Transformations on density operators preserving quantum relative entropy or related quantities

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J. von Neumann (1903-1957):

Mathematical description of quantum mechanics. Basic postulates:

A quantum mechanical system is characterized by a Hilbert space H.

The pure states of the system are represented by unit rays $\underline{\varphi}$ ($\leftrightarrow$ one-dimensional subspaces $\leftrightarrow$ rank-one projections) in H.

Observables are represented by self-adjoint operators A on H.

The expectation value of A in the (pure) state $\underline{\varphi}$ is given by $\langle A\varphi, \varphi\rangle$.

Observables are simultaneously measurable iff the corresponding self-adjoint operators commute.

E. Wigner (1902-1995):

Transition probability: The transition probability between unit rays $\underline{\varphi}$ and $\underline{\psi}$ is $|\langle\varphi,\psi\rangle|^2$.

In the language of one-dimensional subspaces: If L, M are one-dimensional subspaces of H, then the transition probability between them is $\cos^2\theta$, where $\theta = \angle(L, M)$.

In the language of rank-one projections: If P, Q are rank-one projections, then the transition probability between them is tr PQ.

Quantum mechanical symmetry transformation: Bijective map on the set of all pure states which preserves the transition probability.

Wigner's theorem (1931)

Wigner's theorem (for subspaces): Let $\mathcal{L}$ be the set of all one-dimensional subspaces of H. If $T : \mathcal{L} \to \mathcal{L}$ is a bijective map which preserves the angle between the elements of $\mathcal{L}$, then there is a unitary or an antiunitary operator U on H such that

$$T(L) = U[L]$$

holds for every $L \in \mathcal{L}$.

Wigner's theorem (for projections): If $\phi : P_1(H) \to P_1(H)$ is a bijective map which satisfies

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in P_1(H)),$$

then there is either a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \qquad (P \in P_1(H)).$$

First mathematical proofs: in the 60's.

We found an algebraic approach to Wigner's theorem. That applied results of Herstein, Jacobson and Rickart concerning the structure of Jordan homomorphisms on rings.

Wigner-type results for several other structures.

E.g.: Wigner-type result for Grassmann spaces (CMP, 2001) and (PAMS, 2008).

Description of all transformations on the set of all subspaces of a Hilbert space with a fixed (finite) dimension which preserve the so-called principle angles between subspaces. Essentially the same conclusion as in Wigner's theorem.

Very recently, characterization of isometries of Grassmann spaces (JFA, 2013).

Motivation to consider transformations on the space of mixed states that preserve important numerical quantities of two variables.

H: Hilbert space

$\mathcal{S}(H)$: The convex set of all density operators (positive operators with unit trace). They represent states. The extreme points of $\mathcal{S}(H)$ are exactly the rank-one projections (pure states).

In what follows we assume that $\dim H < \infty$ (this is sufficient for the purposes of quantum information theory).

von Neumann entropy:

$$S(A) = -\operatorname{tr} A \log A, \quad A \in \mathcal{S}(H).$$

Relative entropies: measures of distinguishability between states

Umegaki relative entropy: For any pair $A, B \in \mathcal{S}(H)$, the Umegaki relative entropy $S(A||B)$ is defined by

$$S(A||B) = \begin{cases} \operatorname{tr}[A(\log A - \log B)], & \text{if } \operatorname{supp} A \subset \operatorname{supp} B; \\ +\infty, & \text{otherwise.} \end{cases}$$

$S(A||B)$ is always nonnegative, and equals zero if and only if $A = B$.

Relative entropy preservers

Theorem

(JMP, 2008). *Let $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ be a bijective map which satisfies*

$$S(\phi(A)\|\phi(B)) = S(A\|B) \qquad (A, B \in \mathcal{S}(H)).$$

Then there exists either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \qquad (A \in \mathcal{S}(H)).$$

Theorem

with P. Szokol (LAA, 2010). *The condition of bijectivity can be relaxed.*

Jensen-Shannon divergence:

$$D_J(A\|B) = \frac{S\left(A\middle\|\frac{1}{2}(A+B)\right) + S\left(B\middle\|\frac{1}{2}(A+B)\right)}{2} \quad (A, B \in \mathcal{S}(H)).$$

A sort of symmetrization of relative entropy.

$\sqrt{D_J}$ is conjectured to be a true metric $\mathcal{S}(H)$. Proved only if $\dim H = 2$.

Theorem (LM and W. Timmermann (JPA, 2009))

Assume that $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ *is a bijection satisfying*

$$D_J(\phi(A)\|\phi(B)) = D_J(A\|B) \quad (A, B \in \mathcal{S}(H)).$$

Then there is a unitary or an antiunitary operator U on H such that

$$\phi(A) = UAU^* \quad (A \in \mathcal{S}(H)).$$

Further concepts of relative entropy: Assume $0 < q < 1$ and let $A, B \in S(H)$.

(i) Umegaki relative entropy: $S(A\|B) = \operatorname{tr}(A(\log A - \log B))$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S(A\|B) = +\infty$ otherwise;

(ii) Belavkin-Staszewski relative entropy: $S_{BS}(A\|B) = \operatorname{tr}(\sqrt{A} \log \sqrt{A} B^{-1} \sqrt{A})$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S_{BS}(A\|B) = +\infty$ otherwise;

(iii) Tsallis relative entropy: $S_T(A\|B) = (1/(1-q))(1 - \operatorname{tr} A^q B^{1-q})$;

(iv) Quadratic relative entropy: $S_Q(A\|B) = \operatorname{tr} A^{-1}(A - B)^2$ if $\operatorname{supp} B \subset \operatorname{supp} A$, and $S_Q(A\|B) = +\infty$ otherwise;

(v) Jensen-Shannon divergence: $D_{JS}(A\|B) = \frac{1}{2}\left(S\left(A\|\frac{A+B}{2}\right) + S\left(B\|\frac{A+B}{2}\right)\right)$.

By the -1-th power of an element of $S(H)$ we mean the inverse of its restriction onto its support.

The method in the proof of the result by LM and P. Szokol (2010) can also be applied for maps preserving any of the relative entropies (ii),(iv).

Theorem

LM and G. Nagy (LAMA, 2012). *Let $X(.\|.)$ denote any of the relative entropies* (ii)–(v). *Suppose that $\phi : S(H) \to S(H)$ is a transformation such that*

$$X(\phi(A)\|\phi(B)) = X(A\|B)$$

holds for all $A, B \in S(H)$. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

Csiszár's f-divergence in classical information theory:

Let $f\colon [0,\infty[\to \mathbb{R}$ be a convex function and let $P = (p_1, \cdots, p_n)$; $Q = (q_1, \cdots, q_n)$ be probabtility distributions. Then the f-divergence between P and Q is

$$D_f(P\|Q) = \sum_{\{i:q_i\neq 0\}} q_i f\left(\frac{p_i}{q_i}\right) + \alpha \sum_{\{i:q_i=0\}} p_i,$$

where $\alpha = \lim_{x\to\infty} \frac{f(x)}{x}$.

Quasi-entropy and f-divergence

Quantum f-divergences: quantum analogue of Csiszár's f-divergence.

Let $A, B \in PD(H)$ be positive definite operators. Define the linear transformations $\mathfrak{L}_A, \mathfrak{R}_B : B(H) \to B(H)$ by

$$\mathfrak{L}_A X = AX, \quad \mathfrak{R}_B X = XB, \qquad X \in B(H).$$

They are commuting positive operators on $B(H)$ equipped with the standard (Hilbert-Schmidt) inner product $\langle X, Y \rangle = \operatorname{tr} XY^*$.

Let $f : [0, \infty[\to \mathbb{R}$ be a function and $K \in B(H)$ an operator. For any positive semi-definite operator A and $B \in PD(H)$, the quantity

$$S_f^K(A||B) = \operatorname{tr} K^* f(\mathfrak{L}_A {\mathfrak{R}_B}^{-1}) \mathfrak{R}_B K$$

is called quasi-entropy introduced by D. Petz.

Important properties of quasi-entropy :
monotonicity (assuming f is operator monotone and $f(0) \geq 0$);
joint convexity (assuming f is operator convex).

If $K = I$, $f(t) = t \log t$, we have

$$S_f^I(A||B) = S_f(A||B) = \operatorname{tr}[A(\log A - \log B)], \qquad A, B \in PD(H),$$

i.e., the Umegaki relative entropy of A, B.

An important particular case of quasi-entropy: f-divergence for states (possibly singular density operators).

Assume f is continuous on the open interval $]0, \infty[$ and the limit

$$\alpha := \lim_{x \to \infty} \frac{f(x)}{x}$$

exists in $[-\infty, \infty]$. Then the limit

$$\lim_{\epsilon \searrow 0} S_f(A||B + \epsilon I)$$

exists and is called the f-divergence of A and B. The following formula can be derived.

Let $A, B \in \mathcal{S}(H)$ and for any $\lambda \in \mathbb{R}$ denote by P_λ, respectively by Q_λ the projection on H projecting onto the kernel of $A - \lambda I$, respectively onto the kernel of $B - \lambda I$. The f-divergence of A and B is

$$S_f(A||B) = \sum_{a \in \sigma(A)} \left(\sum_{b \in \sigma(B) \setminus \{0\}} bf\left(\frac{a}{b}\right) \operatorname{tr} P_a Q_b + \alpha a \operatorname{tr} P_a Q_0 \right).$$

Quasi-entropy and f-divergence

1. If $f(x) = x \log x$ $(x > 0)$ and $f(0) = 0$, then

$$S_f(A\|B) = S(A\|B) = \begin{cases} \operatorname{tr} A(\log A - \log B), & \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise} \end{cases}$$

   which is the standard Umegaki relative entropy of A w.r.t. B.

2. Let $q \in]0,1[$ and define the functions $f_q \colon [0,\infty[\to \mathbb{R}$ by $f_q(x) = (1 - x^q)/(1 - q)$ $(x \geq 0)$. Then

$$S_{f_q}(A\|B) = \frac{1 - \operatorname{tr} A^q B^{1-q}}{1 - q}$$

   is the quantum Tsallis relative entropy of A w.r.t. B.

3. If $f(x) = (\sqrt{x} - 1)^2$ $(x \geq 0)$, then $S_f(A\|B) = \|\sqrt{A} - \sqrt{B}\|_{HS}^2$, where $\|.\|_{HS}$ stands for the Hilbert-Schmidt norm (which corresponds to the Frobenius norm of matrices).

Quasi-entropy and f-divergence

Theorem

LM, G. Nagy and P. Szokol (QINP, 2013). *Assume that $f : [0, \infty[\to \mathbb{R}$ is a strictly convex function and $\phi \colon \mathcal{S}(H) \to \mathcal{S}(H)$ is a transformation satisfying*

$$S_f(\phi(A)||\phi(B)) = S_f(A||B) \quad (A, B \in \mathcal{S}(H)).$$

Then there is either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \qquad (A \in \mathcal{S}).$$

A really new result:

Theorem

Let $K, L \in B(H)$ be invertible, $f(t) = t \log t$. Assume $\phi : PD(H) \to PD(H)$ is a bijective map satisfying

$$S_f^L(\phi(A)||\phi(B)) = S_f^K(A, B), \quad A, B \in PD(H).$$

Then we have a unitary or antiunitary operator U on H and a positive scalar λ such that

$$\phi(A) = \lambda UAU^*, \qquad A \in PD(H)$$

and $L = (1/\sqrt{\lambda})UKU^*$.

The Holevo bound:
Let $(\lambda_1, \ldots, \lambda_m)$ be a probability distribution. The Holevo bound (or information) of a collection (or ensemble) $(A_1, \ldots, A_m)$ of states is

$$\chi(A_1, \ldots, A_m) = S\left(\sum_{k=1}^{m} \lambda_k A_k\right) - \sum_{k=1}^{m} \lambda_k S(A_k).$$

This quantity plays an important role in quantum communication. According to a fundamental result of Holevo, this provides an upper bound for the information that can be sent over a quantum channel.

The Holevo bound is nonnegative and equals 0 iff $A_1 = \ldots = A_m$. Moreover

$$\chi(A_1, \ldots, A_m) = \sum_{k=1}^{m} \lambda_k S\left(A_k \middle\| \sum_{l=1}^{m} \lambda_l A_l\right).$$

Observe that if $\lambda_1 = \lambda_2 = \frac{1}{2}$, we get the Jensen-Shannon divergence:

$$D_J(A\|B) = \frac{S\left(A \middle\| \frac{1}{2}(A+B)\right) + S\left(B \middle\| \frac{1}{2}(A+B)\right)}{2} \quad (A, B \in \mathcal{S}(H)).$$

Holevo bound

Theorem (LM and G. Nagy (IJTP, 2014))

Let $(\lambda_1, \dots, \lambda_m)$ *be a probability distribution of positive numbers. Assume that* $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ *is a map satisfying*

$$\chi(\phi(A_1), \dots, \phi(A_m)) = \chi(A_1, \dots, A_m) \quad (A_1, \dots, A_m \in \mathcal{S}(H)). \tag{1}$$

Then there is a unitary-antiunitary operator U *on* H *such that*

$$\phi(A) = UAU^* \quad (A \in \mathcal{S}(H)).$$

Equation (1) means that ϕ preserves the Holevo bound only for a fixed probability distribution not for all.

Consequence: structure of transformations on density operators which preserve a fixed convex combination of *n*-tuples.

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