

INVERTIBILITY PRESERVERS

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PŠ, J. Algebra 408 (2014), 42-60.

LR & PŠ, LAA 433 (2010), 2257-2268.

A, B unital algebras

$\phi: A \rightarrow B$ linear

$a \in A$ inv. $\Rightarrow \phi(a) \in B$ inv.

WLOG: $\phi(1) = 1$

$x \mapsto \phi(1)^{-1} \phi(x)$

1959: Marcus & Purves: $M_n(\mathbb{C})$

1967, 8: GKZ

- automorphism (inner-automorphism)

$$X \mapsto aXa^{-1}$$

- anti-automorphism (inner anti-automorphism)

$$X \mapsto aX^t a^{-1}$$

- direct sum: $\mathcal{Y}_1 \oplus \mathcal{Y}_2$

- Jordan homomorphism: $\phi(x^2) = \phi(x)^2$

- Kaplansky (FA setting)

- Aupetit : $\frac{1}{2}$ -simple algebras (*)

bijjective maps (**)

\Downarrow

Jord. isom ϕ

(*) : T_n

(**) : $\mathcal{B}(H) \cong \mathcal{B}(H \oplus H)$

$$A \mapsto \begin{bmatrix} A & * \\ 0 & A^{tr} \end{bmatrix}$$

- Sourour : $\mathcal{B}(X)$

- Anquet : VN algebras

$$\phi: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \cong \mathbb{C}$$

$n = 2, 4, 8$ real case

$$\phi: M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$$

$$A \mapsto TAT^{-1}$$

$$A \mapsto TA^t T^{-1}$$

$$\mathcal{Y} \subset M_n(\mathbb{F})$$

full non-singular subspace

- $\dim \mathcal{Y} = n$

- $A \in \mathcal{Y} \setminus \{0\} \Rightarrow A \text{ inv.}$

$$\psi: \mathbb{F}^n \rightarrow \mathcal{Y} \quad \text{bij, lin}$$

$$\phi: M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$$

$$\begin{aligned} \phi(A) &= \psi(Ax_0) \\ &= \psi(A^h x_0) \end{aligned}$$

CDSP, LAA 433 (2010)

L2 & P3

LOCALIZATION TECHNIQUE

$$\phi: M_n \rightarrow M_n$$

$$\phi(A) = T A S$$

$$x_0: A \mapsto \phi(A)x_0 = \psi_{x_0}(A)$$

$$M_n \rightarrow \mathbb{F}^n$$

$$\psi_{x_0}(A) = T A u \quad (u = S x_0)$$

$$\phi: M_n \rightarrow M_n$$

Each localization $(A \mapsto \phi(A)_{x_0})$

nice:

$$\varphi_{x_0}(A)_z \begin{cases} TA_u \\ TA^k_u \end{cases}$$

$\Downarrow ?$

ϕ nice

LR & PŠ

$$\phi: M_n \rightarrow M_n \text{ linear}$$

inv. preserver

$$X_0: \Psi_{X_0}(A) = \phi(A)X_0$$

$$\text{property: } A \text{ inv.} \Rightarrow \Psi_{X_0}(A) \neq 0$$

$$\Psi_{X_0}: M_n \rightarrow \mathbb{F}^n$$

$$\dim \text{Ker } \Psi_{X_0} \geq n(n-1)$$

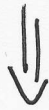
$$A \in \text{Ker } \Psi_{X_0} \Rightarrow A \text{ Sing.}$$



$$\text{Ker } \varphi_{x_0} = \mathcal{N}$$

$$\begin{bmatrix} 0 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \\ 0 & 0 & \dots & 0 \end{bmatrix}$$



φ_{x_0} "nice"

- Central simple algebras

- $M_n(D)$, $\dim_{\mathbb{F}} D < \infty$

- Determinant

- Optimal result