INVERTIBILITY PRESERVERS

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PŠ, J. Algebra 408 (2014), 42-60.

LR & PŠ, LAA 433 (2010), 2257-2268.

$\mathcal{A}, \mathcal{B}$ unital algebras

$\phi : \mathcal{A} \to \mathcal{B}$ linear

$a \in \mathcal{A}$ inv. $\Longrightarrow \phi(a) \in \mathcal{B}$ inv.

WLOG: $\phi(1) = 1$

$x \mapsto \phi(1)^{-1} \phi(x)$

1959: Marcus & Purves: $M_n(\mathbb{C})$

1967,8: GKŻ

- automφ (inner-automφ)

$$x \mapsto axa^{-1}$$

- anti-automφ (inner anti-automφ)

$$x \mapsto ax^{tr}a^{-1}$$

- direct sum : $\varphi_1 \oplus \varphi_2$

- Jordan homφ : $\phi(x^2) = \phi(x)^2$

- Kaplansky (FA setting)
- Aupetit : $\frac{1}{2}$-simple algebras $(*)$

  bijective maps $(**)$

  $\Downarrow$

  Jord. isom^y

$(*)$: T_n

$(**)$: $\mathcal{B}(H) \equiv \mathcal{B}(H \oplus H)$

$$A \mapsto \begin{bmatrix} A & * \\ 0 & A^{tr} \end{bmatrix}$$

- Sourour : $\mathcal{B}(X)$

- Aupetit : VN algebras

$$\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \cong \mathbb{C}$$

$n = 2, 4, 8$ real case

$$\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$$

$$A \mapsto TAT^{-1}$$

$$A \mapsto TA^{tr}T^{-1}$$

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$\mathcal{Y} \subset M_n(\mathbb{F})$

full non-singular subspace

- $\dim \mathcal{Y} = n$
- $A \in \mathcal{Y} \setminus \{0\} \Rightarrow A$ inv.

$$\varphi : \mathbb{F}^n \to \mathcal{S} \quad \text{bij, lin}$$

$$\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$$

$$\Phi(A) = \varphi(Ax_0)$$

$$= \varphi(A^{tr} x_0)$$

CDSP, LAA 433 (2010)

L2&PS

LOCALIZATION TECHNIQUE

$$\phi : M_n \to M_n$$

$$\phi(A) = TAS$$

x_0 : $\quad A \mapsto \phi(A) x_0 = \varphi_{x_0}(A)$

$$M_n \longrightarrow \mathbb{F}^n$$

$$\varphi_{x_0}(A) = TAu \qquad (u = Sx_0)$$

$\phi : M_n \to M_n$

Each localization $(A \mapsto \phi(A)_{x_0})$

nice :

$$\varphi_{x_0}(A) = \begin{cases} TAu \\ TA^t u \end{cases}$$

$\Downarrow$?

ϕ nice

LR & PŠ

$\phi : M_n \to M_n$ linear

inv. preserver

x_0: $\varphi_{x_0}(A) = \phi(A)x_0$

property: A inv. $\Rightarrow \varphi_{x_0}(A) \neq 0$

$\varphi_{x_0} : M_n \to \mathbb{F}^n$

$\dim \operatorname{Ker} \varphi_{x_0} \geq n(n-1)$

$A \in \operatorname{Ker} \varphi_{x_0} \Rightarrow A$ sing.

$$\Downarrow$$

$$\operatorname{Ker} \varphi_{x_0} = \sim$$

$$\begin{bmatrix} 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\Downarrow$$

φ_{x_0} "nice"

- Central simple algebras
- $M_n(\mathbb{D})$, $\dim_{\mathbb{F}} \mathbb{D} < \infty$

- determinant
- Optimal result