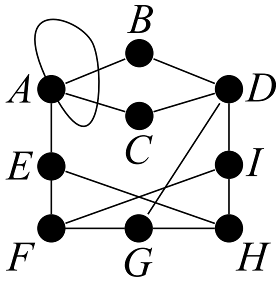
Question 1.

Fill in the following table for the graph to the right:



vertex	A	B	C	D	E	F	G	H	I
degree									

Solution.

vertex	A	B	C	D	E	F	G	H	I
degree	5	2	2	4	3	3	3	3	3

Question 2. *Which of the following are degree sets for simple graphs? For each set that is a degree set for a simple graph, draw such a graph, and for each set that is not a degree set for a simple graph, explain why not.*

(a) $\{2, 2, 2, 3, 4, 6\}$

(b) $\{2, 3, 3, 4\}$

(c) $\{2, 2, 3, 3\}$

(d) $\{0, 1, 2, 2, 2, 5\}$

(e) $\{2, 2, 2, 4, 4\}$

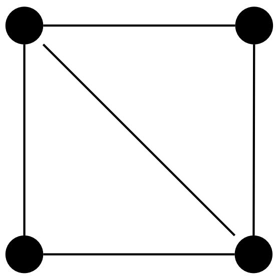
(f) $\{1, 2, 3, 3, 4, 5\}$

Solution.

(a) No. The maximum degree a vertex can have in a simple graph with 6 vertices is 5, and here there is one of degree 6.

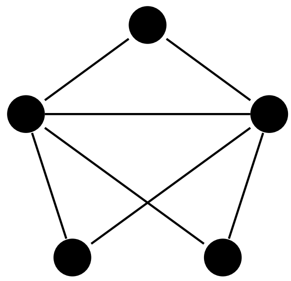
(b) No, for the same reason as above.

(c) Yes:

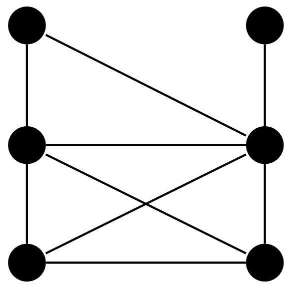


(d) No. The maximum degree a vertex can have in a simple graph with 6 nodes, one of them isolated, is 4. Here there is one node of degree 5.

(e) Yes.



(f) Yes.



Question 3. *Give an example of a simple graph*

(a) having no vertices of odd degree.

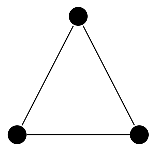
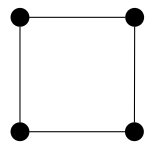
(b) having all vertices of odd degree.

(c) that is connected, has exactly 6 vertices, and exactly 5 edges.

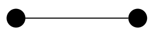
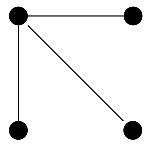
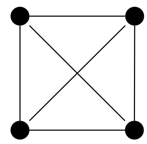
(d) that is connected, but has the property that removing any edge will disconnect the graph.

Solution.

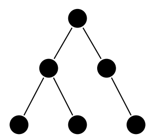
(a) There are an infinite number of correct answers here. Here are a couple:

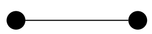
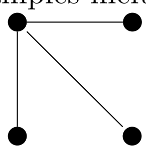
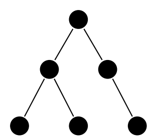
(b) Again, there are infinitely many correct answers. Here are a few:

(c) These are trees on 6 vertices (a tree is a graph containing no circuit), so any tree on 6 vertices will do. For example,



(d) Every tree has this property, so these will again work. In fact, trees are the only graphs for which this will work. Some examples include:

Question 4. *Is it possible for a graph to have 11 edges and each vertex have degree 3? If yes, how many vertices does it have? If not, explain why not.*

Solution.

No. We know that the sum of the degrees in a graph is equal to twice the number of edges. In such a graph as is described, if there are x vertices, then the sum of the degrees is $3x$. Twice the number of edges is 22. Therefore we would have

$$3x = 22$$
$$x = \frac{22}{3},$$

which is not possible, since x has to be an integer. Therefore such a graph cannot exist.

Question 5. *A graph has 24 edges, 4 vertices of degree 5, and all other vertices of degree 2. How many nodes does it have in total?*

Solution.

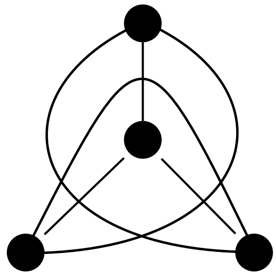
Let x be the number of nodes of degree 2. Then the graph has $x + 4$ nodes in total.

We know that the sum of the degrees is equal to twice the number of edges. Therefore,

$$\begin{aligned} 2(24) &= 5 + 5 + 5 + 5 + 2x \\ &= 20 + 2x \\ 48 &= 20 + 2x \\ 2x &= 28 \\ x &= 14 \end{aligned}$$

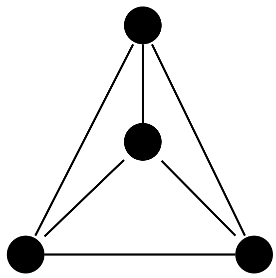
So, the graph has $x + 4 = 14 + 4 = 18$ nodes in total.

Question 6. *Determine if the following graph is planar. If it is, redraw it with no edges crossing.*



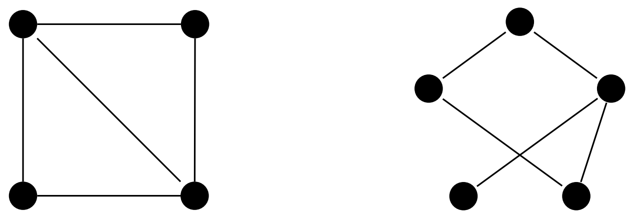
Solution.

This graph is planar, as shown by the following drawing of it:

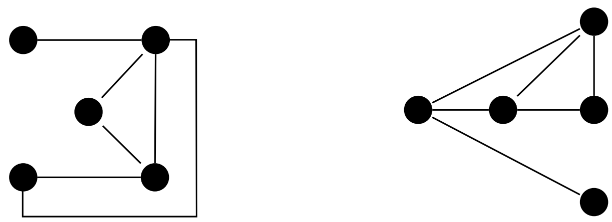


Question 7. *For each of the following pairs of graphs, explain why they are* ***not*** *equivalent.*

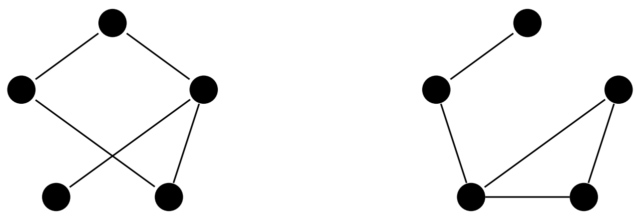
(a)



(b)



(c)



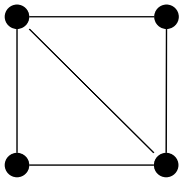
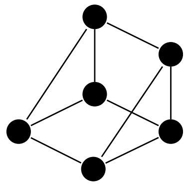
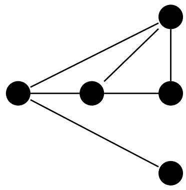
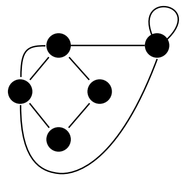
Solution.

(a) These graphs have a different number of nodes (4 on the left and 5 on the right), and therefore cannot be equivalent.

(b) These graphs have different degree sets ($\{1, 2, 2, 3, 4\}$ on the left, and $\{1, 2, 3, 3, 3\}$ on the right). Therefore they cannot be equivalent.

(c) While these graphs do have the same degree sets, in the graph on the left, the vertices of degree 3 and 1 are connected, while on the right, the vertices of degree 3 and 1 are not connected. Therefore these graphs cannot be equivalent.

Question 8. *For each of the following graphs, fill in the below table (check mark if yes, leave blank if no)*

(a)  *(b)*  *(c)*  *(d)* 

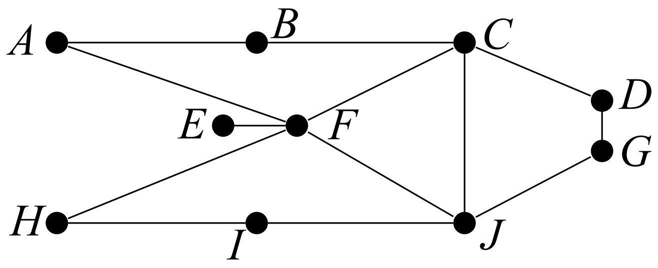
graph	*simple*	*planar*	*connected*	*has an Euler circuit*	*has an Euler path*	*has a Hamilton circuit*
(a)						
(b)						
(c)						
(d)						

Solution.

graph	simple	planar	connected	has an Euler circuit	has an Euler path	has a Hamilton circuit
(a)	✓	✓	✓		✓	✓
(b)	✓	✓	✓			✓
(c)	✓	✓	✓			
(d)		✓	✓	✓	✓	✓

Note that (d) does have an Euler path, since it has an Euler circuit (every Euler circuit *is* an Euler path).

Question 9. *Consider the following graph:*



(a) Does this graph have an Euler circuit? If so, then find one. If not, then explain why not.

(b) Does this graph have an Euler path? If so, then find one. If not, then explain why not.

(c) If the vertex E is removed from the graph (together with the edge attached to it), does the graph then have an Euler circuit? If so, then find one. If not, explain why not.

Solution.

(a) No. Euler's theorem says that a graph has an Euler circuit if and only if every node has even degree, which is not the case here. For example, node E has odd degree.

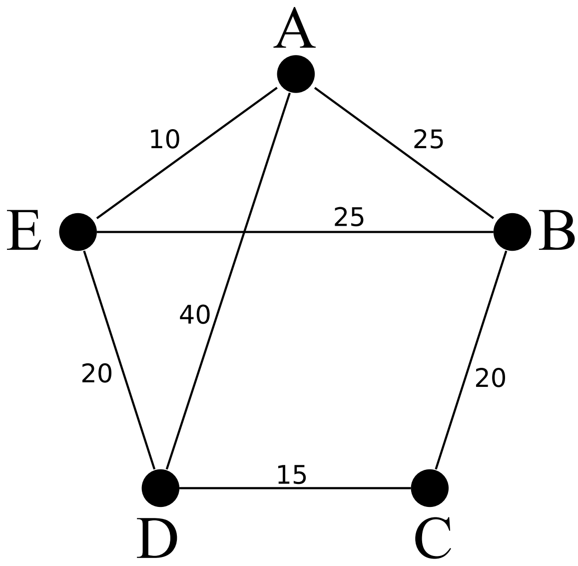
(b) Yes. The corollary to Euler's theorem states that a graph without an Euler circuit contains an Euler path if and only if there are exactly two nodes of odd degree, which is true here (E and F). One such path is:

$$E - F - A - B - C - D - G - J - F - C - J - I - H - F.$$

(c) Yes. This makes every node have even degree, and therefore by Euler's theorem, there is an Euler circuit. One such circuit would be:

$$F - A - B - C - D - G - J - F - C - J - I - H - F.$$

Question 10. *The following graph represents available flights between cities A, B, C, D, and E (there is an edge from one city to another if one can travel either way between them). Each edge is assigned a value measuring how expensive the trip is.*

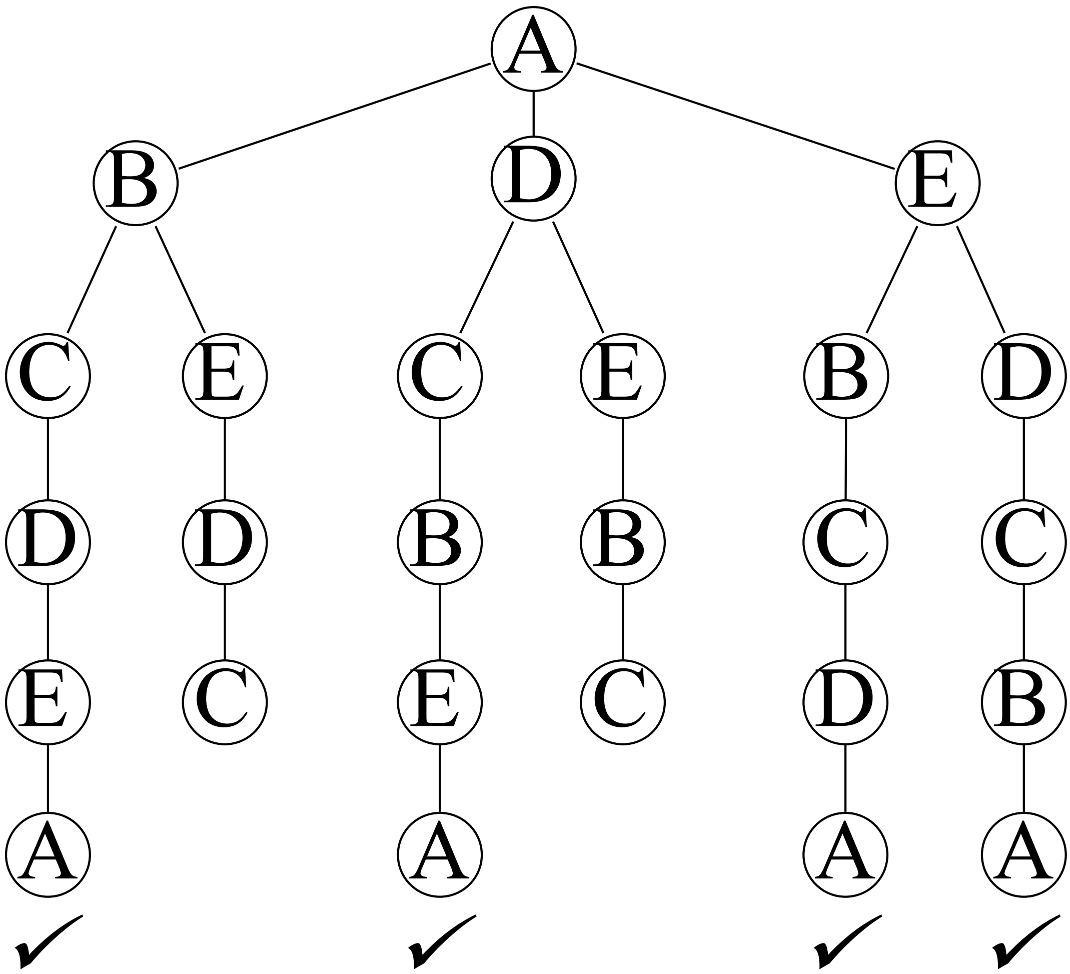


(a) Find all hamilton circuits in this graph.

(b) Which hamilton circuits have the least total cost?

Solution.

One should perform the algorithm discussed in class in order to make sure one finds every hamilton circuit. Here is one possible result:



Therefore the hamilton circuits are:

	cost
$A - B - C - D - E - A$	90
$A - E - D - C - B - A$	90
$A - D - C - B - E - A$	110
$A - E - B - C - D - A$	110

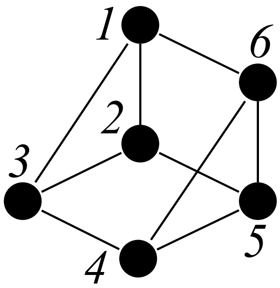
Therefore, the two hamilton circuits with the least cost are

$$A - B - C - D - E - A$$

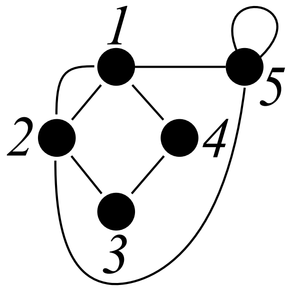
and

$$A - E - D - C - B - A.$$

Question 11. *Find the adjacency matrices for each of the following graphs:*



(a)



(b)

Solution.

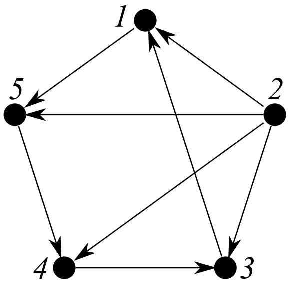
(a)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

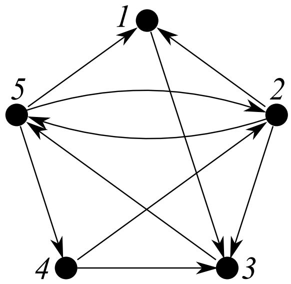
(b)

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Question 12. *Find the adjacency matrices for each of the following digraphs:*



(a)



(b)

Solution.

(a)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

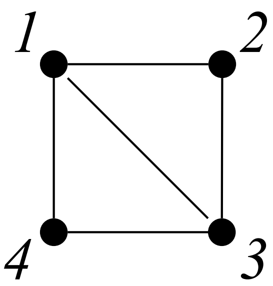
(b)

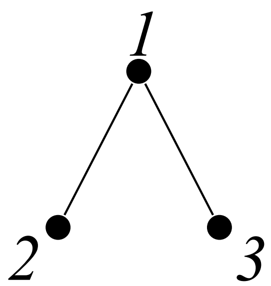
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Question 13. *For each of the following matrices, draw a graph that has the matrix as an adjacency matrix.*

(a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ *(b)* $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution.

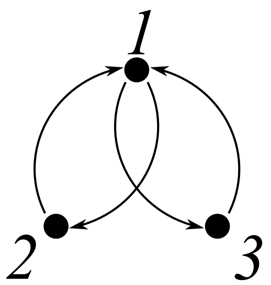
(a) 

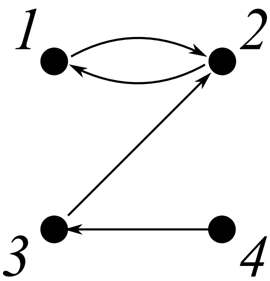
(b) 

Question 14. *For each of the following matrices, draw a digraph that has the matrix as an adjacency matrix.*

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ *(b)* $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Solution.

(a) 

(b) 

Question 15.

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 3 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \end{bmatrix}.$$

The matrix A is the adjacency matrix for a graph with nodes labelled 1, 2, 3, 4, *and* 5.

(a) *How many routes are there from node 5 to node 2 using* ***exactly*** *2 edges?*

(b) *How many routes are there from node 5 to node 2 using* ***exactly*** *3 edges? (Hint: you do not have to find all of A^3).*

(c) *How many routes are there from node 5 to node 2 using* ***at most*** *3 edges?*

Solution.

(a) 1 (the (5,2) entry of A^2)

(b) We need the (5,2) entry of A^3, which can be found by multiplying the fifth row of A times the second column of A^2, to get:

$$1(2) + 1(4) + 0(2) + 0(1) + 0(1) = 6.$$

(c) This is the (5,2) entry of $A + A^2 + A^3$, which in this case is

$$1 + 1 + 6 = 8.$$