Preserversonon tensor product of matrices and vectors

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- Actually, the same is true for other study (and professions).
- One would test a limited number of cases or measurements (of special types), and deduce useful information.

Orthogonal projections

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Lemma [Helton and Rodman, 1985]

A linear map $L : M_N \to M_N$ satisfies $L(P_N) = P_N$ if and only if it has the form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^tU^*.$$

Maps on tensor product

Theorem [Flfsen & Shultz, 2010], [Friedland,Li,Poon,Sze, 2010]

Let $\mathcal{S}$ be the set

$$\{A \otimes B : A \in P_m, B \in P_n\},$$

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where L_1 has the form $A \mapsto UAU^*$ or $A \mapsto UA^tU^*$,
and L_2 has the form $B \mapsto VBV^*$ or $B \mapsto VB^tV^*$.

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(a) $L\left(\otimes_{i=1}^k P_{n_i}\right) = \otimes_{i=1}^k P_{n_i}$.

(b) $L\left(\mathrm{conv}\left(\otimes_{i=1}^k P_{n_i}\right)\right) = \mathrm{conv}\ \left(\otimes_{i=1}^k P_{n_i}\right)$.

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(c) There is a permutation π on $\{1, \ldots, k\}$ and linear maps L_i on H_{n_i} for $i = 1, \ldots k$ such that

$$L\left(\otimes_{i=1}^{k} A_i\right) = \otimes_{i=1}^{k} L_i\left(A_{\pi(i)}\right) \quad \text{for} \quad \otimes_{i=1}^{k} A_k \in \otimes_{i=1}^{k} P_{n_i},$$

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where L_i has the form

$$X \mapsto U_i X U_i^* \quad \text{or} \quad X \mapsto U_i X^t U_i^*,$$

for some unitary $U_i \in M_{n_i}$ and $n_{\pi(i)} = n_i$ for $i = 1, \ldots, k$.

Key lemma, and further extensions

In our proof, we extended a result in [Baruch and Loewy, 1993] to the following.

[Friedland, Li, Poon, Sze, 2010]

A linear map $L : M_r \to M_s$ satisfies $L(P_r) \subseteq P_s$ if and only if one of the following holds.
(a) There is $S \in P_s$ such that L has the form $A \mapsto (\operatorname{tr} A)S$.
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- One may send $A_1 \otimes \cdots \otimes A_k$ into $H_{n_1} \otimes \{B_2\} \otimes \cdots \otimes \{B_k\}$.

Some recent study

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for some unitary U, where $L_i(X) = X$ or $L_i(X) = X^t$.

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(a) $X \mapsto (\mathrm{tr}\, X)R$ for a fixed orthogonal projection.

(b) $A \otimes B \mapsto U((\mathrm{tr}\, A)R \otimes L(B))U^*$, etc.

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- Then $L_A(B)$ is unitarily similar to $U_A(\tau(B))U_A^*$ such that $\tau(B)$ is a direct sum of

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such that $k = r + s + p + q$.

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- Then we will use these local maps to determine the structure of L such that $L(A \otimes B)$ for all $(A, B) \in P_m \otimes P_n$.

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If L maps Hermitian matrices to Hermitian matrices, then S is unitary.

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A linear map $L : M_{mn} \to M_{mn}$ sending the set of Hermitian matrices to itself and satisfies

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We get essentially the same conclusion if we replace $\sigma(X)$ by the spectral radius $r(X)$; the preservers are $\pm$ of spectrum preservers.

Spectrum preservers

Theorem [Fosner, Huang, Li, Sze, 2012]

A linear map $L : M_{mn} \to M_{mn}$ sending the set of Hermitian matrices to itself and satisfies

$$\sigma(L(A \otimes B)) = \sigma(A \otimes B) \quad \text{for all Hermitian } A \in M_m, B \in M_n$$

if and only if there is a unitary $U \in M_n$ such that L has the form

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Conjecture

$$A \otimes B \mapsto S(\tau_1(A) \otimes \tau_2(B))S^{-1}.$$

Determinant preservers

Theorem [Frobenius, 1897], [Marcus and Moyls, 1959]

A linear map $L : M_n \to M_n$ satisfies

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It follows that a linear determinant preserver on $L : H_n \to H_n$ has the form

$$A \mapsto \mu SAS^* \quad \text{or} \quad A \mapsto \mu SAS^*$$

with $\mu \in \{-1, 1\}$ and $S \in M_n$ such that $\det(\mu SS^*) = 1$.

Result on tensor product

Theorem [Xu and Fosner, 2013]

Suppose $L : H_{mn} \to H_{mn}$ is linear and unital. Then

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Example [Sze et al.] The unital assumption is necessary. Otherwise, there are some other maps, say, when $m = n$ is even,

$$A \otimes B \mapsto I_{n/2} \otimes \begin{bmatrix} 0 & AB \\ BA & 0 \end{bmatrix}.$$

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- There are other questions related to numerical ranges, norms, etc. (on matrices or vectors).
- Hope to get your help, and I can tell you more next time.

Thank you for your attention!