MATH 1210 Techniques of Classical and Linear Algebra

Solutions to Assignment 2

1. Let $P(x) = 2x^5 - 9x^4 + 12x^3 - 4x^2 - 8x + 4$.

   (a) Show that $(1+i)$ is a zero of $P(x)$.

Solution: We can compute that $(1+i)^2 = 1 + 2i + i^2 = 2i$, $(1+i)^3 = (1+i)^2(1+i) = 2i(1+i) = -2+2i$, $(1+i)^4 = \left((1+i)^2\right)^2 = (2i)^2 = -4$, and $(1+i)^5 = (1+i)^4(1+i) = -4-4i$.

Alternatively, one can use the polar form $(1+i) = 2\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) i\right)$ or the exponential form $(1+i) = 2e^{\frac{\pi}{4}i}$ to find the above powers.

In any case, we have that

$$\begin{aligned} P(1+i) &= 2(1+i)^5 - 9(1+i)^4 + 12(1+i)^3 - 4(1+i)^2 - 8(1+i) + 4 \\ &= 2(-4-4i) - 9(-4) + 12(-2+2i) - 4(2i) - 8(1+i) + 4 \\ &= -8 - 8i + 36 - 24 + 24i - 8i - 8 - 8i + 4 = 0, \end{aligned}$$

and so $(1+i)$ is a root of $P(x)$.

   (b) Find all zeros of $P(x)$.

Solution: Since $P(x)$ has real coefficients and $x_1 = (1+i)$ is a zero of $P(x)$, we have that $x_2 = \overline{1+i} = 1 - i$ is also a zero of $P(x)$. Therefore both $(x - (1+i))$ and $(x - (1-i))$ must be factors of $P(x)$, which implies that $P(x)$ is divisible by

$$(x - (1+i))(x - (1-i)) = (x - 1 - i)(x - 1 + i) = (x-1)^2 - i^2 = x^2 - 2x + 2.$$

Using either long or synthetic division, we find that $P(x) = (x^2 - 2x + 2)(2x^3 - 5x^2 - 2x + 2)$. So in order to find the remaining zeros of $P(x)$, we need to find the zeros of $Q(x) = 2x^3 - 5x^2 - 2x + 2$. Possible rational zeros are $\pm 1, \pm 2, \pm\frac{1}{2}$. Trial and error shows that $x_3 = \frac{1}{2}$ is a zero of $Q(x)$ $\left(Q\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{5}{4} - 1 + 2 = 0\right)$. Using either long or synthetic division, we can express $Q(x)$ as $Q(x) = (2x-1)(x^2 - 2x - 2)$. We can then find the last 2 zeros of $P(x)$ by solving the quadratic equation $x^2 - 2x - 2 = 0$, which will bring us to $x_{4,5} = 1 \pm \sqrt{3}$.

Answer: $x_{1,2} = 1 \pm i,\ x_3 = \frac{1}{2},\ x_{4,5} = 1 \pm \sqrt{3}$.

2. Consider the equation $5x^7 - 9x^3 + 3x^2 + 4 = 4x^6 + 5x^4 - 4x^3 - 2$.

   (a) Find the possible number of positive and the possible number of negative real solutions of this equation.

   Solution: We first bring all the terms to the left in order to rewrite the equation as $5x^7 - 4x^6 - 5x^4 - 5x^3 + 3x^2 + 6 = 0$ and denote the polynomial in the left hand side by $P(x)$. We can see that there are 2 sign changes in the coefficients of $P(x)$, hence by the Decartes' Rules of Signs the equation has either 2 or no positive solutions.

   $P(-x) = -5x^7 - 4x^6 - 5x^4 + 5x^3 + 3x^2 + 6$ and since there is one sign change in the coefficients of $P(-x)$, Decartes' Rules of Signs imply that the equation has exactly one negative solution.

   (b) Prove that the above equation has at least four non-real solutions.

   Solution: We discovered in part (a) that the equation has at most 2 positive solutions and exactly one negative solution, which means it has at most 3 real solutions. Because the degree of the polynomial P is 7, by the Fundamental Theorem of Algebra, the equation has 7 complex solutions counting multiplicities. If at most 3 of them are real numbers, it means that at least $7 - 3 = 4$ of them are non-real numbers.

   (c) Show that this equation has no solutions in the interval $[-7, -3]$.

   Solution: Using the Bounds Theorem, we find that every root x_0 of $P(x)$ and hence every solution of our equation should satisfy $|x_0| < \frac{6}{5} + 1 = 2.2$. Therefore, there are no solutions to our equation with $|x| \geq 3$ and, in particular, there are no solutions in the interval $[-7, -3]$.

   Alternatively, one can use substitution to find that $x = -1$ is a solution to our equation ($P(-1) = -5 - 4 - 5 + 5 + 3 + 6 = 0$) and because in part (a) we obtained that there is exactly one negative solution, there are no other negative solutions. In particular, there are no solutions in the interval $[-7, -3]$.

3. Let $P(x) = 10x^4 - 9x^3 + 7x^2 + 3x - 2$.

   (a) Use the Rational Roots Theorem to find all possible rational roots of $P(x)$.

   Solution: If $\frac{p}{q}$ is a root of $P(x)$, then according to the Rational Roots Theorem we have that $p|(-2)$ and $q|10$. Therefore, the possible rational roots are

   $$\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{1}{10}$$

(b) Find all roots of $P(x)$.

Solution: Using substitution, we can find that $P\left(-\frac{1}{2}\right) = \frac{10}{16} + \frac{9}{8} + \frac{7}{4} - \frac{3}{2} - 2 = 0$, so that $x_1 = -\frac{1}{2}$ is a root of $P(x)$. Using long or synthetic division, we obtain that $P(x) = (2x+1)(5x^3 - 7x^2 + 7x - 2)$. Substituting into $5x^3 - 7x^2 + 7x - 2$, we find that $x_2 = \frac{2}{5}$ is a root $\left(\frac{8}{25} - \frac{28}{25} + \frac{14}{5} - 2 = 0\right)$, and then using long or synthetic division we obtain that $5x^3 - 7x^2 + 7x - 2 = (5x-2)(x^2 - x + 1)$. Finally, we find the remaining roots of $P(x)$ by solving the quadratic equation $x^2 - x + 1 = 0$, which yields $x_{3,4} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Answer: $x_1 = -\frac{1}{2}$, $x_2 = \frac{2}{5}$, $x_{3,4} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

4. Consider the matrices

$$A = \begin{bmatrix} -1 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}.$$

In parts (a)-(e) find the specified matrix when possible. If not possible, explain why.

Solution: (a) $3A - 4B$ is not possible to evaluate since A is a 2×3 matrix and B is a 3×2 matrix, so their sizes are different.

(b) $AB + 3D = \begin{bmatrix} -1 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 4 \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 24 \\ -1 & -7 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 9 & 0 \end{bmatrix}$

$= \begin{bmatrix} 10 & 21 \\ 8 & -7 \end{bmatrix}$

(c) $BAC = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 10 \\ 8 & -6 & 5 \end{bmatrix}$

$= \begin{bmatrix} -25 & 20 & -5 \\ 40 & -30 & 25 \\ 30 & -20 & 40 \end{bmatrix}$

(d) CAB is not possible to evaluate since C is a $3 \times \underline{3}$ matrix, and A is a $\underline{2} \times 3$ matrix, which makes CA undefined as the inner dimensions don't match.

(e) $2DA - DB^T = D(2A - B^T) = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \left(2 \begin{bmatrix} -1 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ 2 & 4 \end{bmatrix}^T \right)$

$= \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 & 6 \\ 9 & -1 & -8 \end{bmatrix} = \begin{bmatrix} -12 & 3 & 14 \\ -9 & 6 & 18 \end{bmatrix}$

(f) Find a matrix X that satisfies the equation $2X^T + I_2 = D^3$.

Solution: $2X^T + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}^3 = \begin{bmatrix} -2 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -6 & -3 \end{bmatrix}$. Hence $X^T = \frac{1}{2}\left(\begin{bmatrix} -5 & 2 \\ -6 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -3 & 1 \\ -3 & -2 \end{bmatrix}$, and so $X = (X^T)^T = \begin{bmatrix} -3 & -3 \\ 1 & -2 \end{bmatrix}$.

(g) Find the dimensions of a matrix Y that would allow for the product YCA^TY to be defined.

Solution: Let Y be an $m \times n$ matrix. Since C is a 3×3 matrix, for YC to be defined we need that $n = 3$. Note that because A is a 2×3 matrix, A^T is a 3×2 matrix and so CA^T is defined. Finally, for A^TY to be defined, we need that $2 = m$, which means that Y should be a 2×3 matrix.

5. Let $\mathbf{u} = \langle 2, 1, 3 \rangle$ and $\mathbf{v} = \langle 2, -5, -3 \rangle$. Find each of the following.

(a) $|2\mathbf{u} + \mathbf{v}|$

Solution: $|2\mathbf{u} + \mathbf{v}| = |\langle 6, -3, 3 \rangle| = \sqrt{6^2 + (-3)^2 + 3^2} = \sqrt{54} = 3\sqrt{6}$

(b) the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$

Solution: $\mathbf{u} + \mathbf{v} = \langle 4, -4, 0 \rangle$, $\mathbf{u} - \mathbf{v} = \langle 0, 6, 6 \rangle$. If θ is the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, then $\cos(\theta) = \frac{(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}{|\mathbf{u} + \mathbf{v}|\,|\mathbf{u} - \mathbf{v}|} = \frac{-24}{4\sqrt{2} \cdot 6\sqrt{2}} = -\frac{1}{2}$. Therefore, $\theta = \frac{2\pi}{3}$.

(c) the vector of length 3 in the direction opposite to $\mathbf{v}$

Solution: The vector we are looking for is

$$-3\hat{\mathbf{v}} = -3\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3}{\sqrt{2^2 + (-5)^2 + (-3)^2}}\langle 2, -5, -3 \rangle = \left\langle -\frac{6}{\sqrt{38}}, \frac{15}{\sqrt{38}}, \frac{9}{\sqrt{38}} \right\rangle.$$

6. Consider the plane $\pi : 2x + 3y - z = -5$, the line $\ell : x = -1 - t,\ y = 6 + 4t,\ z = 1$, and the point $P(4, -2, 3)$.

   (a) Determine whether the plane π intersects with the line ℓ and in case it does, find the point(s) of intersection.

   Solution: Common points of the plane π and the line ℓ should satisfy both the equations of π and ℓ. So we need to substitute $x = -1 - t$, $y = 6 + 4t$, $z = 1$ into $2x + 3y - z$ and find the value(s) of t (if any) for which it is equal to -5.

   $$2(-1 - t) + 3(6 + 4t) - 1 = -2 - 2t + 18 + 12t - 1 = -5 \;\Rightarrow\; 10t = -20 \;\Rightarrow\; t = -2.$$

   Therefore, the plane π intersects the line ℓ at the point that corresponds to $t = -2$.

   $$x = -1 - (-2) = 1,\ y = 6 + 4(-2) = -2,\ z = 1,$$

   and so the point of intersection is $(1, -2, 1)$.

   (b) Find parametric and, if possible, symmetric equations of the line that is perpendicular to the plane π and passes through the point P.

   Solution: If the line is perpendicular to the plane π, then it is parallel to the normal $\mathbf{n} = \langle 2, 3, -1\rangle$ of π. Because our line should also pass through $P(4, -2, 3)$, its parametric equations are

   $$x = 4 + 2t,\ y = -2 + 3t,\ z = 3 - t.$$

   Since none of the components of $\mathbf{n}$ is zero, we can also find symmetric equations of the line:

   $$\frac{x-4}{2} = \frac{y+2}{3} = \frac{z-3}{-1}.$$

   (c) Find an equation of the plane that is perpendicular to the plane π, parallel to the line ℓ, and passes through the point P.

   Solution: Because our plane should be perpendicular to the plane $\pi : 2x+3y-z = -5$, its normal $\mathbf{n_1}$ has to be perpendicular to the normal vector $\mathbf{n} = \langle 2, 3, -1\rangle$ of π. The condition that the plane is perpendicular to the line $\ell : x = -1 - t,\ y = 6 + 4t,\ z = 1$ implies that $\mathbf{n_1}$ has to be perpendicular to the vector $\mathbf{u} = \langle -1, 4, 0\rangle$ thatis parallel to ℓ. Therefore, we can take $\mathbf{n_1} = \mathbf{n} \times \mathbf{u}$.

   $$\mathbf{n_1} = \langle 2, 3, -1\rangle \times \langle -1, 4, 0\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ -1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{k}}$$
   $$= (0 - (-4))\hat{\mathbf{i}} - (0 - 1)\hat{\mathbf{j}} + (8 - (-3))\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 11\hat{\mathbf{k}} = \langle 4, 1, 11\rangle.$$

   So our plane should pass through the point $P(4, -2, 3)$ and have a normal $\langle 4, 1, 11\rangle$ which leads to the following equation:

   $$4(x - 4) + (y + 2) + 11(z - 3) = 0 \quad\Longleftrightarrow\quad 4x + y + 11z - 47 = 0.$$