Mathematics MATH1300
Vector Geometry and Linear Algebra
Final Examination
December 11, 2013, 6:00-8:00pm

Problem 1 (7%) Let

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 4 & 3 & 2 & 0 & 1 \\ 1 & -3 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Evaluate $\det(A)$.

Solution: Expanding on the first row, $\det(A) = -2\det\left(\begin{bmatrix} 4 & 2 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}\right) = 6\det(\begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}) = 6(4+2-2) = 24$

Problem 2 (14%) Let

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(i). Compute A^{-1} by putting A in reduced row echelon form.

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_2 \\ \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

(ii). Compute $\det(A)$ and $\det(A^{-1})$.

Solution: $\det(A^{-1}) = 1 - \frac{1}{2} = \frac{1}{2}$, and $\det(A) = \frac{1}{\frac{1}{2}} = 2$, or $\det(A) = 8 + 2 + 4 - 4 - 4 - 4 = 2$.

(iii). Compute A^{-1} using the adjoint of A.

Solution: The cofactor matrix is $C = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ -2 & 0 & 2 \end{bmatrix}$ and so $A^{-1} = \frac{1}{2}C^T = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Problem 3 (6%) Let $\mathbf{x} = (1, 2, -1, -2, -1)$, $\mathbf{y} = (0, 1, 1, 0, 1)$ and $\mathbf{z} = (1, 1, 1, 1, 1)$ be vectors in $\mathbb{R}^5$.

(i). Compute the lengths: $\|\mathbf{x}\|$, $\|\mathbf{y}\|$ and $\|\mathbf{z}\|$.

Solution: $\|\mathbf{x}\| = \sqrt{11}$, $\|\mathbf{y}\| = \sqrt{3}$ and $\|\mathbf{z}\| = \sqrt{5}$

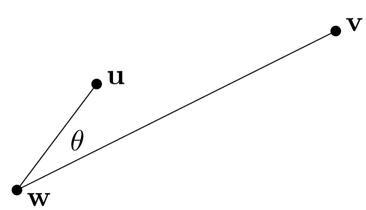
(ii). Evaluate $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{z}$ and $\mathbf{y} \cdot \mathbf{z}$.

Solution: $\mathbf{x} \cdot \mathbf{y} = 0$, $\mathbf{x} \cdot \mathbf{z} = -1$ and $\mathbf{y} \cdot \mathbf{z} = 3$

(iii). Find two vectors from among $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ that are orthogonal.

Solution: $\mathbf{x} \cdot \mathbf{y} = 0$, and so $\theta = \frac{\pi}{2}$

Problem 4 (12%) Let $\mathbf{u} = (4, 3)$, $\mathbf{v} = (7, 12)$ and $\mathbf{w} = (0, 0)$.



(i). Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

Solution: $\|\mathbf{u}\| = 5$ and $\|\mathbf{v}\| = 13$

(ii). Find $\cos(\theta)$, where θ is the angle between the line joining $\mathbf{u}$ with $\mathbf{w}$ and the line joining $\mathbf{v}$ and $\mathbf{w}$.

Solution: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{64}{65}$

(iii). Find the projection of $\mathbf{u}$ onto $\mathbf{v}$, that is, the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution: $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{64}{169}(7, 12)$

(iv). Find the area of the triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution: The area is $\frac{1}{2}\|(4, 3, 0) \times (7, 12, 0)\|$ or $\frac{1}{2}\left|\det \begin{bmatrix} 4 & 3 \\ 7 & 12 \end{bmatrix}\right| = 13\frac{1}{2}$

Problem 5 (10%) Let A and B be square matrices of the same size. Explain why

(i). $(AB)^{-1} = B^{-1}A^{-1}$

> *Solution*: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$, and so $B^{-1}A^{-1}$ is the inverse of AB, that is $(AB)^{-1} = B^{-1}A^{-1}$.

(ii). $(A^{-1})^{-1} = A$

> *Solution*: $A^{-1}A = I$, and so A is the inverse of A^{-1}, that is $(A^{-1})^{-1} = A$.

Problem 6 (9%) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(i). Find all of the eigenvalues of A.

> *Solution*: Since $\det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)(-2\lambda + \lambda^2) = -\lambda(\lambda-3)(\lambda-2)$, the eigenvalues are 3, 2 and 0.

(ii). What is the characteristic polynomial of A?

> *Solution*: $p_A(\lambda) = \lambda(\lambda-2)(\lambda-3) = \lambda^3 - 5\lambda^2 + 6\lambda$

(iii). For each eigenvalue of A, give a corresponding eigenvector.

> *Solution*: For $\lambda = 3$, the eigenvector is a multiple of $(0,0,1)$. For $\lambda = 2$, the eigenvector is a multiple of $(1,1,0)$. For $\lambda = 0$, the eigenvector is a multiple of $(1,-1,0)$.

Problem 7 (6%) Let a system of linear equations have the following reduced row echelon form:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Describe the values of a for which there are

- No solutions: a is ____________

  Solution: $a \neq 0$

- One solution: a is ____________

  Solution: It never happens.

- An infinite number of solutions: a is ____________

  Solution: $a = 0$

Problem 8 (16%) Let $\mathbf{x} = (2,1,1)$, $\mathbf{y} = (1,2,1)$, $\mathbf{z} = (1,1,2)$, $\mathbf{w} = (3,3,3)$ be points in $\mathbb{R}^3$. Let Π be the plane through $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$

(i). Find the equation of the line joining $\mathbf{z}$ and $\mathbf{w}$.

> *Solution*: $(x,y,z) = t(1,1,2) + (1-t)(3,3,3) = (3,3,3) + t(-2,-2,-1)$. Also $\mathbf{w} - \mathbf{z} = (2,2,1)$, so the line also has the equation $(x,y,z) = (3,3,3) + t(2,2,1)$.

(ii). Find the equation of the plane Π.

> *Solution*: $(\mathbf{z} - \mathbf{x}) \times (\mathbf{z} - \mathbf{y}) = (-1,0,1) \times (0,-1,-1) = (1,1,1)$ and so the equation of the plane is $x+y+x+d = 0$. Since $(2,1,1)$ is in the plane, the equation is $x+y+z-4 = 0$, or $x+y+z = 4$.

(iii). Find the (shortest) distance from $\mathbf{w}$ to Π.

> *Solution*: The distance is $\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{5}{\sqrt{3}}$

(iv). Find the equation of the line L that is within the plane Π that passes through $\mathbf{z}$ and is perpendicular to the line joining $\mathbf{0}$ and $\mathbf{x}$.

> *Solution*: The equation of a plane perpendicular to $\mathbf{x} = (2,1,1)$ has equation $2x + y + z + d = 0$. Since this line contains $\mathbf{z} = (1,1,2)$, the equation of the plane perpendicular to $\mathbf{x}$ must be $2x+y+z-5 = 0$. The line of intersection of this plane and Π will be a line perpendicular to $\mathbf{x}$. The desired line is the intersection of these two planes. Solving the two equations gives a line of the form $(x,y,z) = (1,t,3-t) = (1,0,3) + t(0,1,-1)$.

Problem 9 (10%) In each case, put a check in [T] if the statement is true or a check in [F] if the statement is false. The credit given for this problem will be the number of correct answers minus the number of incorrect answers.

✓T	F	If $A\mathbf{x} = 0$ then $A(2\mathbf{x}) = 0$
T	F✓	For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, $\mathbf{x} \times \mathbf{y} = \mathbf{y} \times \mathbf{x}$.
✓T	F	For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
T	F✓	If $\mathbf{x} \cdot \mathbf{y} = 0$ and $\mathbf{y} \cdot \mathbf{z} = 0$ then $\mathbf{x} \cdot \mathbf{z} = 0$.
T	F✓	$\det(A + B) = \det(A) + \det(B)$.
✓T	F	$\det(AB) = \det(A)\det(B)$.
T	F✓	$\det(rA) = r\det(A)$.
T	F✓	If $AB = 0$ and $A \neq 0$, then $B = 0$.
T	F✓	If $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, then $T^2(\mathbf{x}) = T(\mathbf{x}^2)$.
✓T	F	Let λ be an eigenvalue of the matrix A with $\mathbf{x}$ as its corresponding eigenvector. Then $A^2(\mathbf{x}) = \lambda^2\mathbf{x}$.

Problem 10 (10%) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation satisfying

$$T(\mathbf{e}_1) = (1, 1, 1)$$
$$T(\mathbf{e}_2) = (0, 1, 1)$$
$$T(\mathbf{e}_3) = (0, 0, 1)$$

(i). Find $T\big((1, 2, 3)\big)$.

Solution: $(1, 2, 3) = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, and so

$$\begin{aligned} T\big((1,2,3)\big) &= T(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) \\ &= T(\mathbf{e}_1) + 2T(\mathbf{e}_2) + 3T(\mathbf{e}_3) \\ &= (1,1,1) + 2(0,1,1) + 3(0,0,1) \\ &= (1,3,6) \end{aligned}$$

(ii). Find $T^{-1}\big((1, 2, 3)\big)$.

Solution: The matrix representing T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence $T^{-1}\big((1, 2, 3)\big) = (1, 1, 1)$.