Preserving Entangled States

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Outline

- Linear Preservers
- Entangled States
- Linear Preservers of Maximally Entangled States
  - Theorem
  - Outline of Proof

Notation

M_n = space of (complex) $n \times n$ matrices

$H_n = \{A \in M_n : A^* = A\}$ (real space of Hermitian matrices)

$U_n = \{A \in M_n : A^* = A^{-1}\}$ (group of unitary matrices)

$\mathcal{P}_n = \{A \in M_n : A^* = A = A^2\}$ (set of projections)

Rank 1 nonincreasing operators

Theorem (Baruch-Loewy, 1993)

Let $\psi : H_n \to H_n$ be linear. Suppose rank $\psi(A) \leq 1$ whenever rank $A = 1$. Then ψ has one of the following forms:

1. $\psi(A) = \epsilon SAS^*$ *for some* $S \in M_n$, $\epsilon = \pm 1$;
2. $\psi(A) = \epsilon SA^t S^*$ *for some* $S \in M_n$, $\epsilon = \pm 1$; *or*
3. $\psi(A) = L(A)B$ *for some linear functional* $L : H_n \to \mathbb{R}$ *and* $B \in H_n$ *of rank 1.*

Preservers of rank 1 projections

Corollary

Let $\psi : H_n \to H_n$ be linear. Suppose $\psi(A)$ is a rank one projection whenever A is. Then ψ has one of the following forms:

1. $\psi(A) = UAU^*$ *for some unitary* $U \in M_n$;
2. $\psi(A) = UA^tU^*$ *for some unitary* $U \in M_n$; *or*
3. $\psi(A) = (TrA)\,P$ *for some projection* P.

State

- A *state* ρ is a positive linear functional acting on $\mathcal{B}(\mathcal{H})$ whose value at the identity I is one.

- In our finite-dimensional setting:

  complex Hilbert space $\mathcal{H} = \mathbb{C}^k, \quad \mathcal{B}(\mathcal{H}) = M_k,$

  ρ is a positive semidefinite $k \times k$ matrix with trace one

- A state is *pure* if it has rank one; otherwise, it is *mixed*.

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- Entanglement is what makes quantum computing work!

Maximally Entangled State

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- General multipartite case: maximally entangled states depend on measure used (or may not exist).

- Bipartite case: most measures have the same maximally entangled states.

Bipartite case

- Von Neumann entropy: $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$
- Partial trace over subsystem B: linear map defined by

$$\mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \mathrm{Tr}\, \rho_B$$

- Entropy of Entanglement (for bipartite pure state ρ):

$$S(\mathrm{Tr}_B\, \rho) = S(\mathrm{Tr}_A\, \rho)$$

- Maximized when $\mathrm{Tr}_A\, \rho = \mathrm{Tr}_B\, \rho = \frac{1}{n} I$.

Schmidt decomposition

- Every vector $\psi \in \mathbb{C}^n \otimes \mathbb{C}^n$ has a Schmidt decomposition

$$\psi = \sum_{i=1}^{n} c_i u_i \otimes v_i$$

  for some orthonormal bases $\{u_i\}$ and $\{v_i\}$ of $\mathbb{C}^n$, and nonnegative numbers c_i (Schmidt coefficients).

- If $\rho = \psi\psi^*$ is a pure state then the entropy of entanglement

$$S(\mathrm{Tr}_B \rho) = -\sum_{i=1}^{n} c_i^2 \log c_i^2$$

  is maximized when $c_i = 1/\sqrt{n}$ for all i.

MES

- A pure state ρ is a *maximally entangled state* (MES) if $\rho = \psi\psi^*$, where $\psi = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_i \otimes v_i$ for some orthonormal bases $\{u_i\}$ and $\{v_i\}$ of $\mathbb{C}^n$.
- Let $e_1, \ldots, e_n$ be the standard basis vectors, and $E_{ij} = e_i e_j^*$. For unitaries $U, V \in M_n$, define

$$\psi_{U,V} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Ue_i \otimes Ve_i,$$

$$\rho_{U,V} = \psi_{U,V}\psi_{U,V}^* = \frac{1}{n} \sum_{i,j=1}^{n} UE_{ij}U^* \otimes VE_{ij}V^*.$$

Simple Properties of MES

The set of Maximally Entangled States is:

- the orbit of the group action of $U_n \otimes U_n$ on

$$\rho_0 = \frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}$$

  since

$$\rho_{U,V} = (U \otimes V)\rho_0(U \otimes V)^*$$

- compact
- path-connected

Linear Preservers of MES

What linear maps Φ satisfy $\Phi(MES) \subseteq MES$?

1. $\rho \mapsto (U \otimes V)\rho(U \otimes V)^*$ for some unitary U, V
2. $\rho \mapsto \rho^t$
3. $A \otimes B \mapsto B \otimes A$

Since a generic MES is

$$\rho_{U,V} = \psi_{U,V}\psi_{U,V}^* = \frac{1}{n}\sum_{i,j=1}^{n} UE_{ij}U^* \otimes VE_{ij}V^*,$$

clearly any composition of these three maps will preserve MES.

Q) Are there any others?

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Q) Are there any others? A) Yes.

Real linear span

ρ is a MES if and only if $\mathrm{Tr}_A\,\rho = \mathrm{Tr}_B\,\rho = \frac{1}{n}I$. Let

$$\mathcal{S}_n = \{X \in H_n \otimes H_n : \mathrm{Tr}_A\, X = \mathrm{Tr}_B\, X = 0\}.$$

This is a real vector space of dimension $(n^2-1)^2$.

Proposition

The real linear span of MES, denoted by Span(MES), is $\mathbb{R}I + \mathcal{S}_n$.

If $\rho \in$ Span(MES) is a pure state, then $\rho \in$ MES

Since Span(MES) is a proper subspace, one could define a linear preserver $\tilde{\Phi}$ as follows.

Let Φ be one (or a composition) of the three linear preservers just presented. Set $\tilde{\Phi}(X) = \Phi(X)$ for all $X \in$Span (MES), and define $\tilde{\Phi}$ however we like on the orthogonal complement of Span(MES).

Thus we restrict to maps Φ :Span(MES) $\rightarrow$ Span(MES) when searching for preservers of MES.

Main Theorem

Theorem

A linear map $\Phi : Span(MES) \to Span(MES)$ preserves MES if and only if Φ has one of the following forms:

1. $\Phi(A \otimes B) = (U \otimes V)(A \otimes B)^\sigma (U \otimes V)^*$ *for some unitaries* U, V.
2. $\Phi(A \otimes B) = (U \otimes V)(B \otimes A)^\sigma (U \otimes V)^*$ *for some unitaries* U, V.
3. $\Phi(X) = (TrX)\,\rho$ *for some* $\rho \in MES$.

Here the map $A \mapsto A^\sigma$ is either the identity or transpose map.

Outline of proof

Suppose Φ is a linear map preserving MES.

- We may assume that $\Phi(\rho_0) = \rho_0$.
- Reduce redundancy.
- Discern basic linear structure of MES.
- ...

Reduce redundancy

Lemma

Let $U, V, W \in M_n$ be unitaries. Then $\rho_{U,V} = \rho_{I,W}$ if and only if $W = e^{i\phi} V U^t$ for some $\phi \in \mathbb{R}$.

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- Every MES can be expressed as $\rho_{I,W}$ for an appropriate unitary W.
- Since $\rho_{I,V} = \rho_{I,W}$ if and only if $W = e^{i\phi} V$ for some $\phi \in \mathbb{R}$, we have a bijection between U_n/U_1 and MES.

Linear structure

Proposition

Fix $\lambda, \mu \in (0,1)$ *and* $V_1 \in U_n$ *such that* $\rho_{I,V_1} \neq \rho_0$. *Then there exist* $V_2, V_3 \in U_n$ *satisfying*

$$\lambda\rho_0 + (1-\lambda)\rho_1 = \mu\rho_2 + (1-\mu)\rho_3$$

(here $\rho_i = \rho_{I,V_i}$*) if and only if one of the following hold:*

1. $\lambda = \mu$, $\rho_0 = \rho_2$, *and* $\rho_1 = \rho_3$.
2. $\lambda = 1-\mu$, $\rho_0 = \rho_3$, *and* $\rho_1 = \rho_2$.
3. *There are* $\theta, \alpha, \beta, w_1 \in \mathbb{R}$ *and a Hermitian unitary* $H \neq \pm I$ *such that*

$$V_1 = e^{iw_1}((\cos\theta)I + i(\sin\theta)H) \qquad \textit{and}$$

$$\mu e^{i2\alpha} + (1-\mu)e^{i2\beta} = \lambda + (1-\lambda)e^{i2\theta}.$$

- In Case 3, there are w_2, $w_3 \in \mathbb{R}$ such that

$$V_2 = e^{iw_2}((\cos\alpha)I + i(\sin\alpha)H) \quad \text{and}$$
$$V_3 = e^{iw_3}((\cos\beta)I + i(\sin\beta)H).$$

- The equation

$$\lambda\rho_0 + (1-\lambda)\rho_{I,V_1} = \mu\rho_{I,V_2} + (1-\mu)\rho_{I,V_3}$$

has infinitely many solutions (for ρ_{I,V_2} and ρ_{I,V_3}) if and only if $\lambda = \mu = 1/2$ and $V_1 = \xi H$ for some complex unit ξ and some Hermitian unitary H.

Special sets

The structural proposition singles out

$$\mathcal{T}_0 = \{\rho_{I,iH} : H \in U_n \cap H_n\}$$

and

$$\begin{aligned} \mathcal{T} &= \{\rho_{I,xI+iyH} : H \in H_n \cap U_n;\ x, y \in \mathbb{R},\ x^2 + y^2 = 1\} \\ &= \{\rho_{I,U} : U \in U_n \text{ has at most 2 distinct eigenvalues}\} \end{aligned}$$

as special sets which must be preserved by Φ.

Outline of proof

Suppose Φ is a linear map preserving MES.

- We may assume that $\Phi(\rho_0) = \rho_0$.
- Reduce redundancy.
- Discern basic linear structure of MES.

- $\Phi(\mathcal{T}_0) \subseteq \mathcal{T}_0$ and $\Phi(\mathcal{T}) \subseteq \mathcal{T}$. Use structural proposition to show
$$\Phi(\rho_{I,xI+iyH}) = \rho_{I,xI+iyg(H)}$$
for some map $g : H_n \cap U_n \to H_n \cap U_n$.
- Extend g to a linear map on H_n preserving rank one projections.

Φ is now determined on Span($\mathcal{T}$), where

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$\Phi(\rho_{I,xI+iyH}) = \rho_{I,xI+iyg(H)}$ for all Hermitian H, where

1. $g \equiv 0$, or
2. $g(H) = \epsilon UHU^*$ for $\epsilon \in \{-1, 1\}$ and $U \in U_n$, or
3. $g(H) = \epsilon UH^t U^*$ for $\epsilon \in \{-1, 1\}$ and $U \in U_n$.

Comparison

Writing $\Phi(\rho_{I,X}) = \rho_{I,f(X)}$, we have the following correspondences:

Mapping
$f : U_n/U_1 \to U_n/U_1$

1. $f(X) = UX$
2. $f(X) = XV$
3. $f(X) = \overline{X}$
4. $f(X) = X^t$

Linear Preserver
$\Phi : \mathrm{Span}(\mathcal{T}) \to \mathrm{Span}(\mathcal{T})$

1. $\rho \mapsto (I \otimes U)\rho(I \otimes U)^*$
2. $\rho \mapsto (V^t \otimes I)\rho(V^t \otimes I)^*$
3. $A \otimes B \mapsto A^t \otimes B^t$
4. $A \otimes B \mapsto B \otimes A$

Extending beyond Span($\mathcal{T}$)

- May assume $\Phi(\rho) = \rho_0$ for all $\rho \in \mathcal{T}$ (degenerate case), or $\Phi(\rho) = \rho$ for all $\rho \in \mathcal{T}$.

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- For $n > 2$, let

$$\mathcal{T}_+ = \{\rho_{I,U} : U \in U_n \text{ has at most 2 distinct eigenvalues} \\ \text{or is unitarily similar to } \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \oplus I_{n-2}\}.$$

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- Span($\mathcal{T}_+$) = Span(MES)
- Transfer structural proposition to analyze solutions of

$$\lambda\rho_{I,U_0} + (1-\lambda)\rho_{I,U_1} = \mu\rho_{I,U_2} + (1-\mu)\rho_{I,U_3}$$

The End

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- Thank you for your attention!