MATH 1210 Assignment 1 Winter 2020 **Due date: February 3**

Attempt all questions and show all your work. Attach to Honesty Declaration Form.

1. **Use mathematical induction on integer n to prove each of the following:**

   (a) $1(4)+2(5)+3(6)+\cdots+n(n+3) \;=\; \frac{1}{3}(n)(n+1)(n+5)$ **for** $n\geq 1$**;**

   (b) $3^{n+1}(n+2)!\geq 2^n(n+3)!$ **for** $n\geq 0$**;**

   (c) $\big(1-\frac{1}{3^2}\big)\big(1-\frac{1}{4^2}\big)\big(1-\frac{1}{5^2}\big)\,\cdots\,\big(1-\frac{1}{(n+1)^2}\big) \;=\; \frac{2(n+2)}{3(n+1)}$ **for** $n\geq 2$**;**

   (d) $2^{3n+2}+3^{6n+1}$ **is divisible by** 7 **for** $n\geq 1$**.**

Solution:

(a) **Let** $P(n)$ **be the statement** $1(4)+2(5)+3(6)+\cdots+n(n+3) \;=\; \frac{1}{3}(n)(n+1)(n+5)$ **for** $n\geq 1$**.**

If $n=1$**, then** $P(1)$ **is true because** $1(4)=4$ **and** $\frac{1}{3}(1)(2)(6)=4$**.**

We assume that for $n=k$**,** $P(k)$ **is valid, that is**

$$1(4)+2(5)+3(6)+\cdots+k(k+3) \;=\; \frac{1}{3}(k)(k+1)(k+5) \qquad (1)$$

We need to prove that for $n=k+1$**,** $P(k+1)$ **is valid that is**

$$1(4)+2(5)+3(6)+\cdots+(k+1)(k+4) \;=\; \frac{1}{3}(k+1)(k+2)(k+6)\,. \qquad (*)$$

But

$$\begin{aligned}
L.H.S.\,of\,(*) &= 1(4)+2(5)+3(6)+\cdots+(k+1)(k+4)\\
&= \big[1(4)+2(5)+3(6)+\cdots+k(k+3)\big]+(k+1)(k+4)\\
&= \frac{1}{3}(k)(k+1)(k+5)+(k+1)(k+4) \quad \textbf{by } (1)\\
&= \frac{1}{3}(k+1)\big[k(k+5)+3(k+4)\big]\\
&= \frac{1}{3}(k+1)\big[k^2+8k+12\big]\\
&= \frac{1}{3}(k+1)\big[(k+2)(k+6)\big]\\
&= R.H.S..
\end{aligned}$$

Therefore by the principle of mathematical induction $P(n)$ **is valid for all** $n\geq 1$**.**

(b) **Let** $P(n)$ **be the statement "** $3^{n+1}(n+2)!\geq 2^n(n+3)!$ **for** $n\geq 0$ **".**

If $n=0$**, then** $P(0)$ **is true because** $3^{0+1}(0+2)!=3(2)=6$ **and** $2^0(0+3)!=1(6)=6$**.**

We assume that for $n=k$**,** $P(k)$ **is valid, that is**

$$3^{k+1}(k+2)!\geq 2^k(k+3)! \qquad (1)$$

We need to prove that for $n=k+1$**,** $P(k+1)$ **is valid that is**

$$3^{k+2}(k+3)!\geq 2^{k+1}(k+4)!\,. \quad (*)$$

But

$$
\begin{aligned}
3^{k+2}(k+3)! &= 3(3^{k+1})[(k+3)(k+2)!] \\
&= 3(k+3)[3^{k+1}(k+2)!] \\
&\geq 3(k+3)[2^k(k+3)!] \quad \textbf{by } (1) \\
&\geq 2(k+4)[2^k(k+3)!] \quad \textbf{because } 3(k+3) \geq 2(k+4) \textbf{ (see below for the reason)} \\
&= 2(2^k)[(k+4)(k+3)! \\
&= 2^{k+1}(k+4)!\,.
\end{aligned}
$$

Hence $3^{k+2}(k+3)! \geq 2^{k+1}(k+4)!$ **.**
Reason for $3(k+3) \geq 2(k+4)$**:**

$$3(k+3) \geq 2(k+4) \Leftrightarrow 3k+9 \geq 2k+8 \Leftrightarrow k \geq -1 \textbf{ which is true because } k \geq 0\,.$$

Therefore by the principle of mathematical induction $P(n)$ **is valid for all** $n \geq 0$**.**

(c) Let $P(n)$ **be the statement "** $(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})\,\cdots\,(1-\frac{1}{(n+1)^2}) = \frac{2(n+2)}{3(n+1)}$ **for** $n \geq 2$ **" .**
If $n = 2$**, then** $P(2)$ **is true because** $1 - \frac{1}{3^2} = \frac{8}{9}$ **and** $\frac{2(2+2)}{3(2+1)} = \frac{8}{9}$ **.**
We assume that for $n = k$**,** $P(k)$ **is valid, that is**

$$(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})\,\cdots\,(1-\frac{1}{(k+1)^2}) = \frac{2(k+2)}{3(k+1)} \qquad (1)$$

We need to prove that for $n = k+1$**,** $P(k+1)$ **is valid that is**

$$(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})\,\cdots\,(1-\frac{1}{(k+2)^2}) = \frac{2(k+3)}{3(k+2)}$$

But

$$
\begin{aligned}
&(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})\,\cdots\,(1-\frac{1}{(k+2)^2}) \\
&= \big[(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})\,\cdots\,(1-\frac{1}{(k+1)^2})\big]\,\big[1-\frac{1}{(k+2)^2}\big] \\
&= \frac{2(k+2)}{3(k+1)}\,\big[1-\frac{1}{(k+2)^2}\big] \quad \textbf{by } (1) \\
&= \frac{2(k+2)}{3(k+1)}\,\big[\frac{(k+2)^2-1}{(k+2)^2}\big] \\
&= \frac{2(k+2)}{3(k+1)}\,\big[\frac{k^2+4k+3}{(k+2)^2}\big] \\
&= \frac{2(k+2)}{3(k+1)}\,\big[\frac{(k+1)(k+3)}{(k+2)^2}\big] \\
&= \frac{2(k+3)}{3(k+2)}\,.
\end{aligned}
$$

Therefore by the principle of mathematical induction $P(n)$ **is valid for all** $n \geq 2$**.**

(d) **Let $P(n)$ be the statement "$2^{3n+2}+3^{6n+1}$ is divisible by 7 for $n \geq 1$".**
If $n = 1$, then $P(1)$ is true because $2^5 + 3^7 = 32 + 2187 = 2219 = 7(317)$ which is divisible by 7.
We assume that for $n = k$, $P(k)$ is valid, that is $2^{3k+2} + 3^{6k+1}$ is divisible by 7. We need to prove that for $n = k+1$, $P(k+1)$ is valid that is $2^{3k+5}+3^{6k+7}$ is divisible by 7.

But

$$\begin{aligned} 2^{3k+5}+3^{6k+7} &= 2^3(2^{3k+2})+3^6(3^{6k+1}) \\ &= 8(2^{3k+2})+729(3^{6k+1}) \\ &= [8(2^{3k+2})+8(3^{6k+1})]+[721(3^{6k+1})] \\ &= 8[2^{3k+2}+3^{6k+1}]+[7(103)(3^{6k+1})]\,. \end{aligned}$$

Now by induction hypothesis $2^{3k+2}+3^{6k+1}$ is divisible 7 and also $7(103)(3^{6k+1})$ is divisible by 7, therefore the right hand side is divisible by 7, and as a result the left hand side, that is $2^{3k+5}+3^{6k+7}$, is divisible by 7.
Therefore by the principle of mathematical induction $P(n)$ is true for all $n \geq 1$.

2. Simplify as much as possible using properties of sigma notation.

$$\sum_{n=0}^{1000}(n+1)^6 \,-\, \sum_{n=2}^{1000}2(n+1)^6 \,+\, \sum_{n=1}^{999}[(n+2)^6+1]\,.$$

Solution:

First we note that $\displaystyle\sum_{n=0}^{1000}(n+1)^6 = (0+1)^6+(1+1)^6+\sum_{n=2}^{1000}(n+1)^6 = 65+\sum_{n=2}^{1000}(n+1)^6$ **and**

$$\begin{aligned} \sum_{n=1}^{999}[(n+2)^6+1] &= \sum_{n=2}^{1000}[(n+1)^6+1] \quad (\textbf{ by } n \to n-1) \\ &= \sum_{n=2}^{1000}(n+1)^6 \,+\, \sum_{n=2}^{1000}1 \\ &= \sum_{n=2}^{1000}(n+1)^6 \,+\, (1000-2+1) \\ &= \sum_{n=2}^{1000}(n+1)^6 \,+\, 999\,. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=0}^{1000}(n+1)^6 \,-\, \sum_{n=2}^{1000}2(n+1)^6 \,+\, \sum_{n=1}^{999}[(n+2)^6+1] \\ &= 65+\sum_{n=2}^{1000}(n+1)^6 \,-\, 2\sum_{n=2}^{1000}(n+1)^6 \,+\, \sum_{n=2}^{1000}(n+1)^6 \,+\, 999 \\ &= 65+999 \\ &= 1064\,. \end{aligned}$$

3. Identities $\sum_{k=1}^{m} k = \frac{1}{2}[m(m+1)]$, $\sum_{k=1}^{m} k^2 = \frac{1}{6}[m(m+1)(2m+1)]$ **and** $\sum_{k=1}^{m} k^3 = \frac{1}{4}[m^2(m+1)^2]$ **are given. Use the identities to evaluate the sum** $\sum_{\ell=100}^{110}[(\ell-100)^3 + (\ell-99)^2 - 4]$.

Solution:

$$\sum_{\ell=100}^{110}[(\ell-100)^3 + (\ell-99)^2 - 4] \qquad \textbf{(replacying } \ell \textbf{ by } \ell+100\textbf{)}$$
$$= \sum_{\ell=100-100}^{110-100}[(\ell+100-100)^3 + (\ell+100-99)^2 - 4]$$
$$= \sum_{\ell=0}^{10}[\ell^3 + (\ell+1)^2 - 4]$$
$$= \sum_{\ell=0}^{10}\ell^3 + \sum_{\ell=0}^{10}(\ell+1)^2 - \sum_{\ell=0}^{10}4 \qquad \textbf{(replacying } \ell \textbf{ by } \ell-1 \textbf{ in the middle sigma)}$$
$$= \sum_{\ell=0}^{10}\ell^3 + \sum_{\ell=0+1}^{10+1}\ell^2 - \sum_{\ell=0}^{10}4$$
$$= 0^3 + \sum_{\ell=1}^{10}\ell^3 + \sum_{\ell=1}^{11}\ell^2 - 4\sum_{\ell=0}^{10}1$$
$$= 0 + \frac{1}{4}[(10)^2(11)^2] + \frac{1}{6}[(11)(12)(2(11)+1)] - 4(10-0+1)$$
$$= 0 + 3025 + 506 - 44$$
$$= 3487\,.$$

4. Find all solutions of the following equation. Express your answers in polar form.

$$(x^4 + 6x^2 + 9)(x^4 + x^3 + 5x^2 + 4x + 4) = 0\,.$$

Hint: In the right bracket consider $5x^2$ **as** $x^2 + 4x^2$ **and then solve it by factoring.**

Solution: $(x^4 + 6x^2 + 9)(x^4 + x^3 + 5x^2 + 4x + 4) = 0$ **implies either** $x^4 + 6x^2 + 9 = 0$ **or** $x^4 + x^3 + 5x^2 + 4x + 4 = 0$**.**
Now if $x^4 + 6x^2 + 9 = 0$ **then** $(x^2+3)^2 = 0$**, so** $x^2 + 3 = 0$ **which means** $x = \pm\sqrt{3}\,i$**.**
If $x^4 + x^3 + 5x^2 + 4x + 4 = 0$ **then** $x^4 + x^3 + x^2 + 4x^2 + 4x + 4 = 0$ **and by factoring we get** $x^2(x^2 + x + 1) + 4(x^2 + x + 1) = 0$ **and again by factoring we get** $(x^2 + x + 1)(x^2 + 4) = 0$**.**
Therefore either $x^2 + x + 1 = 0$**, which means** $x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$**; or** $x^2 + 4 = 0$ **which means** $x = \pm 2i$**.**
Hence all solutions, in polar form, are:
$\sqrt{3}\,i = \sqrt{3}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, $-\sqrt{3}\,i = \sqrt{3}(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2})$ $\Big($ **or** $\sqrt{3}(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$ $\Big)$,
$\frac{-1}{2} + \frac{\sqrt{3}}{2}i = 1(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$, $\frac{-1}{2} - \frac{\sqrt{3}}{2}i = 1(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3})$ $\Big($ **or** $1(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$ $\Big)$,

$2\,i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$, $-2\,i = 2(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2})$ $\Big($ **or** $2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})\Big)$.

5. Express each of the following in simplified Cartesian form.

(a) $\left(\frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{18}}{2}i\right)^{10}$;

Solution: There are two options; we can use either polar form or exponential form. For this one we use polar form. Let $z = \frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{18}}{2}i$. **Then**

$|z| = \sqrt{(\frac{\sqrt[4]{2}}{2})^2 + (\frac{-\sqrt[4]{18}}{2})^2} = \sqrt{\frac{\sqrt{2}+3\sqrt{2}}{4}} = \sqrt{\sqrt{2}} = \sqrt[4]{2}$ **and**

$\tan\theta = \frac{-\sqrt[4]{18}}{\sqrt[4]{2}} = -\sqrt{3}$ **which means** $\theta = \tan^{-1}(-\sqrt{3}) = \frac{-\pi}{3}$ **(0r** $\frac{5\pi}{3}$**). Hence**

$z = \sqrt[4]{2}\left[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right]$. **Now**

$$\begin{aligned}
z^{10} = \left(\frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{18}}{2}i\right)^{10} &= (\sqrt[4]{2})^{10}[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})]^{10} \\
&= (2)^{\frac{10}{4}}\left[\cos(-\frac{10\pi}{3}) + i\sin(-\frac{10\pi}{3})\right] \quad \textbf{(since } \cos(-\alpha) = \cos\alpha \textbf{ and } \sin(-\alpha) = -\sin\alpha\textbf{)} \\
&= (2)^{\frac{5}{2}}\left[\cos(\frac{10\pi}{3}) - i\sin(\frac{10\pi}{3})\right] \\
&= (2)^{\frac{5}{2}}\left[\cos(2\pi + \frac{4\pi}{3}) - i\sin(2\pi + \frac{4\pi}{3})\right] \\
&= (2)^{\frac{5}{2}}\left[\cos(\frac{4\pi}{3}) - i\sin(\frac{4\pi}{3})\right] \\
&= 4\sqrt{2}[-\frac{1}{2} - i(-\frac{\sqrt{3}}{2})] \\
&= -2\sqrt{2} + 2\sqrt{6}\,i\,.
\end{aligned}$$

(b) $\frac{1}{81}\left(\frac{1}{2i}\right)^{11}(1-i)^8(-\sqrt{3}-3i)^8$.

Solution: There are two options; we can use either polar form or exponential form. For this one we use exponential form. First note that

$$\left(\frac{1}{2i}\right)^{11} = \left(\frac{1}{2i}\cdot\frac{-i}{-i}\right)^{11} = \left(-\frac{i}{2}\right)^{11} = -\frac{1}{2^{11}}(i)^{11} = -\frac{1}{2^{11}}(i^2)^5 i = -\frac{1}{2^{11}}(-1)^5 i = \frac{1}{2^{11}}i\,.$$

Let $z_1 = 1 - i$. **Then** $|z_1| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ **and** $\tan\theta_1 = \frac{-1}{1} = -1$ **which means** $\theta_1 = \tan^{-1}(-1) = \frac{-\pi}{4}$ **(0r** $\frac{7\pi}{4}$**). Hence** $z_1 = 1 - i = \sqrt{2}\,e^{-\frac{\pi}{4}i}$. **So then**

$$z_1^8 = (1-i)^8 = \left[\sqrt{2}\,e^{-\frac{\pi}{4}i}\right]^8 = (\sqrt{2})^8\left[e^{-\frac{\pi}{4}i}\right]^8 = 2^4 e^{-\frac{8\pi}{4}i} = 2^4 e^{-2\pi i}\,.$$

Also let $z_2 = -\sqrt{3} - 3i$. **Then** $|z_2| = \sqrt{(-\sqrt{3})^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$ **and** $\tan\theta_2 = \frac{-3}{-\sqrt{3}} = \sqrt{3}$; **since both** x **and** y **are negative so it is in the third quadrant**

and $\theta_2 = \tan^{-1}(\sqrt{3}) = \dfrac{4\pi}{3}$ **(0r** $-\dfrac{2\pi}{3}$**). Hence** $z_2 = -\sqrt{3} - 3i = 2\sqrt{3}\, e^{\frac{4\pi}{3}i}$ **. So then**

$$z_2^8 = (-\sqrt{3} - 3i)^8 = \big[2\sqrt{3}\, e^{\frac{4\pi}{3}i}\big]^8 = (2\sqrt{3})^8 \Big[e^{\frac{4\pi}{3}i}\Big]^8 = 2^8(3^4)e^{\frac{32\pi}{3}i}.$$

(Note that since $(-\sqrt{3} - 3i)^8 = (\sqrt{3} + 3i)^8$ **, so another method is working with** $(\sqrt{3} + 3i)^8$ **instead.)**
Now by substitution in the given expression we get:

$$\begin{aligned}
\frac{1}{81}\Big(\frac{1}{2i}\Big)^{11}(1-i)^8(-\sqrt{3}-3i)^8 &= \frac{1}{81}\Big[\frac{1}{2^{11}}i\Big]\Big[2^4 e^{-2\pi i}\Big]\Big[2^8(3^4)e^{\frac{32\pi}{3}i}\Big] \\
&= \frac{1}{81}\Big[\frac{1}{2^{11}}i\Big]\Big[2^4(2^8)(3^4)\Big]\Big[e^{-2\pi i}e^{\frac{32\pi}{3}i}\Big] \\
&= (2i)\Big[e^{\left(-2\pi+\frac{32\pi}{3}\right)i}\Big] \\
&= (2i)\Big[e^{\frac{26\pi}{3}i}\Big] \\
&= (2i)\Big[e^{(8\pi+\frac{2\pi}{3})i}\Big] \\
&= (2i)[e^{\frac{2\pi}{3}i}] \\
&= (2i)[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}] \\
&= (2i)[-\frac{1}{2} + i(\frac{\sqrt{3}}{2})] \\
&= -i + \sqrt{3}i^2 \\
&= -\sqrt{3} - i\,.
\end{aligned}$$

6. **Find all solutions of the equation** $2x^5 + \dfrac{1}{16} = 0$**. Express all solutions in polar form, simplified as much as possible.**

Solution: $x^5 = -\dfrac{1}{32} = \dfrac{1}{32}(-1) = \dfrac{1}{32}e^{\pi i} = \dfrac{1}{32}e^{(2k\pi+\pi)i} = \dfrac{1}{32}e^{(2k+1)\pi i}$**. Therefore all fifth roots are of form** $z_k = (\frac{1}{32}e^{(2k+1)\pi i})^{\frac{1}{5}} = (\dfrac{1}{32})^{\frac{1}{5}}(e^{(2k+1)\pi i})^{\frac{1}{5}} = \dfrac{1}{2}\, e^{\frac{(2k+1)\pi}{5}i}$, **where** $k = 0, 1, 2, 3, 4\,.$

If $k = 0$ **then** $z_0 = \dfrac{1}{2}e^{\frac{\pi}{5}i} = \dfrac{1}{2}(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5})$**.**

If $k = 1$ **then** $z_1 = \dfrac{1}{2}e^{\frac{3\pi}{5}i} = \dfrac{1}{2}(\cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5})$**.**

If $k = 2$ **then** $z_2 = \dfrac{1}{2}e^{\frac{5\pi}{5}i} = \dfrac{1}{2}(\cos\pi + i\sin\pi) = \dfrac{1}{2}(-1 - i(0)) = -\dfrac{1}{2}$**.**

If $k = 3$ **then** $z_3 = \dfrac{1}{2}e^{\frac{7\pi}{5}i} = \dfrac{1}{2}(\cos\dfrac{7\pi}{5} + i\sin\dfrac{7\pi}{5})$**.**

If $k = 4$ **then** $z_4 = \dfrac{1}{2}e^{\frac{9\pi}{5}i} = \dfrac{1}{2}(\cos\dfrac{9\pi}{5} + i\sin\dfrac{9\pi}{5})$**.**