# Preservers and converters of immanant functions

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# Linear preservers of the determinant

$M_n(\mathbb{F})$ n-square matrices over a field $\mathbb{F}$

Frobenius (1897) solved the first L.P.P.

$\mathbb{F} = \mathbb{C}$

$$T:\ M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$$

T linear

$$\det T(X) = \det X \ \text{ for all } \ X \in M_n(\mathbb{C})$$

$T(X) = MXN$ for all X ϵ $M_n(\mathbb{C})$

$or$

$T(X) = MX^TN$ for all X ϵ $M_n(\mathbb{C})$

where M and N are nonsigular matrices satisfying

$$\det(MN) = 1.$$

# Linear preservers of the permanent

$$\det(X) = \sum_{\boldsymbol{\sigma} \in S_n} \varepsilon(\boldsymbol{\sigma}) \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

$$\operatorname{per}(X) = \sum_{\boldsymbol{\sigma} \in S_n} \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

# The permanent function,
# M. Marcus and F. C. May, Canad. J.Math. (1962)

$T\colon\ M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$T$ *linear*

$\operatorname{per} T(X) = \operatorname{per} X$ *for all* $X \in M_n(\mathbb{C})$

$$T(X) = D_1\, P(\pi) X P(\rho)\, D_2 \quad \text{for all } X \in M_n(\mathbb{C})$$

*or*

$$T(X) = D_1\, P(\pi) X^T P(\rho)\, D_2 \quad \text{for all } X \in M_n(\mathbb{C})$$

*where* $D_1, D_2$ *are diagonal matrices satisfying*

$$\det(D_1 D_2) = 1,$$

and $P(\pi), P(\rho)$ *are permutation matrices*.

# Linear converters of the permanent into the determinant

M. Marcus, H. Minc , Illinois. J. Math(1961)
P. Botta , Canad. Math. Bull. (1968)

$T\colon\ M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$T$ *linear*

$\operatorname{per} T(X) = \det X$ *for all* $X \in M_n(\mathbb{C})$

The answer is NO if $n \geq 3$.

# With an exception for n=2

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$\det(X) = x_{11}x_{22} - x_{12}x_{21}$$
$$\text{per}(X) = x_{11}x_{22} + x_{12}x_{21}$$

$$\mathsf{T}\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} x_{11} & -x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$perT(X) = \det(X)$$

# Linear preservers of the permanent on symmetric matrices

M. H. Lim and Hoch Ong , LAMA (1979)

$H_n(\mathbb{C})$, the n-square symmetric matrices

$T\colon H_n(\mathbb{C}) \longrightarrow H_n(\mathbb{C})$

$T$ *linear*

$\operatorname{per} T(X) = \operatorname{per} X$ *for all* $X \in H_n(\mathbb{C})$

# Linear preservers and linear converters

V is a subspace of $M_n(\mathbb{F})$

$$T : V \longrightarrow V \quad \textit{a linear map}$$

$f, g$ *are functions* $\quad f, g : V \longrightarrow \mathbb{F}$

$$f\big(T(X)\big) = f(X) \quad \text{for all } X \in V \qquad \text{(preserver)}$$

$$f\big(T(X)\big) = g(X) \quad \text{for all } X \in V \qquad \text{(converter)}$$

$\chi$ irreducible complex character of $S_n$

the **immanant** of an n-square matrix $X \in M_n(\mathbb{C})$

$$d_\chi(X) = \sum_{\boldsymbol{\sigma} \in S_n} \chi(\boldsymbol{\sigma}) \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

$$d_{\chi}(X) = \sum_{\boldsymbol{\sigma} \in S_n} \chi(\boldsymbol{\sigma}) \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

if $\chi(\boldsymbol{\sigma}) = \varepsilon(\boldsymbol{\sigma})$, the immanant is the determinant

$$det(X) = \sum_{\boldsymbol{\sigma} \in S_n} \varepsilon(\boldsymbol{\sigma}) \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

$$d_{\chi}(X) = \sum_{\boldsymbol{\sigma} \in S_n} \chi(\boldsymbol{\sigma}) \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

if $\chi(\boldsymbol{\sigma}) = 1, \ \forall \sigma \in S_n$, the immanant is the permanent

$$per(X) = \sum_{\boldsymbol{\sigma} \in S_n} \prod_{i=1}^{n} x_{i\boldsymbol{\sigma}(i)}$$

# Linear preservers of an immanant

M. Antonia Duffner L.A.A. (1993)

$\chi$ is an irreducible complex characters of $S_n$

$T\colon\ M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$T$ linear

$d_{\chi}(T(X)) = d_{\chi}(X)$ for all $X \in M_n(\mathbb{C})$

$$T(X) = C * P(\pi) X P(\rho) \quad \text{for all} \quad X \in M_n(\mathbb{C}),$$

*or*

$$T(X) = C * P(\pi) X^T P(\rho) \quad \text{for all} \quad X \in M_n(\mathbb{C}),$$

where $P(\pi), P(\rho)$ are permutation matrices satisfying

$$\chi(\sigma) \prod_{i=1}^{n} c_{i\,\sigma(i)} = \chi(\pi\sigma\rho), \ \forall \boldsymbol{\sigma} \in S_n$$

*Proof*

(i)T is bijective

(ii) $\mathrm{T}(\mathrm{E}_{\mathrm{ij}})$,

where $\left(E_{ij}\right)_{kl}$ is the standard basis of $M_n(\mathbb{C})$.

$\chi = \varepsilon$ or $\chi \equiv 1$, are the unique linear characters of $S_n$, and

$\varepsilon(\sigma) = 1$ or $\varepsilon(\sigma) = -1$

and if $\chi \equiv 1$, $\chi(\sigma) = 1$

And for all other (nonlinear) characters

$$\exists\, \sigma \in S_n \ : \ \chi(\sigma) = 0.$$

We identify the character $\chi$ with the Young

Diagram associated with the partition $[\chi]$

We use the Murnagham-Nakayama Rule to

determine if for a certain permutation $\sigma$,

$$\chi(\sigma) \neq 0 \quad or \quad \chi(\sigma) = 0.$$

$$\det(AB) = \det(A)\det(B) \qquad \forall\ \mathrm{A}, \mathrm{B} \in M_n$$

However

$$\mathrm{per}(AB) \neq \mathrm{per}(\mathrm{A})\mathrm{per}(B)$$

$$d_{\boldsymbol{\chi}}(AB) \neq d_{\boldsymbol{\chi}}(A)\, d_{\boldsymbol{\chi}}(B)$$

$$\det(P(\pi)AP(\rho)) = \pm\det(A) \qquad \forall A \in M_n$$

$$per(P(\pi)AP(\rho)) = per(A) \qquad \forall A \in M_n$$

However

$$d_{\chi}(P(\pi)AP(\rho)) \neq d_{\chi}(A)$$

But

$$d_{\chi}(P(\pi)AP(\pi^{-1})) = d_{\chi}(A) \qquad \forall A \in M_n$$

# Linear converters of immanants

M. P. Coelho and M. A. Duffner , L.A.A. (1998)

X, λ are irreducible complex characters of $S_n$

$$\chi \neq \lambda$$

$T$: $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

T linear

$d_{\chi}(T(X)) = d_{\lambda}(X)$ for all X ϵ $M_n(\mathbb{C})$.

The answer is NO with an exception for $n = 3$.

# Linear preservers of an immanant on the skew-symmetric matrices

M. Purificação Coelho and M. Antónia Duffner (L.A.A. 2012)

X is an irreducible complex characters of $S_n$

$T$: $Q_n(\mathbb{C}) \longrightarrow Q_n(\mathbb{C})$

$T$ linear

$d_{\chi}(T(X)) = d_{\chi}(X)$ for all $X \in Q_n(\mathbb{C})$

# (Linear) preservers and (linear) converters

$\mathrm{V}$ is a subspace of $\mathrm{M_n}(\mathbb{F})$

$T : V \longrightarrow V$ a ***multiplicative*** *map*

$f, g$ *are functions* $f, g : V \longrightarrow \mathbb{F}$

$f\big(T(X)\big) = f(X)$ for all $X \in V$

$f\big(T(X)\big) = g(X)$ for all $X \in V$

# Multiplicative preservers and induced operators

W. S. Cheung, M. A. Duffner, Chi Kwong Li
(L. A. A. 2005)

$T : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

a multiplicative map

$d_{\chi}(T(X)) = d_{\chi}(X)$ for all X $\in M_n(\mathbb{C})$

# Determinant preserving maps on matrix algebras

Dolinar and P. Semrl, L.A.A. (2002)

$T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$

$\det(T(A) + \lambda T(B)) = \det(A + \lambda B)$

$\forall\, A, B \in M_n(\mathbb{C}),\ \forall\, \lambda \in \mathbb{C}$

$+$

$T\ surjective$

$T$ is **linear** and bijective

$$\det(T(A)) = \det(A), \quad \forall\, A \in M_n(\mathbb{C})$$

$T$ is a linear preserver of the determinant.

# On determinant preserver problems

Victor Tan, Tei Wang  L.A.A. (2003)

$T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$

~~T surjective~~

$$\det(T(A) + \lambda T(B)) = \det(A + \lambda B)$$

$\forall\ A, B \in M_n(\mathbb{C}), \qquad \forall\ \lambda \in \mathbb{C}$

# Immanant preserving and immmanant converting maps

M. P. Coelho and M. A. Duffner (L.A.A. 2006)

$T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ , T surjective

$$d_\chi(T(A) + \lambda T(B)) = d_{\chi'}(A + \lambda B)$$

$\forall\, A, B \in M_n(\mathbb{C}),\qquad \forall\, \lambda \in \mathbb{C}$

$T$ ~~surjective~~ (B. Kuzma)

If $\chi = \chi'$

$T$ is a **linear** preserver of $d_\chi$

If $\chi \neq \chi'$

$T$ is a **linear** converter of $d_\chi$ in $d_{\chi'}$

# Immanant conversion on symmetric matrices

M. Purificação Coelho, M. A. Duffner, Alexander Guterman , Special Matrices (2014)

$$T: H_n(\mathbb{C}) \to H_n(\mathbb{C})$$

$$d_\chi(T(A) + \lambda T(B)) = d_{\chi'}(A + \lambda B)$$

$$\forall\, A, B \in H_n(\mathbb{C}), \qquad \forall\, \lambda \in \mathbb{C}$$

T is a linear and bijective map

$$\chi = \chi'$$

$T$ is a linear preserver of an immanant on the subspace of the symmetric matrices , M. P. Coelho and M. A. Duffner

$$\chi \neq \chi'$$

M. P. Coelho and M. A. Duffner

There is no linear T such that

$d_{\chi}(T(X)) = d_{\chi'}(X)$ for all X ϵ $H_n(\mathbb{C})$

With an exception for $n = 4$.

# Permanent preservers on the space of doubly stochastic matrices

H. M. Moyls, Marvin Marcus, Henryk Minc, Can. J. Math (1962)

A permanent preserver on DS(n) is a map

$T: DS(n) \to DS(n)$, such that

$$T(\alpha A + \beta B) = \alpha\, T(A) + \beta\, T(B)$$

$$perT(A) = per(A)$$

for all $A, B \in DS(n)$, *and* for all real numbers $\alpha, \beta$ *such that*

$$0 \le \alpha \le 1, \quad 0 \le \beta \le 1, \ \alpha + \beta = 1$$

Immanant preservers on the set (convex polyedron) of doubly stochastic matrices

Rosário Fernandes and M. Antónia Duffner

(in preparation)

An immanant preserver on DS(n) is a map

$T: DS(n) \rightarrow DS(n)$, such that :

$$T(\alpha A + \beta B) = \alpha\, T(A) + \beta\, T(B)$$

$$d_\chi T(A) = d_\chi(A)$$

for all $A, B \in DS(n)$, *and* for all real numbers

$\alpha, \beta$ *such that*

$$0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1, \; \alpha + \beta = 1$$

Let $\chi$ be an irreducible nonlinear character of $S_n$,

T is an immanant preserver on DS($n$)

$T\ surjective$

if and only if

$$T(A) = P(\pi)A\,P(\pi^{-1})$$

$or$

$$T(A) = P(\pi)A^T P(\pi^{-1})$$

for all A ϵ DS($n$), where

$P(\pi)\ is\ a\ permutation\ matrix$

Proof

(i) $\left|d_{\chi}(S)\right| \leq \chi(id)per(S)$ ∀ S∈ DS(n)

and since

$per(S) \leq 1$, ∀ S∈ DS(n)

$$\left|d_{\chi}(S)\right| \leq \chi(id), \quad \forall\ S \in \mathrm{DS(n)}$$

(ii) $T(I_n) = I_n$ , with an exception for n=4

(iii) $T$ is injective

(iv) $T(P(\sigma)) = P(\pi)\, P(\sigma)\, P(\pi^{-1})$