UNIVERSITY OF MANITOBA

DATE: February 26, 2015 — MIDTERM TITLE PAGE

EXAMINATION: Vector Geometry and Linear Algebra — TIME: 1 hour

COURSE: MATH 1300 — EXAMINER: Kalajdzievski, Moghaddam, Zhao

NAME: (Print in ink) Midterm Math 1300

STUDENT NUMBER: __________

SEAT NUMBER: __________ W2015

SIGNATURE: (in ink) __________

(I understand that cheating is a serious offense)

Please place a check mark (✓) for your section.

- ☐ A01 10:30–11:20 AM MWF (204 Armes) Xiangui Zhao
- ☐ A02 10:00–11:15 AM TR (208 Armes) Sasho Kalajdzievski
- ☐ A03 1:30–2:20 PM MWF (204 Armes) G. I. Moghaddam

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 7 pages of questions and also 1 blank page for rough work. Please check that you have all the pages. You may remove the blank page if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 70 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	10	
2	11	
3	4	
4	6	
5	9	
6	16	
7	6	
8	8	
Total:	70	

UNIVERSITY OF MANITOBA

DATE: February 26, 2015 MIDTERM

EXAMINATION: Vector Geometry and Linear Algebra TIME: 1 hour

COURSE: MATH 1300 EXAMINER: Kalajdzievski, Moghaddam, Zhao

[10] 1. Consider the linear system

$$\begin{array}{rrrrcl} & & x_3 & -3x_4 & = & 4 \\ 3x_1 & +6x_2 & -3x_3 & & = & 9\,. \\ 3x_1 & +6x_2 & -2x_3 & -3x_4 & = & 13 \end{array}$$

(a) Find the augmented matrix of this system.

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -3 & 4 \\ 3 & 6 & -3 & 0 & 9 \\ 3 & 6 & -2 & -3 & 13 \end{array}\right]$$

(b) Find the reduced row echelon form of the augmented matrix.

$$R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{cccc|c} 3 & 6 & -3 & 0 & 9 \\ 0 & 0 & 1 & -3 & 4 \\ 3 & 6 & -2 & -3 & 13 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 4 \\ 3 & 6 & -2 & -3 & 13 \end{array}\right] R_3 \to -3R_1 + R_3$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & -3 & 4 \end{array}\right] \begin{array}{l} R_1 \to R_2 + R_1 \\ \\ R_3 \to -R_2 + R_3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 7 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

(c) Write **all** of the solutions, if there are any.

$$\begin{array}{l} x_1 + 2x_2 \quad -3x_4 = 7 \\ x_3 - 3x_4 = 4 \\ 0 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = 7 - 2x_2 + 3x_4 \\ x_3 = 4 + 3x_4 \end{array}$$

let $x_2 = t$, $x_4 = r$, then

$$\begin{array}{l} x_1 = 7 - 2t + 3r \\ x_2 = t \\ x_3 = 4 + 3r \\ x_4 = r \end{array} \qquad t, r \in \mathbb{R}$$

[11] 2. Let $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 0 \\ 1 \\ 9 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -1 \\ 6 & 1 \end{bmatrix}$.

(a) Indicate if the expression is defined or undefined by placing a check mark (✓) in the appropriate column. If it is defined, then indicate the size of the resulting matrix.

EXPRESSION	UNDEFINED	DEFINED	SIZE
$C(2A+3B)$	✓		
$A^T(C-4C^T)$		✓	4x2
$A(BB^T)$		✓	2x4

(b) Evaluate $AA^T + C^2$.

$$AA^T + C^2 = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -1 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 12 \\ 12 & 18 \end{bmatrix} + \begin{bmatrix} 19 & -6 \\ 36 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 34 & 6 \\ 48 & 13 \end{bmatrix}$$

[4] 3. Let $A = \begin{bmatrix} a-4 & 0 \\ 1 & a+4 \end{bmatrix}$. Find all values of a for which the matrix A is invertible.

$|A| = (a-4)(a+4) - 0(1) = a^2 - 16$

A is invertible if $|A| \neq 0$ so $a^2 - 16 \neq 0$ i.e $a \neq 4, a \neq -4$

So all values of "a" except ± 4, make A invertible.

4. Calculate the following determinants.

[3] (a) $\begin{vmatrix} 4 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & -1 \end{vmatrix}$ $R_2 \leftrightarrow R_3$

$= -1 \begin{vmatrix} 4 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$= (-1)(4(2)(3)(-1)) = 24$

[3] (b) $\begin{vmatrix} a & a & b \\ 1 & 1 & 1 \\ a+b & a+b & 2b \end{vmatrix}$ $R_3 \rightarrow -R_1 + R_3$

$= \begin{vmatrix} a & a & b \\ 1 & 1 & 1 \\ b & b & b \end{vmatrix}$

$= b \begin{vmatrix} a & a & b \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= b(0) = 0$

5. Let A and B be 2×2 matrices such that $\det(A) = 3$ and $AB^T = A^2$.

[4] (a) Find $\det(B)$.

$AB^T = A^2 \Rightarrow \det(AB^T) = \det(A^2) \Rightarrow \det(A)\det(B^T) = (\det(A))^2$

$\Rightarrow 3\det(B^T) = 3^2 \Rightarrow \det(B^T) = 3$

since $\det(B) = \det(B^T)$ so $\det(B) = 3$

[or say $\det(A) = 3 \neq 0$ so A is invertible so $A^{-1}(AB^T) = A^{-1}(A^2)$ i.e.
$B^T = A$ which means $\det(B^T) = \det(A) = 3$ so $\det(B) = \det(B^T) = 3$]

[5] (b) Find $\det(A(A + B^T))$.

$A(A+B^T) = A^2 + AB^T = A^2 + A^2 = 2A^2$

$\det(A(A+B^T)) = \det(2A^2) = 2^2\det(A^2) = 4(\det(A))^2$
$= 4(3)^2$
$= 36$

6. Let $A = \begin{bmatrix} 7 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

[6] (a) Given that $adj(A) = \begin{bmatrix} 1 & -3 & a \\ -2 & 7 & 7 \\ -2 & b & 7 \end{bmatrix}$; find the values of a and b.

$$a = C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} = +1(-3-0) = -3$$

$$b = C_{23} = (-1)^{2+3} \begin{vmatrix} 7 & 3 \\ 2 & 0 \end{vmatrix} = (-1)(0-6) = +6$$

[6] (b) Find A^{-1} by using the $adj(A)$.

$$|A| = \begin{vmatrix} 7 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 7 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 7(1-0) - 3(0+2) = 7-6 = 1$$

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{1} \begin{bmatrix} 1 & -3 & -3 \\ -2 & 7 & 7 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 7 & 7 \\ -2 & 6 & 7 \end{bmatrix}$$

[4] (c) Use A^{-1} to solve the linear system $\begin{array}{rrrcr} 7x & +3y & & = & -1 \\ & y & -z & = & 1 \\ 2x & & +z & = & 0 \end{array}$.

No mark will be given for any other method.

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & -3 \\ -2 & 7 & 7 \\ -2 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 9 \\ 8 \end{bmatrix}$$

i.e $x = -4$
$y = 9$
$z = 8$

[6] 7. Consider the linear system $\begin{array}{rrrcl} ax & +2y & +z & = & 2 \\ bx & +y & -z & = & 3 \\ cx & +2y & -3z & = & 7 \end{array}$. It is known that $det(A) = 3$, where A is the coefficient matrix of the system. Find the value of x **only**.

$$|A_1| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & -1 \\ 7 & 2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 7 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix}$$

$$= 2(-3+2) - 2(-9+7) + 1(6-7)$$

$$= -2 + 4 - 1 = 1$$

$$x = \frac{|A_1|}{|A|} = \frac{1}{3}$$

[8] 8. Let $A = \begin{bmatrix} 2 & -8 \\ 0 & 1 \end{bmatrix}$. First find elementary matrices E_1 and E_2 such that $E_2E_1A = I$, and then express A as a product of elementary matrices.

$A = \begin{bmatrix} 2 & -8 \\ 0 & 1 \end{bmatrix}$ $R_1 \to \frac{1}{2}R_1$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_1 \to \frac{1}{2}R_1 \Rightarrow E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ $R_1 \to 4R_2 + R_1$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_1 \to 4R_2 + R_1 \Rightarrow E_2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ RREF

So $E_2 E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1}$

but $E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ so

$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$