Q1 (1) False: $A^2 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix} \neq \begin{bmatrix} a^2 & a^2 \\ a^2 & a^2 \end{bmatrix}$

(2) True: $\begin{bmatrix} 12 & -8 \\ -9 & 18 \end{bmatrix} = 2\begin{bmatrix} 3 & -4 \\ 6 & 0 \end{bmatrix} + 3\begin{bmatrix} 2 & 0 \\ -7 & 6 \end{bmatrix}$

(3) False: for instance $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2$ but $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q2 $A^2 - 3A = B$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} - 3\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2-3a & 0 \\ 0 & b^2-3b \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

so $a^2 - 3a = 4$ and $b^2 - 3b = 0$. But

$a^2 - 3a = 4 \Rightarrow a^2 - 3a + 4 = 0 \Rightarrow (a-4)(a+1) = 0$

$\Rightarrow a = -1,\ a = 4$

Also $b^2 - 3b = 0 \Rightarrow b(b-3) = 0 \Rightarrow b = 0,\ b = 3$

so all possible answers for A are:

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Q3 $A^2 + 4A = 0$

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + 4\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2+4a & 0 \\ ab+4b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

continued ⇒

$\Rightarrow$ Q3: so $a^2+4a=0$ and $ab+4b=0$

so $a(a+4)=0$ and $b(a+4)=0$.

Now if $a(a+4)=0$, then $a=0$ or $a=-4$. so if $a=0$, then

$b(0+4)=0 \Rightarrow 4b=0 \Rightarrow b=0$.

If $a=-4$, then $b(-4+4)=0 \Rightarrow 0=0$ ✓

therefore either $a=0$, $b=0$, or $a=-4$ and b arbitrary.

[in fact $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -4 & 0 \\ b & 0 \end{bmatrix}$ (for any b) all the possible answers for A]

Q4 (1) $A^2-2A+I_2 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix}$$

(2) $-2A(B+C^T) = -2\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 5 \\ 1 & -1 & 0 \end{bmatrix}\right)$

$$= \begin{bmatrix} -2 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 6 & -1 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & -6 & -14 \\ 12 & -2 & 10 \end{bmatrix}$$

(3) C is 3×2 and B is 2×3 so CB is 3×3.

therefore $(CB)^2$ is 3×3.

$\Rightarrow$ continued

$\Rightarrow$ Q4 part (3)

$$(CB)^2 = \left(\begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \right)^2$$

$$= \begin{bmatrix} 9 & -1 & 1 \\ -1 & -3 & -1 \\ 10 & -5 & 0 \end{bmatrix} \begin{bmatrix} 9 & -1 & 1 \\ -1 & -3 & -1 \\ 10 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 92 & -11 & 10 \\ -16 & 15 & 2 \\ 95 & 5 & 15 \end{bmatrix}$$

Q5 (a) $(5,4)$ entry of $B^T A^T = (5^{th}$ row of $B^T) \times (4^{th}$ column of $A^T)$

$$= \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$= 3(-4) + (-1)(5) = -17$$

(b) First we notice that $(1,2)$-entry of $(A^TA + BB^T)$ is equal to the $(1,2)$-entry of A^TA plus the $(1,2)$ entry of BB^T. But

$(1,2)$-entry of $A^TA = (1^{st}$ row of $A^T) \cdot (2^{nd}$ column of $A)$

$$= \begin{bmatrix} 1 & -1 & 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \\ -2 \end{bmatrix} = 2 + 3 - 20 = -15$$

Also $(1,2)$-entry of $BB^T = (1^{st}$ row of $B) \cdot (2^{nd}$ column of $B^T)$

$$= \begin{bmatrix} 2 & -1 & 0 & 1 & 3 & -6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \\ -1 \\ 5 \\ -6 \end{bmatrix} = 2 - 3 - 4 - 3 - 30 + 6 = -32$$

Hence $(1,2)$-entry of $(A^TA + BB^T) = -15 + (-32) = -47$.

Q6 $AB=\begin{bmatrix} 1 & x & 2 \\ y & 3 & -1 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ z \end{bmatrix} = \begin{bmatrix} 1-x+2z \\ y-3-z \\ -1-4z \end{bmatrix}$

since $AB=\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$ so must $\begin{bmatrix} 1-x+2z \\ y-3-z \\ -1-4z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$

which means:

$$\begin{aligned} 1-x+2z&=-3 \\ y-3-z&=5 \\ -1-4z&=7 \end{aligned} \implies \begin{aligned} -x+2z&=-4 \\ y-z&=8 \\ -4z&=8 \end{aligned}$$

so from $-4z=8$ we get $z=-2$, then back substitution

gives $y-(-2)=8 \implies y=6$. putting ~~z=-2~~

in the first equation gives $-x+2(-2)=-4 \implies x=0$

therefore $x=0$, $y=6$, $z=-2$.

Q7 (a) $\left[\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ -3 & -1 & -1 & -9 \\ 3 & -1 & 5 & 15 \\ 0 & -2 & 4 & 6 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -3 & -1 & -1 & -9 \\ 3 & -1 & 5 & 15 \\ 0 & -2 & 4 & 6 \end{array}\right] \begin{array}{l} R_2 \to 3R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{array} \implies$

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & -6 \\ 0 & -4 & 8 & 12 \\ 0 & -2 & 4 & 6 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & -4 & 8 & 12 \\ 0 & -2 & 4 & 6 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ R_3 \to 4R_2 + R_3 \\ R_4 \to 2R_2 + R_4 \end{array} \implies$

$\Rightarrow$ Q7 (a)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$$\begin{aligned} x \quad + z &= 4 \\ y - 2z &= -3 \\ 0 &= 0 \checkmark \\ 0 &= 0 \checkmark \end{aligned} \Rightarrow \text{let } z = t \text{ then } \begin{array}{l} x = 4 - t \\ y = -3 + 2t \\ z = t \end{array}, \quad t \in \mathbb{R}$$

(b)
$$\left[\begin{array}{cccc|c} -2 & 0 & -4 & -2 & -20 \\ -1 & 1 & 1 & -1 & -5 \\ -1 & -1 & -5 & -2 & -21 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right] R_1 \to \frac{1}{-2} R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ -1 & 1 & 1 & -1 & -5 \\ -1 & -1 & -5 & -2 & -21 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right] \begin{array}{l} R_2 \longrightarrow R_1 + R_2 \\ R_3 \longrightarrow R_1 + R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & -1 & -3 & -1 & -11 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right] \begin{array}{l} R_3 \to R_2 + R_3 \\ R_4 \to -R_2 + R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 \longrightarrow -R_3$$

$\Rightarrow$ Continued

$\Rightarrow$ Q7 (b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to -R_3 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad RREF$$

$$\begin{aligned} x_1 \quad + 2x_3 \quad &= 4 \\ x_2 + 3x_3 \quad &= 5 \\ x_4 &= 6 \\ 0 &= 0 \end{aligned}$$

Let $x_3 = t$, then

$x_1 = 4 - 2t$

$x_2 = 5 - 3t$ $\qquad t \in \mathbb{R}$

$x_3 = t$

$x_4 = 6$

(c)

$$\left[\begin{array}{cccc|c} -3 & -1 & 1 & 0 & -2 \\ 2 & 1 & 2 & -1 & 1 \\ -7 & -1 & 13 & -4 & -2 \end{array}\right] R_1 \to R_2 + R_1 \Rightarrow \left[\begin{array}{cccc|c} -1 & 0 & 3 & -1 & -1 \\ 2 & 1 & 2 & -1 & 1 \\ -7 & -1 & 13 & -4 & -2 \end{array}\right] R_1 \to -R_1 \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ -7 & -1 & 13 & -4 & -2 \end{array}\right] \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to 7R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 8 & -3 & -1 \\ 0 & -1 & -8 & 3 & 5 \end{array}\right] R_3 \to R_2 + R_3$$

$\Rightarrow$ continued

⇒ Q7 (c)

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & 8 & -3 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

In the third row we get $0=4$ which is impossible, therefore this system has no solution at all.