solutions, Assignment 2, Math 1500

Q1)

1) $$\lim_{x\to-\infty} \frac{\sqrt{x^4+x}}{(x+1)(2x-1)} = \lim_{x\to-\infty} \frac{\sqrt{x^4(1+\frac{1}{x^3})}}{x(1+\frac{1}{x})\,x(2-\frac{1}{x})} = \lim_{x\to-\infty} \frac{|x^2|\sqrt{1+\frac{1}{x^3}}}{x^2(1+\frac{1}{x})(2-\frac{1}{x})}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^2}\sqrt{1+\frac{1}{x^3}}}{\cancel{x^2}(1+\frac{1}{x})(2-\frac{1}{x})} = \frac{\sqrt{1+0}}{(1+0)(2-0)} = \frac{1}{2}$$

2) $$\lim_{x\to-\infty} \frac{x\sqrt{3x^2-1}}{x^2+2} = \lim_{x\to-\infty} \frac{x|x|\sqrt{3-\frac{1}{x^2}}}{x^2(1+\frac{2}{x^2})} = \lim_{x\to-\infty} \frac{-x^2\sqrt{3-\frac{1}{x^2}}}{x^2(1+\frac{2}{x^2})}$$

$: x<0 \Rightarrow |x|=-x$

$$= \lim_{x\to-\infty} \frac{-\sqrt{3-\frac{1}{x^2}}}{1+\frac{2}{x^2}} = \frac{-\sqrt{3-0}}{1+0} = -\sqrt{3}$$

3) $$\lim_{x\to(-1)^+} \frac{x-1}{x^2(x+1)} = -\infty$$ (b/c $x\to(-1)^+ \Rightarrow x>-1 \Rightarrow x+1>0 \Rightarrow x^2(x+1)>0$)

4) $$\lim_{x\to1} \frac{x^2-x-3}{(x-1)^2} = -\infty$$ (b/c $x\to1 \Rightarrow (x-1)^2>0$ and numerator becomes -3)

5) $$\lim_{x\to-\infty} (2x+\sqrt{4x^2-x}) = \lim_{x\to-\infty} \frac{(2x+\sqrt{4x^2-x})(2x-\sqrt{4x^2-x})}{(2x-\sqrt{4x^2-x})} = \lim_{x\to-\infty} \frac{4x^2-(4x^2-x)}{2x-\sqrt{4x^2-x}}$$

$$= \lim_{x\to-\infty} \frac{x}{2x-|x|\sqrt{4-\frac{1}{x}}} = \lim_{x\to-\infty} \frac{x}{x(2+\sqrt{4-\frac{1}{x}})}$$

$(x<0 \Rightarrow |x|=-x)$

$$= \lim_{x\to-\infty} \frac{1}{2+\sqrt{4-\frac{1}{x}}} = \frac{1}{2+\sqrt{4-0}} = \frac{1}{4}$$

6) $$\lim_{x\to0} \left(\sqrt{\frac{1}{x^2}+1} - \sqrt{\frac{1}{x^2}}\right) = \lim_{x\to0} \frac{\left(\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}}\right)\left(\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}}\right)}{\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}}} =$$

$\longrightarrow$

6) $\cdots = \lim_{x\to 0} \dfrac{\frac{1}{x^2}+1-\frac{1}{x^2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}}} = \lim_{x\to 0} \dfrac{1}{\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}}} = 0$

(b/c when $x\to 0$, denominator tends to ∞)

7) $\lim_{x\to\infty}(\sqrt{4x+x^2}-x) = \lim_{x\to\infty} \dfrac{(\sqrt{4x+x^2}+x)(\sqrt{4x+x^2}-x)}{\sqrt{4x+x^2}+x} = \lim_{x\to\infty} \dfrac{4x+x^2-x^2}{\sqrt{4x+x^2}+x}$

$$= \lim_{x\to\infty} \frac{4x}{|x|\sqrt{\frac{4}{x}+1}+x} = \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{\frac{4}{x}+1}+1\right)}$$

$x>0 \Rightarrow |x| = x$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{\frac{4}{x}+1}+1} = \frac{4}{\sqrt{0+1}+1} = 2$$

8) $\lim_{x\to 0} \dfrac{3x}{\sin 2x} = \lim_{x\to 0} \dfrac{3}{\frac{2\sin 2x}{2x}} = \frac{3}{2}\left(\dfrac{1}{\lim_{x\to 0}\frac{\sin 2x}{2x}}\right) = \frac{3}{2}\left(\frac{1}{1}\right) = \frac{3}{2}$

9) $\lim_{t\to 0} \dfrac{\sin t}{\sin 4t} = \lim_{t\to 0} \dfrac{\frac{\sin t}{t}}{\frac{\sin 4t}{t}} = \lim_{t\to 0} \dfrac{\frac{\sin t}{t}}{4\frac{\sin 4t}{4t}} = \frac{1}{4} \dfrac{\lim_{t\to 0}\frac{\sin t}{t}}{\lim_{t\to 0}\frac{\sin 4t}{4t}} = \frac{1}{4}\left(\frac{1}{1}\right) = \frac{1}{4}$

Q2) (1) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{3(x+h)-(x+h)^2-(3x-x^2)}{h}$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{x^2}-2xh-h^2\cancel{-3x}+\cancel{x^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}(3-2x-h)}{\cancel{h}}$$

$$= \lim_{h\to 0}(3-2x-h) = 3-2x-0 = 3-2x$$

Q2) (2) $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}\right] = \lim_{h\to 0} \frac{-1}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}$

$= \frac{-1}{(\sqrt{x}\sqrt{x+0})(\sqrt{x}+\sqrt{x+0})} = \frac{-1}{2x\sqrt{x}}$

(3) $h'(x) = \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2+3} - \frac{1}{x^2+3}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+3)-[(x+h)^2+3]}{[(x+h)^2+3](x^2+3)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{x^2}+\cancel{3}-\cancel{x^2}-2xh-h^2-\cancel{3}}{[(x+h)^2+3](x^2+3)}\right]$

$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(-2x-h)}{[(x+h)^2+3](x^2+3)}\right] = \frac{-2x-0}{[(x+0)^2+3](x^2+3)} = \frac{-2x}{(x^2+3)^2}$

Q3) (1) $y' = \frac{9}{4} x^{\frac{9}{4}-1} + 0 - e^{\pi^2}(1) = \frac{9}{4} x^{\frac{5}{4}} - e^{\pi^2}$

(2) $y' = 4\left(\frac{1}{3}\right) x^{\frac{1}{3}-1} + \sec x \tan x - \frac{2x}{x^4} + 0 = \frac{4}{3} x^{-\frac{2}{3}} + \sec x \tan x - \frac{2}{x^3}$

(3) $y' = \frac{(-\sin x)(1+\sqrt{x}) - \left(\frac{1}{2\sqrt{x}}\right)\cos x}{(1+\sqrt{x})^2}$

(4) $f'(x) = (\tan x)'\left(\frac{x^4+5}{e^x}\right) + \left(\frac{x^4+5}{e^x}\right)'(\tan x)$

$\longrightarrow$

Q(3) (4) ... $= (\sec^2 x)\left(\frac{x^4+5}{e^x}\right) + \left[\frac{4x^3e^x - e^x(x^4+5)}{e^{2x}}\right](\tan x)$

(5) $g'(x) = 2(x-x^2)^{2-1}(x-x^2)' + 0 = 2(x-x^2)(1-2x)$

(6) $h'(x) = (\sin^2 x)'\sqrt{x^2+e^x} + (\sqrt{x^2+e^x})'(\sin^2 x)$

$= (2\sin x\cos x)\sqrt{x^2+e^x} + \left(\frac{2x+e^x}{2\sqrt{x^2+e^x}}\right)(\sin^2 x)$

(7) $k'(x) = -[6-\tan(2x)]'\sin[6-\tan(2x)] + (e^{-x})'\sqrt{5+x} + (\sqrt{5+x})'e^{-x}$

$= -[0-2\sec^2(2x)]\sin[6-\tan(2x)] - e^{-x}\sqrt{5+x} + \frac{1}{2\sqrt{5+x}}e^{-x}$

$= 2\sec^2(2x)\sin[6-\tan(2x)] - \frac{\sqrt{5+x}}{e^x} + \frac{1}{2e^x\sqrt{5+x}}$

(8) $l'(x) = 10\left[\sec(4x) - \frac{1}{x} + x^{\pi^2}\right]^{10-1}\left[\sec(4x) - \frac{1}{x} + x^{\pi^2}\right]'$

$= 10\left[\sec(4x) - \frac{1}{x} + x^{\pi^2}\right]^9\left[4\sec(4x)\tan(4x) + \frac{1}{x^2} + \pi^2 x^{\pi^2-1}\right]$

Q4) (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+b) = 0+b = b$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\cos x = \cos 0 = 1$

If $b=1$ (i.e $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$) then

$\lim_{x\to 0} f(x)$ exists.

Q4) (b) $f(0) = \cos 0 = 1$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+b) = b$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \cos x = \cos 0 = 1$$

If $b=1$ (i.e $\lim_{x \to 0} f(x) = f(0)$) then $f(x)$ is continuous at $x=0$.

(c) $f'(0)$ exists if $f'(0^+)$ and $f'(0^-)$ both exist and equal to each other; but

$f'(0^+)$ means right derivative at $x=0$ i.e $x>0$ so

$f(x) = \cos x$, so $f'(x) = (\cos x)' = -\sin x$

so $f'(0^+) = -\sin 0 = 0$

$f'(0^-)$ means left derivative at $x=0$, i.e. when $x<0$, $f(x) = x+b$ and $f'(x) = 1+0 = 1$

so $f'(0^-) = 1$.

Therefore, there is no value of b for which $f'(0^+) = f'(0^-)$ because $f'(0^+) = 0$ and $f'(0^-) = 1$

i.e $f'(0)$ does not exist.