UNIVERSITY OF MANITOBA
COURSE: MATH 1210
DATE & TIME: March 9, 2020, 17:45–19:00
MIDTERM EXAMINATION
DURATION: 75 minutes

I understand that cheating is a serious offence:

Signature (*In Ink*): ______________________

Solutions

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 17 pages including this cover page and one blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 60 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

V. Please do not call or e-mail your instructor to inquire about grades. They will be available shortly after they have been marked.

VI. If the QR codes on your exam paper are deliberately defaced, your exam may not be marked.

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[8] 1. (a) Use mathematical induction to prove that

$$1(2) + 2(3) + 3(4) + \cdots + (2n)(2n+1) = 8(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

for every positive integer n.

Let $P(n)$ be the above statement.

- For $n=1$, $P(1)$ is true because $1(2)+2(3)=8$ and $8(1^2)=8$.

- Assume that for $n=k$, $P(k)$ is true, that is

$$1(2)+2(3)+\cdots+(2k)(2k+1) = 8(1^2+2^2+\cdots+k^2), \qquad (1)$$

Then for $n=k+1$, we need to prove that $P(k+1)$ is true that is

$$1(2)+2(3)+\cdots+(2k+2)(2k+3) = 8(1^2+2^2+\cdots+(k+1)^2)$$

But $1(2)+2(3)+\cdots+(2k+2)(2k+3)$

$= [1(2)+2(3)+\cdots+(2k)(2k+1)] + (2k+1)(2k+2) + (2k+2)(2k+3)$

$= 8(1^2+2^2+\cdots+k^2) + (2k+1)(2k+2) + (2k+2)(2k+3)$, by (1)

$= 8(1^2+2^2+\cdots+k^2) + (2k+2)[(2k+1)+(2k+3)]$

$= 8(1^2+2^2+\cdots+k^2) + (2k+2)(4k+4)$

$= 8(1^2+2^2+\cdots+k^2) + 8(k+1)^2$

$= 8(1^2+2^2+\cdots+k^2+(k+1)^2)$

$=$ R.H.S.

Therefore by the P.M.I., $P(n)$ is true for all $n \geq 1$.

[3] (b) Write $1(2) + 2(3) + 3(4) + \cdots + (2n)(2n+1) = 8(1^2 + 2^2 + 3^2 + \cdots + n^2)$ in sigma notation.

$$\sum_{i=1}^{2n} i(i+1) = 8 \sum_{i=1}^{n} i^2$$

[8] 2. Write the following complex expression in **Cartesian form**. Simplify as much as possible.

$$\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{50}\cdot\frac{(\sqrt{15}-\sqrt{5}\,i)^{18}}{2^{18}(5^9)\,e^{\frac{2\pi}{3}i}}$$

First for $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$, $|z_1| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1/2 + 1/2} = \sqrt{1} = 1$

$\tan\theta_1 = \dfrac{1/\sqrt{2}}{1/\sqrt{2}} = 1 \Rightarrow \theta_1 = \pi/4 \Rightarrow z_1 = 1(\cos \pi/4 + i \sin \pi/4) = e^{\pi/4\, i}$

so then $z_1^{50} = (e^{\pi/4\, i})^{50} = e^{\frac{50\pi}{4} i} = e^{\frac{25\pi}{2} i} = e^{(12\pi + \pi/2) i} = e^{\pi/2\, i}$.

Let $z_2 = \sqrt{15} - \sqrt{5}\, i$, then $|z_2| = \sqrt{(\sqrt{15})^2 + (-\sqrt{5})^2} = \sqrt{15+5} = \sqrt{20} = 2\sqrt{5}$

$\tan\theta_2 = \dfrac{-\sqrt{5}}{\sqrt{15}} = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3} \Rightarrow \theta_2 = e^{-\pi/6\, i} \Rightarrow z_2 = 2\sqrt{5}\left(\cos(-\pi/6) + i\sin(-\pi/6)\right) = 2\sqrt{5}\, e^{-\pi/6\, i}$

so then $z_2^{18} = (2\sqrt{5})^{18}(e^{-\pi/6\, i})^{18} = 2^{18}(\sqrt{5})^{18} e^{-3\pi i} = 2^{18}(5^9)\, e^{-\pi i}$.

Now $\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} i\right)^{50} \cdot \dfrac{(\sqrt{15} - \sqrt{5}\, i)^{18}}{2^{18}(5^9)\, e^{\frac{2\pi}{3} i}}$

$= e^{\frac{\pi}{2} i} \cdot \dfrac{2^{18}(5^9)\, e^{-\pi i}}{2^{18}(5^9)\, e^{2\pi/3\, i}}$

$= e^{\pi/2\, i} \cdot e^{(-\pi - 2\pi/3) i}$

$= e^{\pi/2\, i} \cdot e^{-5\pi/3\, i}$

$= e^{(\pi/2 - 5\pi/3) i}$

$= e^{-\frac{7\pi}{6} i}$

$= \cos(-7\pi/6) + i \sin(-7\pi/6)$

$= \cos 7\pi/6 - i \sin 7\pi/6$

$= -\sqrt{3}/2 - i\,(-1/2)$

$= -\sqrt{3}/2 + 1/2\, i$

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[7] 3. Find all roots of $x^8 + 2x^4 + 1 = 0$.
Hint: First let $z = x^4$.

Let $z = x^4$, then $z^2 + 2z + 1 = 0 \Rightarrow (z+1)^2 = 0 \Rightarrow z + 1 = 0 \Rightarrow z = -1$

So $x^4 = -1$ That is we need to find all 4^{th} roots of -1:

$-1 = 1e^{\pi i}$ (because $e^{\pi i} = \cos\pi + i\sin\pi = (-1) + i(0) = -1$)

so then

$$z_k = 1^{1/4} e^{\pi/4\, i} = 1\, e^{\frac{2k\pi + \pi}{4} i} = e^{\frac{(2k+1)\pi}{4} i}, \quad k = 0, 1, 2, 3;$$

if $k=0$, then $z_0 = e^{\pi/4\, i} = \cos \pi/4 + i \sin \pi/4 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$;

if $k=1$, then $z_1 = e^{\frac{3\pi}{4} i} = \cos 3\pi/4 + i \sin 3\pi/4 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$;

if $k=2$, then $z_2 = e^{5\pi/4\, i} = \cos 5\pi/4 + i \sin 5\pi/4 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$;

if $k=3$, then $z_3 = e^{7\pi/4\, i} = \cos 7\pi/4 + i \sin 7\pi/4 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

we note that each root is of multiplicity 2.

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4. Let $P(x) = 2x^3 - 7x^2 + 5x - 6$.

[2] (a) Use the Rational Roots Theorem to find all possible rational roots of $P(x)$.

If $r = \frac{p}{q}$ is a rational root of $P(x)$ then $p \mid -6$ and $q \mid 2$ that is p is $\pm 1, \pm 2, \pm 3, \pm 6$ and q is ± 1 ~~or~~ ± 2, therefore a possible root is $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

[3] (b) Use the Decartes' Rules of Signs to find the possible number of positive and the possible number of negative roots of $P(x)$.

Since sign changes 3 times in $P(x)$ so the number of positive real roots is either 3 or 1.

$P(-x) = -2x^3 - 7x^2 - 5x - 6$,
since there is no sign change in $P(-x)$ so there is no negative real root for $P(x)$.

[5] (c) Use the results of parts (a) and (b) to find all roots of $P(x)$.

Since there is no negative real root so in part (a) if we only check positive roots, we get $P(3) = 2(27) - 7(9) + 15 - 6 = 54 - 63 + 15 - 6 = 69 - 69 = 0$

Then by long division we get $P(x) = (x-3)(2x^2 - x + 2)$

Now $2x^2 - x + 2 = 0$ gives $x = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(2)}}{2(2)} = \frac{1}{4} \pm \frac{\sqrt{15}}{4} i$

Hence all roots of $P(x)$ are

3, $\frac{1}{4} + \frac{\sqrt{15}}{4} i$ and $\frac{1}{4} - \frac{\sqrt{15}}{4} i$.

5. Consider the matrices

$$A = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 & 4 \\ 2 & 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 3 \\ 2 & 0 \\ 1 & 6 \end{bmatrix}.$$

When possible, find the matrices specified below. If not possible, explain why.

[4] (a) $2AB - D^T$

$$= 2\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & -3 & 4 \\ 2 & 1 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 0 \\ 1 & 6 \end{bmatrix}^T$$

$$= 2\begin{bmatrix} 2 & 4 & -6 \\ 4 & -7 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & -12 \\ 8 & -14 & 16 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 3 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 & -13 \\ 5 & -14 & 10 \end{bmatrix}$$

[4] (b) A^2C^TC

$$= \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix}^T\begin{bmatrix} 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -7 \\ -2 & 1 \end{bmatrix}$$

[6] 6. Let $\mathbf{u} = \langle 2, -1, -2\rangle$ and $\mathbf{v} = \langle 4, 1, -1\rangle$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

But $\vec{u}\cdot\vec{v} = \langle 2,-1,-2\rangle \cdot \langle 4,1,-1\rangle = 2(4)+(-1)(1)+(-2)(-1) = 8-1+2 = 9$

$\|\vec{u}\| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

$\|\vec{v}\| = \sqrt{4^2+1^2+(-1)^2} = \sqrt{16+1+1} = \sqrt{18} = 3\sqrt{2}$

so then

$$\cos\theta = \frac{9}{3(3\sqrt{2})} = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

which means $\theta = \pi/4$.

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7. Consider the line $\ell : x = 2t,\ y = 2 - 4t,\ z = -1 + 3t$ and the points $P(-1, 5, -3)$ and $Q(3, 1, -1)$.

[4] (a) Find parametric equations of the line that passes through the points P and Q.

$\vec{v} = \vec{PQ} = (3, 1, -1) - (-1, 5, -3) = \langle 4, -4, 2 \rangle$

So $\vec{v} = \langle \overset{a}{4}, \overset{b}{-4}, \overset{c}{2} \rangle$ and a point like $P(\overset{x_0}{-1}, \overset{y_0}{5}, \overset{z_0}{-3})$ then

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -1 + 4t \\ y &= 5 - 4t \\ z &= -3 + 2t \end{aligned} \qquad t \in \mathbb{R}$$

[6] (b) Find an equation of the plane that passes through the point P and contains the line ℓ. (**Hint**: Use a specific point on the line ℓ.)

Consider a point on the line ℓ, like $A(0, 2, -1)$

Then the vector $\vec{u} = \vec{AP}$ is on the plane that is

$\vec{u} = \vec{AP} = (-1, 5, -3) - (0, 2, -1) = \langle -1, 3, -2 \rangle$

Also $\vec{v} = \langle 2, -4, 3 \rangle$ is a vector parallel to the line ℓ.

Now $\vec{u} \times \vec{v} = \vec{n}$ is a normal vector to the new plane,

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -2 \\ 2 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -4 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} \hat{k}$$

$$= 1\hat{i} - 1\hat{j} - 2\hat{k} = \langle 1, -1, -2 \rangle$$

So let $\vec{n} = \langle \overset{a}{1}, \overset{b}{-1}, \overset{c}{-2} \rangle$ and choose one point on plane like $P(\overset{x_0}{-1}, \overset{y_0}{5}, \overset{z_0}{-3})$

then $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$1(x+1) - 1(y-5) - 2(z+3) = 0$

$x - y - 2z + 1 + 5 - 6 = 0$

$$\boxed{x - y - 2z = 0}$$