Q1 [30] (a) since $(0,0,0)$ is not in the set $(0+0-0\neq 2)$ so it is <u>not</u> a subspace.

(b) let $W=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \mid 2a-c=0,\ 3b-d=0\right\}$. But $2a-c=0$, so $c=2a$, and $3b-d=0$, so $d=3b$, i.e.

$$W=\left\{\begin{bmatrix} a & b \\ 2a & 3b\end{bmatrix} \mid a,b\in\mathbb{R}\right\}$$

since $\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$ is in W so $W\neq\emptyset$.

(1) W is closed under addition: for any $\begin{bmatrix} a_1 & b_1 \\ 2a_1 & 3b_1\end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ 2a_2 & 3b_2\end{bmatrix}$ in W, we get:

$$\begin{bmatrix} a_1 & b_1 \\ 2a_1 & 3b_1\end{bmatrix}+\begin{bmatrix} a_2 & b_2 \\ 2a_2 & 3b_2\end{bmatrix}=\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 2(a_1+a_2) & 3(b_1+b_2)\end{bmatrix}$$

which is in W.

(2) W is closed under scalar multiplication:

For any $\begin{bmatrix} a & b \\ 2a & 3b\end{bmatrix}$ in W and c in $\mathbb{R}$, we get

$$c\begin{bmatrix} a & b \\ 2a & 3b\end{bmatrix}=\begin{bmatrix} ac & bc \\ 2ac & 3bc\end{bmatrix}$$

which is in W.

Hence W is a subspace of M_{22}.

(c) Let $W=\{A\in M_{22} \mid A+A^T=0\}$

since $0+0^T=0$ so $0\in W$ i.e $W\neq\emptyset$

$\Longrightarrow$

$\Rightarrow$ Q1 (c) (1) W is closed under addition:

For any A and B in W since $A+A^T=0$, $B+B^T=0$

so we get

$$(A+B)+(A+B)^T = A+B+A^T+B^T$$
$$=(A+A^T)+(B+B^T)$$
$$=0+0$$
$$=0$$

therefore $A+B$ is in W.

(2) W is closed under scalar multiplication:

For any A in W (i.e $A+A^T=0$) and

c in $\mathbb{R}$, we get

$$(cA)+(cA)^T = cA+cA^T = c(A+A^T) = c0 = 0$$

so cA is also in W.

Hence W is a subspace of M_{22}.

(d) Let $W = \{at^3+bt^2+ct \mid a, b, c \in \mathbb{R}\}$,

since $0t^3+0t^2+0t=0$ is in W so $W \neq \emptyset$.

(1) W is closed under addition:

For $p(t)=a_1t^3+b_1t^2+c_1t$ and $q(t)=a_2t^3+b_2t^2+c_2t$

in W, we get

$$p(t)+q(t) = (a_1+a_2)t^3+(b_1+b_2)t^2+(c_1+c_2)t$$

which is in W.

$\Longrightarrow$

$\Rightarrow$ Q(1) (d) (2) W is closed under scalar multiplication:

for any $p(t) = at^3 + bt^2 + ct$ in W and r in $\mathbb{R}$, we get

$r\,p(t) = (ar)t^3 + (br)t^2 + (cr)t$

which is in W.

Hence W is a subspace of P_3.

(e) since $0(t) = 0$ (i.e the zero object) is not in this set ($0 \neq t^3 + a$ for any a) so this set is not a subspace of P_3

(or say let $p(t) = t^3 + 1$, $q(t) = t^3 + 2$ in W then $p(t) + q(t) = 2t^3 + 3$ is not in W, which means W is not closed under addition.)

(f) 0 object is not in this set because for $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $0 + 0 + 0 \neq 3$. Therefore the set is not a subspace of M_{33}.

[or say for two matrices $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in this set $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which is not in this set ($4 + 1 + 1 = 6 \neq 3$). So it is not a subspace.]

Q 2 [10] we must find c_1, c_2, c_3 such that

$$c_1(1,0,1) + c_2(1,1,3) + c_3(0,1,2) = (3,4,11)$$

so $(c_1 + c_2,\ c_2 + c_3,\ c_1 + 3c_2 + 2c_3) = (3,4,11)$

$$\begin{aligned} c_1 + c_2 \quad &= 3 \\ c_2 + c_3 &= 4 \\ c_1 + 3c_2 + 2c_3 &= 11 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 1 & 3 & 2 & 11 \end{array}\right] \xrightarrow{R_3 \to -R_1 + R_3} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 8 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ R_3 \to -2R_2 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

so $$\begin{aligned} c_1 - c_3 &= -1 \\ c_2 + c_3 &= 4 \\ 0 &= 0 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -1 + c_3 \\ c_2 &= 4 - c_3 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -1 + t \\ c_2 &= 4 - t \\ c_3 &= t \end{aligned} \quad t \in \mathbb{R}$$

therefore yes $(3,4,11)$ is in span S.

(for instance for $t=0$, $c_1 = -1$, $c_2 = 4$, $c_3 = 0$ i.e $(3,4,11) = (-1)(1,0,1) + 4(1,1,3)$)

For $(1,-1,1)$: must

$$c_1(1,0,1) + c_2(1,1,3) + c_3(0,1,2) = (1,-1,1)$$

therefore like above must $\begin{cases} c_1 + c_2 = 1 \\ c_2 + c_3 = -1 \\ c_1 + 3c_2 + 2c_3 = 1 \end{cases}$ so

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 3 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to -R_1 + R_3} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ R_3 \to -2R_2 + R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right] \Rightarrow \begin{aligned} c_1 - c_3 &= 2 \\ c_2 + c_3 &= -1 \\ 0 &= 2 \end{aligned} \text{ not possible}$$

i.e. there is no solution; which means $(1,-1,1)$ is <u>not</u> in span S.

Q3 [18] (a) We offer two solutions for this part:

sol. (1): since $\dim(\mathbb{R}^3)=3$ and there are 4 vectors in S therefore S is a linearly dependent set.

sol. (2) If $c_1(1,-2,1)+c_2(2,-1,3)+c_3(0,1,2)+c_4(0,1,0)=(0,0,0)$, then:

$$(c_1+2c_2,\ -2c_1-c_2+c_3+c_4,\ c_1+3c_2+2c_3)=(0,0,0)$$

$$\begin{aligned} c_1+2c_2 &= 0 \\ -2c_1-c_2+c_3+c_4 &= 0 \\ c_1+3c_2+2c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to 2R_1+R_3 \\ R_3 \to -R_1+R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right] R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \to -2R_2+R_1 \\ R_3 \to -3R_2+R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \end{array}\right] R_3 \to \frac{1}{-5}R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 \end{array}\right] \begin{array}{l} R_1 \to 4R_3+R_1 \\ R_2 \to -2R_3+R_2 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{4}{5} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = \frac{4}{5}t \\ c_2 = -\frac{2}{5}t \\ c_3 = \frac{1}{5}t \\ c_4 = t \end{array} \quad t \in \mathbb{R}$$

therefore S is a linearly independent set in $\mathbb{R}^3$.

(b) If $c_1(t^2-1)+c_2(3t+2)+c_3(t^2+1)=0$, then:

$(c_1+c_3)t^2+3c_2t+(-c_1+2c_2+c_3)=0$ which means:

$$\begin{aligned} c_1+c_3 &= 0 \\ 3c_2 &= 0 \\ -c_1+2c_2+c_3 &= 0 \end{aligned} \Rightarrow \boxed{c_2=0} \Rightarrow \begin{aligned} c_1+c_3 &= 0 \\ -c_1+c_3 &= 0 \end{aligned} \Rightarrow \boxed{c_1=0},\ \boxed{c_3=0}$$

i.e S is a linearly independent set in P_2.

$\Rightarrow$

$\Rightarrow$ Q(3) (c) If $c_1\begin{bmatrix}1 & -2\\0 & 6\end{bmatrix}+c_2\begin{bmatrix}0 & 1\\0 & -3\end{bmatrix}+c_3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}+c_4\begin{bmatrix}1 & -2\\0 & 7\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, then:

$$\begin{bmatrix}c_1+c_4 & -2c_1+c_2-2c_4\\ c_3 & 6c_1-3c_2+7c_4\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$ which means:

$$\begin{aligned} c_1 \qquad\qquad &+c_4=0\\ -2c_1+c_2 \qquad &-2c_4=0\\ c_3 &\qquad=0\\ 6c_1-3c_2 \qquad &+7c_4=0\end{aligned} \Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ -2 & 1 & 0 & -2 & 0\\ 0 & 0 & 1 & 0 & 0\\ 6 & -3 & 0 & 7 & 0\end{array}\right] \begin{array}{l}R_2\to 2R_1+R_2\\ \\ \\ R_4\to -6R_1+R_6\end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & -3 & 0 & 1 & 0\end{array}\right] R_4\to 3R_2+R_4 \Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right] R_1\to -R_4+R_1 \Rightarrow$$

$$\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right] \Rightarrow \begin{array}{l}c_1=0\\ c_2=0\\ c_3=0\\ c_4=0\end{array}$$

so yes S is a linearly independent set in M_{22}.

Q(4) (a) $W=\{A\in M_{22} \mid AA^T=I\}$

[12]

since for $A=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ in W, we get

$\left(AA^T=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I \text{ and } BB^T=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I\right)$

$A+B=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}$; but $A+B=\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}$ is <u>not</u> in W because $\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}^T=\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & -4\end{bmatrix}\neq\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

so W is <u>not</u> closed under addition.

$\Rightarrow$

$\Rightarrow$ Q(4) (b) W is <u>not</u> closed under scalar multiplication:

For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in W and c=2 in $\mathbb{R}$, we get

$2A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ but 2A is not in W because

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) W is <u>not</u> a subspace of M_{22} because it is neither closed under addition, nor closed under scalar multiplication.

Q(5) [6] (a) S is a linearly independent set because the two polynomials in S are not scalar multiple of each other.

(b) S spans P_1: because for any polynomial $p(t) = at+b$ in P_1 we need to find c_1 and c_2 such that

$$c_1(t+1) + c_2(t-1) = at+b$$

but $(c_1+c_2)t + (c_1-c_2) = at+b$

which means:

$$\begin{matrix} c_1 + c_2 = a \\ c_1 - c_2 = b \end{matrix} \Rightarrow c_1 = \tfrac{1}{2}(a+b),\ c_2 = \tfrac{1}{2}(a-b)$$

therefore $p(t) = at+b$ is a linear combination of the two polynomials in S.

Hence S forms a basis for P_1.

Q(6) [24] (a) $W=\{(a,b,c,d) \mid c=2a+b,\ d=a-3b\}$ which means:

$$W=\{(a,\ b,\ 2a+b,\ a-3b) \mid a,\ b \in \mathbb{R}\}$$

Now since $(a,\ b,\ 2a+b,\ a-3b) = (a,0,2a,a)+(0,b,b,-3b)$

$$= a(1,0,2,1)+b(0,1,1,-3)$$

let $S=\{(1,0,2,1),(0,1,1,-3)\}$, so S, spans W.

Also S is a linearly independent set because the two vectors in S are not scalar multiple of each other.

Thus S forms a basis for W and $\dim W = 2$.

(b) $W=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a=b+c,\ d=b-2c\right\}$ which means;

$$W=\left\{\begin{bmatrix} b+c & b \\ c & b-2c \end{bmatrix} \middle| b,c \in \mathbb{R}\right\}$$

since $\begin{bmatrix} b+c & b \\ c & b-2c \end{bmatrix} = \begin{bmatrix} b & b \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ c & -2c \end{bmatrix}$

$$= b\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$$

so $S=\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\right\}$ spans W. But S is also linearly independent (because the two matrices in S are not scalar multiple of each other). Therefore S forms a basis for W, i.e. $\dim(W)=2$.

$\Longrightarrow$

Q(6) (c) $W = \{at^2+bt+c \mid a-2b=0,\ c+b=0\}$ which means

$W = \{at^2+bt+c \mid a=2b,\ c=-b\}$ so

$W = \{2bt^2+bt-b \mid b \in \mathbb{R}\}$. But since

$2bt^2+bt-b = b(2t^2+t-1)$

so $S = \{p(t) = 2t^2+t-1\}$ spans W. But obviously S (which contains only one object) is linearly independent. Therefore S forms a basis for W ~~and~~ $\dim(W)=1$.

(d) $W = \{at^3+bt^2+ct+d \mid a-4b=0,\ c-d=0\}$ which means

$W = \{4bt^3+bt^2+ct+c \mid b, c \in \mathbb{R}\}$. But since

$4bt^3+bt^2+ct+c = b(4t^3+t^2)+c(t+1)$, so

$S = \{4t^3+t^2,\ t+1\}$ spans W. On the other hand, S is also linearly independent (because the two polynomials in S are not scalar multiple of each other), therefore S forms a basis for W, which means $\dim(W)=2$. (because there are exactly two polynomials in S)

Total = 100 marks