Question 1. *Let*

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} \qquad E = \begin{bmatrix} -1 & 0 & 0 \\ 7 & 0 & -1 \\ 2 & -2 & 3 \end{bmatrix}.$$

Evaluate each of the following expressions or explain why it is not defined.

(a) $3B^T - 2C$

(b) $A^2 + BC$

(c) $CB + 6ED$

(d) $(B^T + C)A$

Solution.

(a)

$$\begin{aligned} 3B^T - 2C &= 3\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -1 \end{bmatrix}^T - 2\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 6 & 8 \\ -2 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 \\ 6 & 0 \\ 12 & -3 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 6 & 8 \\ -2 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -8 \\ 0 & -8 \\ 14 & -13 \end{bmatrix}. \end{aligned}$$

(b)

$$\begin{aligned} A^2 + BC &= \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}^2 + \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 29 \\ -3 & -7 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 26 \\ -5 & -1 \end{bmatrix}.$$

(c) $CB + 6ED$ cannot be done, since CB is 3×3, while $6ED$ is 3×1.

(d)

$$(B^T + C)A = \left(\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -1 \end{bmatrix}^T + \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \right) \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
$$= \left(\begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \right) \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -1 \\ 5 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -5 & 9 \\ 3 & 15 \\ 5 & 9 \end{bmatrix}.$$

Question 2. *Let* $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 4 \\ 2 & 1 \\ -5 & 0 \end{bmatrix}$, *find the matrix* X *such that*

$$AB - X = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}.$$

Solution.

$$AB - X = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix} \implies X = AB - \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}.$$

$$\begin{aligned} X &= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 1 \\ -5 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -9 & 7 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 5 \\ 0 & 3 \end{bmatrix}. \end{aligned}$$

Question 3. *The row reduced echelon form of the augmented matrix associated with a system of linear equations is going to be*

$$\left[\begin{array}{ccccc|c} 1 & -4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) What is the number of equations of this system?

(b) What is the number of variables of this system?

(c) What is the number of solutions of this system? Write all solutions.

Solution.

(a) Number of equations = number of rows = 4.

(b) Number of variables = number of columns - 1 = 5

(c) Number of solutions = infinitely many.

$$\begin{aligned} &x_1 = -1 + 4x_2 \\ &x_3 = 3 + 2x_4 - 6x_5 \\ &x_2 \text{ arbitrary} \\ &x_4 \text{ arbitrary} \\ &x_5 \text{ arbitrary} \end{aligned}$$

Question 4. *Use augmented matrices and Gauss-Jordan elimination to find all solutions of the following systems of equations. Clearly indicate what elementary operations you are using.*

(a)
$$\begin{array}{rcrcrcrcr} x & + & 3y & - & z & + & u & = & 1 \\ x & + & 4y & + & 2z & - & 2u & = & 3 \\ 2x & + & 7y & + & z & - & u & = & 4 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcr} & - & x_2 & - & 3x_3 & & & = & -5 \\ 2x_1 & & & + & 4x_3 & + & 2x_4 & = & 20 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & = & 5 \\ -x_1 & - & x_2 & - & 5x_3 & - & 2x_4 & = & -21 \end{array}$$

(c)
$$\begin{array}{rcrcrcrcr} 5x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & 3 \\ -2x_1 & - & x_2 & - & 2x_3 & + & x_4 & = & -1 \\ -7x_1 & - & x_2 & + & 13x_3 & - & 4x_4 & = & -2 \end{array}$$

Solution.

(a)

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 1 & 4 & 2 & -2 & 3 \\ 2 & 7 & 1 & -1 & 4 \end{array}\right]$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$
$$R_1 \to R_1 - 3R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -10 & 10 & -5 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x = -5 + 10z - 10u$

$y = 2 - 3z + 3u$

z, u arbitrary

(b)

$$\left[\begin{array}{cccc|c} 0 & -1 & -3 & 0 & -5 \\ 2 & 0 & 4 & 2 & 20 \\ 1 & -1 & -1 & 1 & 5 \\ -1 & -1 & -5 & -2 & -21 \end{array}\right]$$

$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 2 & 0 & 4 & 2 & 20 \\ 0 & -1 & -3 & 0 & -5 \\ -1 & -1 & -5 & -2 & -21 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$
$$R_4 \to R_4 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 2 & 6 & 0 & 10 \\ 0 & -1 & -3 & 0 & -5 \\ 0 & -2 & -6 & -1 & -16 \end{array}\right]$$

$$R_3 \to -R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 2 & 6 & 0 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & -2 & -6 & -1 & -16 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 5 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 2 & 6 & 0 & 10 \\ 0 & -2 & -6 & -1 & -16 \end{array}\right]$$

$$R_1 \to R_1 + R_2$$
$$R_3 \to R_3 - 2R_2$$
$$R_4 \to R_4 + 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -6 \end{array}\right]$$

$$R_4 \to -R_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$$R_3 \leftrightarrow R_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 4 - 2x_3$$
$$x_2 = 5 - 3x_3$$
$$x_4 = 6$$
$$x_3 \text{ arbitrary}$$

(c)

$$\left[\begin{array}{cccc|c} 5 & 2 & 1 & -1 & 3 \\ -2 & -1 & -2 & 1 & -1 \\ -7 & -1 & 13 & -4 & -2 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ -2 & -1 & -2 & 1 & -1 \\ -7 & -1 & 13 & -4 & -2 \end{array}\right]$$

$$R_2 \to R_2 + 2R_1$$
$$R_3 \to R_3 + 7R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & -1 & -8 & 3 & 1 \\ 0 & -1 & -8 & 3 & 5 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & -1 & -8 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

NO SOLUTION

Question 5. *Find the inverse of each of the following matrices or explain why it does not exist.*

(a) $A = \begin{bmatrix} 3 & -5 \\ 11 & -3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & -2 \\ -2 & -1 & -4 \end{bmatrix}$

(d) $D = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 2 & 4 \\ -5 & 3 & 6 \end{bmatrix}$

Solution.

(a)

$$\left[\begin{array}{cc|cc} 3 & -5 & 1 & 0 \\ 11 & -3 & 0 & 1 \end{array}\right]$$
$$R_1 \to 4R_1$$

$$\left[\begin{array}{cc|cc} 12 & -20 & 4 & 0 \\ 11 & -3 & 0 & 1 \end{array}\right]$$
$$R_1 \to R_1 - R_2$$

$$\left[\begin{array}{cc|cc} 1 & -17 & 4 & -1 \\ 11 & -3 & 0 & 1 \end{array}\right]$$
$$R_2 \to R_2 - 11R_1$$

$$\left[\begin{array}{cc|cc} 1 & -17 & 4 & -1 \\ 0 & 184 & -44 & 12 \end{array}\right]$$
$$R_2 \to \frac{R_2}{184}$$

$$\left[\begin{array}{cc|cc} 1 & -17 & 4 & -1 \\ 0 & 1 & \frac{-11}{46} & \frac{3}{46} \end{array}\right]$$
$$R_1 \to R_1 + 17R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{-3}{46} & \frac{5}{46} \\ 0 & 1 & \frac{-11}{46} & \frac{3}{46} \end{array}\right]$$

Therefore,

$$\begin{bmatrix} 3 & -5 \\ 11 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-3}{46} & \frac{5}{46} \\ \frac{-11}{46} & \frac{3}{46} \end{bmatrix}.$$

(b)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$
$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \end{array}\right]$$
$$R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -2 \end{array}\right]$$
$$R_3 \to -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 2 \end{array}\right]$$
$$R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & -3 & 1 & 0 & 1 & 0 \end{array}\right]$$
$$R_1 \to R_1 - R_2$$
$$R_3 \to R_3 + 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 & 1 & 6 \end{array}\right]$$

$$R_1 \to R_1 - 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -2 & -13 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 & 1 & 6 \end{array}\right]$$

Therefore,

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -2 & -13 \\ -1 & 0 & 2 \\ -3 & 1 & 6 \end{bmatrix}.$$

(c)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ -2 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 + R_1$$

$$R_3 \to R_3 + 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -5 & -4 & 2 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 + 2R_2$$

$$R_3 \to R_3 + 5R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & -14 & 7 & 5 & 1 \end{array}\right]$$

$$R_3 \to \frac{R_3}{-14}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{-5}{14} & \frac{-1}{14} \end{array}\right]$$

$$R_1 \to R_1 + 4R_3$$

$$R_2 \to R_2 + 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{8}{14} & \frac{-4}{14} \\ 0 & 1 & 0 & 0 & \frac{4}{14} & \frac{-2}{14} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{-5}{14} & \frac{-1}{14} \end{array}\right]$$

Therefore,

$$\begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & -2 \\ -2 & -1 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{4}{7} & \frac{-2}{7} \\ 0 & \frac{2}{7} & \frac{-1}{7} \\ \frac{-1}{2} & \frac{-5}{14} & \frac{-1}{14} \end{bmatrix}.$$

(d)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 6 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 3 & 6 & 1 & 0 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \to -R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & -1 & 0 \\ 2 & 3 & 6 & 1 & 0 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 + 5R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & -1 & 0 \\ 0 & 7 & 14 & 1 & 2 & 0 \\ 0 & -7 & -14 & 0 & -5 & 1 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & -1 & 0 \\ 0 & 7 & 14 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{array}\right]$$

Therefore D has no inverse.