Pseudospectra of special operators and Pseudosectrum preservers

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Relationship between $\sigma(A)$ and $\sigma_\varepsilon(A)$:

$$\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A)$$

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$$d(\Lambda, \Delta) = \max\left\{\sup_{s\in\Lambda} \inf_{t\in\Delta} |s-t|, \sup_{t\in\Delta} \inf_{s\in\Lambda} |s-t|\right\}$$

in the co-domain, where Λ and Δ are two sets in $\mathbb{C}$.

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if and only if there is a unitary operator $U \in \mathcal{B}(H)$ such that Φ has the form

$$A \mapsto UAU^* \qquad \text{or} \qquad A \mapsto UA^tU^*.$$

Furthermore, when H is finite dimensional, the surjectivity assumption on Φ can be removed.

Theorem 8 Let $\varepsilon > 0$. Then a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies

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Similar result holds with $\mathcal{B}(H)$ replaced by $\mathcal{B}_s(H)$, the set of all self-adjoint operators on H.

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Thank you!