c_1 N \leqslant m \leqslant c_2 N$, therefore the following estimate is obtained.

$$\begin{aligned}
\sigma_{\min}^2(Q) &\leqslant \sum_{k=0}^{m-2} (\gamma_{n+2k} y_{n+2k} - \gamma_{n+2k+2} y_{n+2k+2})^2 + \gamma_{n+2(m-1)}^2 y_{n+2(m-1)}^2 \\
&= \sum_{k=0}^{m-2} \left(\frac{2}{2(n+2k+2)+1} y_{(n+2k)+2} - \frac{2}{2(n+2k)+1} y_{n+2k} \right)^2 \\
&\qquad + \frac{4}{(2(n+2(m-1))+1)^2} \cdot \frac{1}{m^2}
\end{aligned}$$



(a) Minimum eigenvalue of $\mathfrak{M}^{-1}\Upsilon_h$. (b) Minimum singular value of $\tilde{Q}$.

Fig. 12. Numerical results for Appendix B.

$$
\begin{aligned}
&\leqslant \sum_{k=0}^{m-2}\left(\frac{2}{2n+4k+5}-\frac{2}{2n+4k+1}\right)^2 y_{n+2k}^2+\frac{c}{N^4} \qquad (\text{since } y_{n+2i}\geqslant y_{n+2j}, \text{for all } 0\leqslant i\leqslant j\leqslant m-1)\\
&\leqslant 64\sum_{k=0}^{m-2}\left(\frac{1}{(2n+4k+1)(2n+4k+5)}\right)^2\frac{2(m-k)+1}{m^2}+\frac{c}{N^4}\\
&\leqslant \frac{c}{N^4}.
\end{aligned}
$$

Hence, the claim is proved, thus the result is obtained by following the proof of Lemma 4 from (18). □

Define $\tilde{Q} := Q(n : N-1, :)$, that is, the sub-matrix of Q obtained by removing the first $n-1$ rows of Q, then Fig. 12b verifies the upper bound on $\sigma_{\min}(Q)$, obtained by considering the vector y in the above proof.

Appendix C. Proof of Theorem 4

Let us estimate $|G_t E|_\infty$, which is given by (44). To this end, (37) implies $|\epsilon|_\infty \leqslant cN^2e^{-cN}$, it remains to estimate the infinity-norm of R_i for $1\leqslant i\leqslant 2$. Note that $0\leqslant m,n\leqslant N-2$ and $1\leqslant k\leqslant N$ throughout this section, unless otherwise stated.

For R_1, the non-zero entries of R_1^u are of the form

$$
\begin{aligned}
&(\mathscr{T}_N f_1^k + ((\mathscr{T}_N u)_t - \Delta(\mathscr{T}_N u))(x,y,t_k), \phi_m(x)\phi_n(y))\\
&=: s_f^k + s_1(t_k) + s_2(t_k).
\end{aligned}
$$

Firstly, $|s_f^k| \leqslant \|\mathscr{T}_N f_1^k\|_0 \|\phi_m(x)\phi_n(y)\|_0 \leqslant c\|\mathscr{T}_N f_1^k\|_0$, by Theorem 5.12 in [23, p. 248] for the Legendre truncation error estimate,

$$
|s_f^k| \leqslant ce^{-CN}.
$$

Assume that $s_1(t_k) = z'(t_k)$, where $z(t) = ((\mathscr{T}_N u)(x,y,t), \phi_m(x)\phi_n(y))$, for some $0\leqslant m,n\leqslant N-2$. The interpolant of $z(t)$ is given as $\mathcal{I}_N z(t) = \sum_{i=1}^{N} z(t_i)\ell_i(t) + z(-1)\ell_0(t)$, then

$$
\begin{aligned}
s_1(t_k) = z'(t_k) = (\mathcal{I}_N z)'(t_k) + \varepsilon_k &= ([D](\mathcal{I}_N(z)(t_h)))_k + z(-1)\ell_0'(t_k) + \varepsilon_k\\
&= ([D]\cdot z(t_h))_k + z(-1)\ell_0'(t_k) + \varepsilon_k\\
&= \sum_{i=1}^{N} d_{ki} z(t_i) + z(-1)\ell_0'(t_k) + \varepsilon_k, \qquad (C.1)
\end{aligned}
$$

where the error $|\varepsilon_k| \leqslant cN^2e^{-CN}$ as derived in [30]. To estimate $s_1(t_k)$, note that for $1\leqslant i\leqslant N$,

$$z(t_i) = ((\mathscr{T}_N u)(x, y, t_i), \phi_m(x)\phi_n(y)) \leqslant c\|(\mathscr{T}_N u)(x, y, t_i)\|_0 \leqslant ce^{-CN},$$

where we have used Theorem 5.12 in [23, p. 248] for the Legendre truncation error estimate, i.e., $(\mathscr{T}_N u)(x, y, t_i)$. Also, $z(-1) = ((\mathscr{T}_N u)(x, y, -1), \phi_m(x)\phi_n(y)) \leqslant c\|(\mathscr{T}_N u_0)(x, y)\|_0 \leqslant ce^{-CN}$. Recall from the proof of Lemma 6 that $\|[D]\|_\infty \leqslant cN^2$, implying $|d_{ki}| \leqslant cN^2$ and $|d_{k0}| \leqslant cN^2$ for all $1 \leqslant i, k \leqslant N$. Hence, these results along with (C.1) give $|s_1(t_k)| \leqslant cN^3 e^{-CN}$.

Since $s_2(t_k) = ((\mathscr{T}_N u)(x, y, t_k), -\Delta(\phi_m(x)\phi_n(y)))$, thus

$$|s_2(t_k)| \leqslant c\|(\mathscr{T}_N u)(x, y, t_k)\|_0 \leqslant ce^{-CN}.$$

Therefore, $|R_1^u|_\infty \leqslant cN^3 e^{-CN} + 2ce^{-CN} \leqslant cN^3 e^{-CN}$. Similar estimate holds for R_1^v, hence the following estimate is obtained

$$|R_1|_\infty \leqslant cN^3 e^{-CN}. \tag{C.2}$$

For R_2, its components consist of as r_2^u, r_2^v, and r_2^p. We estimate the entries of r_2^u by using the same Legendre truncation error result, which gives

$$|r_2^u(t_k)| = |((\mathscr{T}_{N-2} p)(x, y, t_k), \phi_m'(x)\phi_n(y))| \leqslant c\|(\mathscr{T}_{N-2} p)(x, y, t_k)\|_0 \leqslant ce^{-CN},$$

similar result holds for r_2^v, and finally for $0 \leqslant m, n \leqslant N-2$ with $m + n > 0$ and $1 \leqslant k \leqslant N$,

$$\begin{aligned} |r_2^p(t_k)| &= |(L_m(x)L_n(y), \left((\mathscr{T}_N u)_x + (\mathscr{T}_N v)_y\right)(x, y, t_k))| \\ &\leqslant |(L_m'(x)L_n(y), (\mathscr{T}_N u)(x, y, t_k))| + |(L_m(x)L_n'(y), (\mathscr{T}_N v)(x, y, t_k))| \\ &\leqslant c\| (\mathscr{T}_N u)(x, y, t_k)\|_0 \leqslant ce^{-CN}, \end{aligned}$$

hence, $|R_2|_\infty \leqslant ce^{-CN}$. This estimate, (C.2), and (44) yield the following result

$$|G_t E|_\infty \leqslant |\epsilon|_\infty + \sum_{i=1}^{2} |R_i|_\infty \leqslant cN^3 e^{-CN}. \tag{C.3}$$

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