Preserver problems on convex combinations of positive elements*

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Outline

1. Introduction
2. Preservers on positive semi-definite operators in Schatten-p class
3. Preservers on positive definite matrices in M_n

Linear isometries on M_n

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Definition: Let X, Y be normed linear spaces w.r.t. $\|\cdot\|_X$ and $\|\cdot\|_Y$, resp.. A map $\phi : X \to Y$ is an isometry (or distance preserving) if for any $a, b \in X$,

$$\|\Phi(a) - \Phi(b)\|_Y = \|a - b\|_X.$$

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Theorem (Schur, 1925)

Let $\Phi : M_n \to M_n$ be a linear isometry w.r.t. operator norm. Then $\exists$ unitary matrices U and V s.t. Φ has the following standard form:

$$\Phi(A) = UAV \qquad (\forall\, A \in M_n) \text{ or } \qquad \Phi(A) = UA^TV \qquad \forall\, A \in M_n.$$

Surjective isometries on normed linear space

Theorem (Mazur-Ulam, 1932)

Let X_1, X_2 be normed linear spaces and $\Phi : X_1 \to X_2$ be a surjective isometry. Then Φ is affine, i.e.,

$$\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y \in X_1, 0 \le t \le 1.$$

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Question:

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Supp. $\Phi : X \to Y$ is surjective and preserves norm of convex combinations, i.e.,

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in X,\ 0 \leq t \leq 1.$$

Then what can we say ?

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Then what can we say ?

Note: Supp. X, Y: linear space. Then it implies that Φ: isometric.

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$\mathcal{F}(H)$: finite rank operators on H.
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$\mathcal{S}_p(H) = \{A \in \mathcal{K}(H) : \operatorname{tr}|A|^p < +\infty\}$: Schatten-$p$ class operators with $\|A\|_p = (\operatorname{tr}\ |A|^p)^{1/p}$, where $|A| = \sqrt{A^*A}$ for $1 \leq p < +\infty$.

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with trace class norm $\|\cdot\|_1$.
(2) $p = 2$, $\mathcal{S}_2(H)$: Hilbert-Schmidt class operators
with Hilbert-Schmidt norm $\|\cdot\|_2$ (or Frobenius norm).
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$\mathcal{S}_p^+(H) = \{\rho \geq 0 : \rho \in \mathcal{S}_p(H)\}$: positive operators in $\mathcal{S}_p(H)$ $(\langle \rho h, h\rangle \geq 0)$
$\mathcal{S}_p^+(H)_1 = \{\|\rho\|_p = 1 : \rho \in \mathcal{S}_p^+(H)\}$: unit p-norm in $\mathcal{S}_p^+(H)$
$P_1(H)$: the set of rank one projections in $\mathcal{S}_p^+(H)$

Linear isometries on $\mathcal{T}(H)$, $\mathcal{S}_p(H)$, $\mathcal{S}_p^+(H)_1$

Theorem (Russo, 1969)

Let $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ be linear surjective isometry.
Then $\exists$ U, V: unitaries on H, s.t. Φ has the following form

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Let $\Phi : \mathcal{S}_p(H) \to \mathcal{S}_p(H)$ be linear surjective isometry, $1 \leq p \leq +\infty$, $p \neq 2$. Then $\exists$ U, V: unitaries on H, s.t. Φ has the following form

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Theorem (Molnár 2007, Nagy 2013)

For $1 \le p < +\infty$. Supp. $\Phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$: isometry w.r.t. $\|\cdot\|_p$, which will be assume to be surjective when $\dim H = +\infty$.

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Then $\exists\ U$: unitary on H, s.t. Φ has the following form

$$\Phi(A) = UAU^* \qquad \text{or} \qquad \Phi(A) = UA^T U^* \qquad \forall A \in \mathcal{S}_p^+(H)_1.$$

Theorem (Kuo, Tsai, Wong, Zhang, 2014)

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$\Leftrightarrow$

$\exists\, U$: unitary on H s.t. $\forall \rho \in \mathcal{S}_p^+(H)$

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Main ideal:

1. $\|\cdot\|_p$ is Frechet differentiable.
2. Wigner Theorem on $P_1(H)$.

Generalization on $\mathcal{S}_p^+(H)$

Theorem (Nagy, 2014)

Let $1 < p < +\infty$ and α, $\beta \in \mathbb{R} \setminus \{0\}$. Supp. a map $\Phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$ with $\|\alpha\Phi(\rho) + \beta\Phi(\sigma)\|_p = \|\alpha\rho + \beta\sigma\|_p \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)$.

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$\exists\ U$: unitary on H, s.t. $\Phi(\rho) = U\rho U^*$ or $\Phi(\rho) = U\rho^T U^*$;

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(a) $\alpha + \beta \neq 0$, $\exists\ U$: unitary on H, s.t. $\Phi(\rho) = U\rho U^*$ or $\Phi(\rho) = U\rho^T U^*$;

(b) $\alpha + \beta = 0$, $\exists\ U$: unitary on H, and $X \in \mathcal{S}_p^+(H)$,
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Lemma: Fix $\alpha, \beta \in \mathbb{R}$. Supp. $\rho = \rho^*, \sigma = \sigma^* \in \mathcal{S}_p(H)$ with $\|\rho\| = \|\sigma\|$.
Then $\rho = \sigma \Leftrightarrow \|\alpha\rho + \beta P\|_p = \|\alpha\sigma + \beta P\|_p \qquad \forall\, P \in P_1(H)$.

Generalization on $\mathcal{S}_p^+(H)$

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Lemma: Fix $\alpha, \beta \in \mathbb{R}$. Supp. $\rho = \rho^*, \sigma = \sigma^* \in \mathcal{S}_p(H)$ with $\|\rho\| = \|\sigma\|$.
Then $\rho = \sigma \Leftrightarrow \|\alpha\rho + \beta P\|_p = \|\alpha\sigma + \beta P\|_p \quad \forall\, P \in P_1(H)$.

Remark: Fix arbitrary a $P \in P_1(H)$, $S = S^*$ on H and $t \in \mathbb{R}$, consider $P(t) = e^{itS}Pe^{-itS} \in P_1(H)$.

Riemannian structure

Recall:

$B(H)^+ = \{A \geq 0 : A \in B(H)\}$, $B(H)^+_{-1} = \{A \in B(H)^+ : A$ is inv. $\}$.

$B(H) = M_n$, $H_n = \{A = A^* : A \in M_n\}$, $P_n = \{A > 0 : A \in M_n\}$.

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Question:
Supp. $\Phi : P_n \to P_n$ preserves norm of convex combinations, i.e.,

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in P_n, \ 0 \leq t \leq 1.$$

Then what does this mean in Riemannian geometry and what can we say ?

Riemannian metric

(F. Hiai and D. Petz; 2009) For any $D \in P_n$: Riemannian manifold, the tangent space at D can be identified with H_n.

A Riemannian metric $K_D : H_n \times H_n \to [0, \infty)$ is a family of inner products on H_n depending smoothly on D.

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A Riemannian metric $K_D : H_n \times H_n \to [0, \infty)$ is a family of inner products on H_n depending smoothly on D.

If $\psi(x, y)$ is a positive kernel function on $(0, \infty) \times (0, \infty)$ and D has the spectral decomposition $\sum_{i=1}^{n} \lambda_i P_i$, then a Riemannian metric K^{ψ} can be defined as

$$K_D^{\psi}(H, K) := \sum_{i,j=1}^{n} \psi(\lambda_i, \lambda_j)^{-1} \operatorname{tr} \, P_i H P_j K, \qquad \forall \, D \in P_n,$$

where H, K: tangent vectors in H_n.

geodesic

Supp. $\rho : [0, 1] \to P_n$ is a differential curve (or a continuous and piecewise differential curve), the length of ρ w.r.t. the metric K^ψ is given by

$$L(\rho) := \int_0^1 \sqrt{K^\psi_{\rho(t)}(\rho'(t), \rho'(t))}\, dt.$$

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The geodesic distance $\delta(A,B)$ between $A, B \in P_n$ is defined as

$$\delta(A,B) = \inf\{L(\rho) \mid \rho \text{ is a differentiable path from A to B}\}.$$

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$$\delta(A,B) = \inf\{L(\rho) | \ \rho \text{ is a differentiable path from A to B}\}.$$

We know that this infimum is attained by a path uniquely determined by A and B. A geodesic joining two given points A and B, is a curve γ from A to B such that $L(\gamma) = \delta(A,B)$.

$\psi_1(x,y) = 1,\ \psi_2(x,y) = \left(\dfrac{x-y}{\log x - \log y}\right)^2$

Ex: the kernel function $\psi_1(x,y) = 1$. For any $D = \sum_{i=1}^{n} \lambda_i P_i \in P_n$, then the Riemannian metric K^{ψ_1} is the Hilbert-Schmidt inner product

$K_D^{\psi_1}(H,K) = \sum_{i,j=1}^{n} \operatorname{tr} P_i H P_j K = \operatorname{tr} H^* K = \langle H, K\rangle_{\text{HS}} \qquad \forall\, H, K \in H_n.$

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In this case, the unique geodesic joining $A, B \in P_n$ is the segment

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The midpoint of the geodesic is the geometric mean of A, B,

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

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Proposition:

(1) $A\#B = B\#A$, $(A\#B)^T = A^T\#B^T$, $(A\#B)^{-1} = A^{-1}\#B^{-1}$.

(2) $\forall\ S$: inv. bdd. linear or conjugate-linear on H,
we have $S(A\#B)S^* = (SAS^*)\#(SBS^*)$.

(3) $A\#B$ is the unique positive solution of $B = X^*A^{-1}X$.

(4) $A\#B = \max\{X \geq 0 : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\}, \qquad A, B \in P_n$.

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Note: Let $A > 0$. Then $\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \Leftrightarrow B \geq XA^{-1}X$

$\Rightarrow A^{-1/2}BA^{-1/2} \geq (A^{-1/2}XA^{-1/2})^2 \Rightarrow A\#B \geq X$.

preserve $\|\cdot\|_P$ of geodesics $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$

Theorem (Szokol, Tsai, Zhang)

Let $\Phi : P_n \to P_n$ be a bijective transformation defined on the different Riemannian metrics K^{ψ_1}, K^{ψ_2}, K^{ψ_3}. Then T.F.A.E.

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(1) For $p > 1$. Φ preserves $\|\cdot\|_P$ of geodesics under (P_n, K^{ψ_1}), i.e.,
$$\|((1-t)A + tB)\|_p = \|((1-t)\Phi(A) + t\Phi(B))\|_p, \forall\, 0 \le t \le 1, A, B \in P_n.$$

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Main tools

Lemma

Supp. $X(t) \in P_n$ for each t and conti. diff. w.r.t. t, then

$$\frac{d}{dt}\operatorname{tr}[X(t)^p] = p\operatorname{tr}[X(t)^{p-1}\frac{d}{dt}X(t)], \qquad p \geq 1.$$

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Let $\Phi : B(H)^+_{-1} \to B(H)^+_{-1}$ be a bijective map.

(1) Supp. $A \leq B \Leftrightarrow \Phi(A) \leq \Phi(B) \qquad \forall\, A, B \in B(H)^+_{-1}$.

Then $\exists\, R$: inv. bdd. linear or conjugate-linear on H, s.t.

$$\Phi(A) = RAR^*, \qquad \forall\, A \in B(H)^+_{-1}.$$

(2) Supp. $\log A \leq \log B \Leftrightarrow \log\Phi(A) \leq \log\Phi(B) \qquad \forall\, A, B \in B(H)^+_{-1}$.

Then $\exists\, S$: inv. bdd. linear or conjugate-linear, $X = X^*$ on H, s.t.

$$\Phi(A) = e^{S(\log A)S^* + X}, \qquad \forall\, A \in B(H)^+_{-1}.$$

Connection between geodesic and entropy

Recall: $M(\mathbb{C}^n) = \{A > 0 : A \in M_n \ \text{tr}\, A = 1\}$, $\gamma_2(t) = e^{(1-t)\log A + t\log B}$.
$\gamma_3(t) = A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$, $0 \le t \le 1$

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(1) The Umegaki relative entropy $S_U(A||B)$ is defined by

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$\therefore$ a map ϕ preserves the trace norm of $\gamma_3(t)$

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Theorem (Szokol, Tsai, Zhang)

Supp. the Riemannian metric on P_n $(n \geq 3)$ is defined by the kernel function $\psi_3(x, y) = xy$. Let $\Phi : P_n \to P_n$ be a bijective continuous map. Then for any $A, B \in P_n$, T.F.A.E.

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(1) Φ maps geodesic joining A, B onto geodesic joining $\Phi(A)$, $\Phi(B)$,
i.e. $\Phi(A\#_t B) = \Phi(A)\#_t\Phi(B)$;

(2) Φ preserves the geometric mean of A and B,
i.e. $\Phi(A\#B) = \Phi(A)\#\Phi(B)$;

(3) $\exists\, S$: inv. bdd. linear on $\mathbb{C}^n$ s.t. Φ is of one of the forms
$\Phi(A) = (\det A)^c SAS^*$, $\Phi(A) = (\det A)^c SA^T S^*$, $c \in \mathbb{R}$, $c \ne -\frac{1}{n}$
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Supp. (P_n, K^{ψ_2}) is a Riemannian manifold with Riemannian metric defined by the kernel function $\psi_2 = \left(\frac{x-y}{\log x - \log y}\right)^2$.

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Then $\Phi : P_n \to P_n$ satisfying $\Phi(\gamma^{\psi_2}_{A,B}(t)) = \gamma^{\psi_2}_{\Phi(A),\Phi(B)}(t)$,

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$\Leftrightarrow\ \exists\, \delta_i \in \mathbb{R}, M_i \in B(H), i = 1, \ldots, k \;\&\; N = N^*$ s.t.

$\Phi(A) = e^{\left(\sum_{i=1}^k \delta_i M_i (\log A) M_i^* + N\right)}, \qquad \forall\, A \in P_n.$

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Remark: Supp. Riemannian metric on P_n is defined by the kernel function $\psi(x,y) = 1$. Then the map satisfying that

$\Phi(\gamma_1(t)) = \Phi((1-t)A + tB) = (1-t)\Phi(A) + t\Phi(B)$: affine

Problem

Recall

Theorem: Let $\Phi : P_n \to P_n$ be a bijective transformation defined on the different Riemannian metrics K^{ψ_i}. Then Φ preserves $\| \cdot \|_P$ of geodesics $\gamma_{A,B}$ under (P_n, K^{ψ_i}), i.e.,

$\|\phi(\gamma^{\psi_i}_{A,B})(t)\|_p = \|\gamma^{\psi_i}_{\phi(A),\phi(B)}(t)\|_p, \forall\, 0 \le t \le 1, A, B \in P_n$., for some i

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(2) " $\forall\ 0 \le t \le 1$ " is replaced with " for some $0 < t < 1$ " ?

(3) $\|\cdot\|_p$ is replaced with unitarily invariant norm on $\mathcal{S}^+_p(H)$ or P_n?

(**Note.** $\|\cdot\|$: unitarily invariant norm if $\|UAV\| = \|A\|, \forall$ unitaries U, V)

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Thanks for your attention !

preserve length of differentiable path in (P_n, K^{ψ_3})

Theorem (Szokol, Tsai, Zhang)

Supp. (P_n, K^{ψ_3}) is a Riemannian manifold with Riemannian metric defined by the kernel function $\psi_3(x,y) = xy$. Let $\Phi : P_n \to P_n$ be a bijective transformation. Then Φ preserves the length of all differentiable paths

$\Leftrightarrow$

(1) $n = 2$: Φ is of one of the forms

$$\Phi(A) = SAS^*,\ SA^TS^*,\ SA^{-1}S^*, S(A^T)^{-1}S^*.$$

(2) $n \geq 3$: Φ is of one of above forms or of below forms

$$\Phi(A) = (\det A)^{-\frac{2}{n}}SAS^*, (\det A)^{-\frac{2}{n}}SA^TS^*, (\det A)^{\frac{2}{n}}SA^{-1}S^*, (\det A)^{\frac{2}{n}}S(A^T)^{-1}S^*$$

for all $A \in P_n$. Here, S: inv. in M_n.

$\psi_{1,\alpha}(x,y) = \left(\alpha \frac{x-y}{x^\alpha - y^\alpha}\right)^2$, $\psi_2(x,y) = \left(\frac{x-y}{\log x - \log y}\right)^2$

Ex: the kernel function $\psi_{1,\alpha}(x,y) = \left(\alpha \frac{x-y}{x^\alpha - y^\alpha}\right)^2$ with $\alpha \neq 0$.

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$$\gamma_{1,\alpha}(t) = \left((1-t)A^\alpha + tB^\alpha\right)^{\frac{1}{\alpha}}, \quad 0 \leq t \leq 1 \quad \text{if } \alpha \neq 0.$$

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$$\gamma_2(t) = e^{(1-t)\log A + t\log B}, \qquad 0 \leq t \leq 1.$$

$\psi_{3,\kappa}(x,y) = \left(\kappa(xy)^{\frac{\kappa}{2}}\frac{x-y}{x^\kappa - y^\kappa}\right)^2$

Ex. the kernel function $\psi_{3,\kappa}(x,y) = \left(\kappa(xy)^{\frac{\kappa}{2}}\frac{x-y}{x^\kappa - y^\kappa}\right)^2$ for $\kappa > 0$.

For every $A, B \in P_n$, $\exists$ a unique geodesic from A to B given by

$$\gamma_{3,\kappa}(t) = (A^\kappa \#_t B^\kappa)^{1/\kappa} := (A^{\kappa/2}(A^{-\kappa/2}B^\kappa A^{-\kappa/2})^t A^{\kappa/2})^{1/\kappa},\ 0 \le t \le 1,\ \kappa > 0.$$

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Special case:
the kernel function $\psi_3(x,y) = xy$ for $\kappa = 1$. It is related to the geometric mean of x, y. In this case, the unique geodesic joining $A, B \in P_n$ is given by

$$\gamma_3(t) = A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \qquad 0 \le t \le 1.$$

The midpoint of the geodesic is just the geometric mean of A, B,

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Theorem (Szokol, Tsai, Zhang)

Let $\Phi : P_n \to P_n$ be a bijective transformation defined on the different Riemannian metrics $K^{\psi_{1,\alpha}}$, K^{ψ_2}, $K^{\psi_{3,\kappa}}$. Then T.F.A.E.

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(1) For $p \geq 1, \alpha > 0$, $\alpha \neq p$. Φ preserves $\|\cdot\|_P$ of geodesics under $(P_n, K^{\psi_{1,\alpha}})$, i.e., for all $0 \leq t \leq 1, A, B \in P_n$, $\|((1-t)A^{\alpha} + tB^{\alpha})^{\frac{1}{\alpha}}\|_p = \|((1-t)\Phi(A)^{\alpha} + t\Phi(B)^{\alpha})^{\frac{1}{\alpha}}\|_p$.

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(1) For $p \geq 1, \alpha > 0$, $\alpha \neq p$. Φ preserves $\|\cdot\|_P$ of geodesics under $(P_n, K^{\psi_{1,\alpha}})$, i.e., for all $0 \leq t \leq 1, A, B \in P_n$, $\|((1-t)A^\alpha + tB^\alpha)^{\frac{1}{\alpha}}\|_p = \|((1-t)\Phi(A)^\alpha + t\Phi(B)^\alpha)^{\frac{1}{\alpha}}\|_p$.

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$\Phi(A) = UAU^*, \quad or \quad \Phi(A) = UA^TU^*, \qquad \forall\, A \in P_n$.

length of differentiable path in (P_n, K^{ψ_3})

Recall:
(P_n, K^{ψ_3}): a Riemannian manifold with Riemannian metric $K_D^{\psi}(H,K) := \operatorname{tr} D^{-1}HD^{-1}K$, where the kernel function $\psi_3(x,y) = xy$.
Supp. $\rho\colon [0,1] \to P_n$ is a differentiable path joining A, B, i.e. $\rho(0) = A, \rho(1) = B$, then the length of ρ can be defined as

$$L(\rho) = \int_0^1 \sqrt{K_{\rho(t)}^{\psi_2}(\rho'(t), \rho'(t))}\, dt = \int_0^1 \|\rho^{-\frac{1}{2}}(t)\rho'(t)\rho^{-\frac{1}{2}}(t)\|_{\text{HS}}\, dt.$$

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Definition:

A map $\Phi : P_n \to P_n$ preserving the length of all differentiable paths means that given any differentiable path ρ joining A, B, the composition $\Phi \circ \rho$ is a path joining $\Phi(A), \Phi(B)$, for which $L(\Phi \circ \rho) = L(\rho)$. That is,

$$\int_0^1 \|(\Phi\circ\rho)^{-\frac{1}{2}}(t)(\Phi\circ\rho)'(t)(\Phi\circ\rho)^{-\frac{1}{2}}(t)\|_{\mathrm{HS}}\, dt = \int_0^1 \|\rho^{-\frac{1}{2}}(t)\rho'(t)\rho^{-\frac{1}{2}}(t)\|_{\mathrm{HS}}\, dt$$

for any $A, B \in P_n$ and differentiable path ρ joining A, B.

preserve geodesic $\gamma_{3,\kappa}(t)$

Recall:

$\gamma_{3,\kappa}(t) = (A^\kappa \#_t B^\kappa)^{1/\kappa} = (A^{\kappa/2}(A^{-\kappa/2}B^\kappa A^{-\kappa/2})^t A^{\kappa/2})^{1/\kappa}, 0 \le t \le 1, \kappa > 0.$

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Corollary

For $n \ge 3$. Let $\Phi : P_n \to P_n$ be a continuous bijective map. Then

$$\Phi((A^\kappa \#_t B^\kappa)^{1/\kappa}) = (\Phi(A)^\kappa \#_t \Phi(B)^\kappa)^{1/\kappa} \qquad \forall\, t \in [0,1],\ A, B \in P_n.$$

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$\Leftrightarrow$

$\exists\, S$: inv. bdd. linear on $\mathbb{C}^n$ s.t. Φ is of one of the forms

$\Phi(A) = (\det A)^c (SA^\kappa S^*)^{1/\kappa}$, $\Phi(A) = (\det A)^c (S(A^T)^\kappa S^*)^{1/\kappa}$, $c \in \mathbb{R} \setminus \frac{-1}{n}$

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continuity condition

Remark: In above, condition of "**continuity**" cannot be omitted.

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Example:

Let $f:]0,\infty[\to]0,\infty[$ be a multiplicative, non-continuous function, then s.t. Φ is of one of the forms

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Theorem (Kuo, Tsai, Wong, Zhang, 2014)

For $1 < p < +\infty$. Supp. $\Phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$, which will be assumed to be surjective when $\dim\ H = +\infty$. Then T.F.A.E. (the following are equivalent).

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Differentiable

Lemma (Abatzoglou, 1979)

$\forall \rho, \sigma \in \mathcal{S}_p(H)$, $\rho \neq 0$ with $\rho = U|\rho|$: polar decomposition, the norm of $\mathcal{S}_p(H)$ is Fréchet differentiable at ρ and

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Theorem (Wigner, 1931)

Let $\Phi : P_1(H) \to P_1(H)$ be a (resp., bijective) map satisfying

$$\mathrm{tr}\,\Phi(P)\Phi(Q) = trPQ \qquad (P, Q \in P_1(H))$$

$\Leftrightarrow \exists\, U$: linear isometry (resp., unitary) on H s.t. $\Phi(P) = UPU^*$, $P \in P_1(H)$.

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Recall: Theorem (Mazur-Ulam, 1932)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and $\Phi : X \to Y$ be surjective isometry, i.e.,

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Then $\quad \Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y \in X,\ 0 \le t \le 1.$

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For $1 < p < +\infty$. Supp. $\Phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$ (resp., $\Phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$), which will be assumed to be surjective when $\dim\ H = +\infty$. Then T.F.A.E. (the following are equivalent).

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Generalization on $\mathcal{S}_p^+(H)_1$

Theorem (Nagy, 2014)

Let $1 < p < +\infty$ and nonzero $\alpha, \beta \in \mathbb{R}$. Supp. $\Phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ is a map satisfying

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proof by hermitian preserving, geodesic $\gamma_1(t)$

Proof: Define $\psi : H_n \to H_n$ by $\psi(T) = \log \Phi(e^T)$ for $T \in H_n$. Then
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Theorem (Hill, 1973)

Let $\Phi : M_n \to M_n$ be a linear map. Then $\Phi(H_n) \subseteq H_n$
$\Leftrightarrow$
Φ is of the form

$$\Phi(A) = \sum_{j=1}^{r} \epsilon_j V_j^* A V_j, \qquad \text{for some } \epsilon_j = \pm 1, V_1, \ldots, V_r \in M_n.$$

Counterexample

Let $\{e_i\}_{i=1}^n$: an o.n.b.(orthonormal basis) of $\mathbb{C}^n$ and $1 \le p < +\infty$.
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Let $\rho_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\rho_2 = I_2$. Then $\Phi(\rho_1) = 2^{1-(1/p)} I_2$, $\Phi(\rho_2) = I_2$.

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(2) $p = 1$: In fact, $\Phi(\rho) = \sum_{i=1}^n P_i\rho P_i$. Hence $\forall \rho, \sigma \in \mathcal{S}_1^+(H)$, $t \in [0,1]$
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Let $\{e_i\}_{i=1}^n$: an o.n.b.(orthonormal basis) of $\mathbb{C}^n$ and $1 \leq p < +\infty$.
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$$\Phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\ \dfrac{\|\rho\|_p}{\|\sum_{i=1}^n P_i \rho P_i\|_p} \displaystyle\sum_{i=1}^n P_i \rho P_i, & \text{if } \rho \neq 0, \end{cases} \quad \text{where } P_i = e_i e_i^*, \quad \forall\, i.$$

Let $\rho_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\rho_2 = I_2$. Then $\Phi(\rho_1) = 2^{1-(1/p)} I_2$, $\Phi(\rho_2) = I_2$.

(1) $1 < p < +\infty$: $\|\Phi(\rho)\|_p = \|\rho\|_p$, but we may have
$\|t\Phi(\rho_1) + (1-t)\Phi(\rho_2)\|_p \neq \|t\rho_1 + (1-t)\rho_2\|_p$ for some $t \in (0,1)$.

(2) $p = 1$: In fact, $\Phi(\rho) = \sum_{i=1}^n P_i \rho P_i$. Hence $\forall \rho, \sigma \in \mathcal{S}_1^+(H)$, $t \in [0,1]$
$\|t\rho + (1-t)\sigma\|_1 = \|t\Phi(\rho) + (1-t)\Phi(\sigma)\|_1$.
But $\nexists\, U$: unitary, s.t. $\Phi(\rho) = U\rho U^*$ or $\Phi(\rho) = U\rho^T U^*$, $\forall \rho \in \mathcal{S}_1^+(H)$.

Counterexample

Example

Let H be a separable Hilbert space with an o.n.b. $\{e_n : n = 1, 2, \ldots\}$

Let S be the unilateral shift on H defined by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$.

Let Φ be defined by $\Phi(\rho) = S\rho S^* \quad \forall\, \rho \in \mathcal{S}_p^+(H)$.

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Theorem (Kuo, Tsai, Wong, Zhang, 2014)

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Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear surjective isometry.

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Theorem (Kadison, 1951)

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear surjective isometry. Then $\exists$ a Jordan $*$-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and a unitary $U \in \mathcal{B}$, s.t.

$$\Phi(A) = U \cdot J(A).$$

Recall: A norm $\|\cdot\|$ is called unitarily invariant norm if

$$\|UAV\| = \|A\| \qquad \forall \text{ unitaries } U\ V.$$

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Then $\exists$ U, V: unitaries on H, s.t. Φ has the following form:

$$\Phi(A) \equiv UAV \qquad \text{or} \qquad \Phi(A) \equiv UA^TV.$$

Lemma (Abatzoglou, 1979)

Let $1 < p < +\infty$ and ρ in $\mathcal{S}_p^+(H)$ be nonzero. The norm of $\mathcal{S}_p^+(H)$ is Fréchet differentiable at ρ. For any σ in $\mathcal{S}_p^+(H)$ we have

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Lemma

Supp. $\rho, \sigma \in \mathcal{S}_p^+(H)$ $(1 < p < +\infty)$. Then T.F.A.E.

(1) $\rho = \sigma$.

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Supp. $\rho, \sigma \in \mathcal{S}_p^+(H)$ for $1 < p < +\infty$. Then T.F.A.E.

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Lemma

Let $1 < p < +\infty$. Supp. Φ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$ preserving the Schatten p-norms of convex combinations. Then

$$\mathrm{tr}(\sigma^{p-1}\rho) = \mathrm{tr}(\Phi(\sigma)^{p-1}\Phi(\rho)).$$

Proposition

Supp. $\Phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfies that

$$\mathrm{tr}(\sigma^{p-1}\rho) = \mathrm{tr}(\Phi(\sigma)^{p-1}\Phi(\rho)), \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1.$$

Then (1) Φ preserves orthogonality in both directions, that is

$$\rho\sigma = 0 \Leftrightarrow \Phi(\rho)\Phi(\sigma) = 0, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1.$$

(2) When $\dim H < +\infty$, $\Phi(P_1(H)) \subseteq P_1(H)$. This holds when $\dim H = +\infty$ and Φ is surjective.

(3) When $\dim H < +\infty$, we have

$$\mathrm{tr}\, PQ = \mathrm{tr}\, \Phi(P)\Phi(Q), \quad \forall P, Q \in P_1(H).$$

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Remark: $\forall \rho, \sigma \in \mathcal{S}_p(H)$, $\rho \neq 0$ with $\rho = U|\rho|$: polar decomposition,

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Theorem (Uhlhorn, 1963)

Supp. $\dim H \geq 3$ and $\Phi : P_1(H) \to P_1(H)$ is a bijective map. Then Φ satisfies

$$PQ = 0 \;\Leftrightarrow \Phi(P)\Phi(Q) = 0 \qquad (P, Q \in P_1(H))$$

$\Leftrightarrow \exists\, U$: linear unitary on H s.t. $\Phi(P) = UPU^*$, $P \in P_1(H)$.

geometric mean

Geometric mean of positive operators: Pusz and Woronowicz (1973) and later Ando (1978).

$$A\#B = \max\{X \geq 0 : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\}, \qquad A, B \in B(H)^+.$$

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Properties of the geometric mean:
(G1) $A\#B = B\#A$.
(G2) If $A \leq C$ and $B \leq D$, then $A\#B \leq C\#D$.
(G3) (Transfer property) $\forall$ S: inv. bdd. linear or conjugate-linear on H, we have $S(A\#B)S^* = (SAS^*)\#(SBS^*)$.
(G4) Supp. $A_1 \geq A_2 \geq \cdots \geq 0$, $B_1 \geq B_2 \geq \cdots \geq 0$ and $A_n \to A$, $B_n \to B$ strongly. Then $A_n\#B_n \to A\#B$ strongly.
(G5) $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ if A is invertible.

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Recall: Geometric mean of positive operators:

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proof by Jordan triple automorphisms on P_n

Theorem (Molnár, 2013)

Let $\Phi : P_n \to P_n$ $(n \geq 3)$ be a continuous Jordan triple automorphism, i.e., Φ is a continuous bijective map which satisfies

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proof

Recall: the geodesic distance between A and B w.r.t. K^{ψ_3} is

$$\delta(A,B) = \|\log(A^{-1/2}BA^{-1/2})\|_{\text{HS}}.$$

Theorem (Molnár, 2013)

Let $\Phi : P_n \to P_n$ ($n \geq 2$) be a surjective isometry w.r.t d_N, where N: unitarily invariant norm on M_n. Then $\exists$ invertible $T \in M_n$, s.t. for

(1) $\Phi(A) = TAT^*$,
(2) $\Phi(A) = TA^{-1}T^*$
(3) $\Phi(A) = TA^{tr}T^*$,
(4) $\Phi(A) = TA^{tr-1}T^*$
(5) $\Phi(A) = (\det A)^{-2/n}TAT^*$
(6) $\Phi(A) = (\det A)^{2/n}TA^{-1}T^*$
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a. $n = 2$, Φ is of one of (1)-(4).
b. $n \geq 3$, N: scalar of the Hilbert-Schmidt norm, Φ is of one of (1)-(8).
c. $n \geq 3$, N: not scalar of the Hilbert-Schmidt norm. Then Φ is of one of (1)-(4) if $n \neq 4$ (resp., (1)-(8) if $n = 4$).

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