Solutions Assignment 3, Math 1310 (W2009)

Q1 [16]

(a)
$$T = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} A & B & C & \\ \end{array}$$
$$T = \begin{array}{l} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{16} & \frac{7}{16} & \frac{1}{4} \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

(b)
$$I - T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{16} & \frac{7}{16} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{16} & \frac{15}{16} & -\frac{3}{8} \\ -\frac{3}{16} & -\frac{7}{16} & \frac{3}{4} \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{2} & -\frac{3}{8} & 0 \\ -\frac{5}{16} & \frac{15}{16} & -\frac{3}{8} & 0 \\ -\frac{3}{16} & -\frac{7}{16} & \frac{3}{4} & 0 \end{array}\right] \begin{array}{l} R_1 \to 2R_1 \\ R_2 \to 16R_2 \\ R_3 \to 16R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -\frac{3}{4} & 0 \\ -5 & 15 & -6 & 0 \\ -3 & -7 & 12 & 0 \end{array}\right] \begin{array}{l} R_2 \to 5R_1 + R_2 \\ R_3 \to 3R_1 + R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -\frac{3}{4} & 0 \\ 0 & 10 & -\frac{39}{4} & 0 \\ 0 & -10 & \frac{39}{4} & 0 \end{array}\right] R_3 \to R_2 + R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -\frac{3}{4} & 0 \\ 0 & 10 & -\frac{39}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to \tfrac{1}{10} R_2 \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -\frac{3}{4} & 0 \\ 0 & 1 & -\frac{39}{40} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_2 + R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{69}{40} & 0 \\ 0 & 1 & -\frac{39}{40} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so
$$P_1 - \frac{69}{40}P_3 = 0$$
$$P_2 - \frac{39}{40}P_3 = 0 \quad \Rightarrow \quad P_1 = \frac{69}{40}P_3, \quad P_2 = \frac{39}{40}P_3 \quad \Rightarrow \quad \text{let } P_3 = 40t \text{ then } \begin{array}{l} P_1 = 69t \\ P_2 = 39t \\ P_3 = 40t \end{array}$$
$$0 = 0$$

i.e. $\vec{P} = t \begin{bmatrix} 69 \\ 39 \\ 40 \end{bmatrix}$

(c) if $P_3 = 4000$ then $40t = 4000 \Rightarrow t = 100$

so $P_1 = 69(100) = 6900$ \$ department A

$P_2 = 39(100) = 3900$ \$ " B

Q2 [16]

(a)
$$C = \begin{array}{cc} & \begin{array}{cc} E & P \end{array} \\ & \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \begin{array}{c} E \\ P \end{array} \end{array}$$

(b)
$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.2 \\ -0.1 & 0.7 \end{bmatrix}$$

now

$$\left[\begin{array}{cc|cc} 0.6 & -0.2 & 1 & 0 \\ -0.1 & 0.7 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to 10R_1 \\ R_2 \to -10R_2 \end{array} \Rightarrow \left[\begin{array}{cc|cc} 6 & -2 & 10 & 0 \\ 1 & -7 & 0 & -10 \end{array}\right] R_1 \leftrightarrow R_2 \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -7 & 0 & -10 \\ 6 & -2 & 10 & 0 \end{array}\right] R_2 \to -6R_1 + R_2 \Rightarrow \left[\begin{array}{cc|cc} 1 & -7 & 0 & -10 \\ 0 & 40 & 10 & 60 \end{array}\right] R_2 \to \tfrac{1}{40} R_2 \Rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & -7 & 0 & -10 \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \end{array}\right] R_1 \to 7R_2 + R_1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{4} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \end{array}\right]$$

since $(I - C)^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}$ and all entries are positive, so

the economy is productive.

(c) $\vec{d} = \begin{bmatrix} 320 \\ 280 \end{bmatrix} \begin{array}{c} E \\ P \end{array}$ and $\vec{x} = (I - C)^{-1} \vec{d}$ i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 320 \\ 280 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 320 \\ 280 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2240 + 560 \\ 320 + 1680 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2800 \\ 2000 \end{bmatrix} = \begin{bmatrix} 700 \\ 500 \end{bmatrix}$$

therefore $x = \$700$ and $y = \$500$

Q3
[18] (a) $\frac{\|\vec{u}\|}{\|\vec{w}\|}(\vec{u}\cdot\vec{w}) = \frac{\sqrt{25+64+9+2}}{\sqrt{16+4+4+1}}\left((5,8,3,\sqrt{2})\cdot(4,2,-2,1)\right)$

$= \frac{\sqrt{100}}{\sqrt{25}}(20+16-6+\sqrt{2})$

$= \frac{10}{5}(30+\sqrt{2})$

$= 60+2\sqrt{2}$

(b) $\|\vec{w}+\vec{w}\|^2$ is a number but $(\|\vec{v}\|\|\vec{w}\|)\vec{u}$ is a vector and one can not add a vector and a number. Hence it is not defined.

(c) $\vec{w}\cdot\vec{v} = (4,2,-2,1)\cdot(-1,0,-2,0) = -4+0+4+0 = 0$

$\vec{v}\cdot\vec{u} = (-1,0,-2,0)\cdot(5,8,3,\sqrt{2}) = -5+0-6+0 = -11$

so $(\vec{w}\cdot\vec{v})\vec{u} - 2(\vec{v}\cdot\vec{u})\vec{v} = 0\vec{u} - 2(-11)\vec{v} = \vec{0} + 22\vec{v}$

$= 22\vec{v} = 22(-1,0,-2,0) = (-22,0,-44,0)$

Q4 (a) $\vec{u} = \overrightarrow{BC} = (3,0,2)-(2,1,2) = (1,-1,0)$
[18]
$\vec{v} = \overrightarrow{BA} = (2,0,2)-(2,1,2) = (0,-1,0)$

$\vec{w} = \overrightarrow{CA} = (2,0,2)-(3,0,2) = (-1,0,0)$

(b) $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{(1,-1,0)\cdot(0,-1,0)}{\|(1,-1,0)\|\|(0,-1,0)\|} = \frac{0+1+0}{(\sqrt{1+1})(\sqrt{1})}$

$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$ (or $\theta = 45^0$)

$\Longrightarrow$

$\Rightarrow$ Q4 (c)

$$\cos\theta = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\|\vec{w}\|} = \frac{(1,-1,0)\cdot(-1,0,0)}{(\sqrt{1+1})(\sqrt{1})} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta = 3\pi/4 \quad (\text{or } \theta = 135^\circ)$$

(d) $$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{(0,-1,0)\cdot(-1,0,0)}{(\sqrt{1})(\sqrt{1})} = \frac{0}{1} = 0$$

$$\Rightarrow \theta = \pi/2 \quad (\text{or } \theta = 90^\circ) \quad \text{i.e. } \vec{v} \perp \vec{w}$$

Q5 [8] $$\|(\sqrt{3}, a, 2\sqrt{a})\| = \sqrt{(\sqrt{3})^2 + a^2 + (2\sqrt{a})^2} = \sqrt{3 + a^2 + 4a}$$

so must $\sqrt{a^2+4a+3} = 2\sqrt{2} \Rightarrow a^2+4a+3 = (2\sqrt{2})^2 = 8$

$$\Rightarrow a^2 + 4a - 5 = 0 \Rightarrow (a+5)(a-1) = 0 \Rightarrow \boxed{a=1},\ \underset{NA}{a=-5}$$

Q6 [8] $$\|\vec{u}\| = \sqrt{(-\sqrt{5})^2 + (\sqrt{2})^2} = \sqrt{5+2} = \sqrt{7}$$

so $\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{7}}(-\sqrt{5}, \sqrt{2}) = \left(-\frac{\sqrt{5}}{\sqrt{7}}, \frac{\sqrt{2}}{\sqrt{7}}\right)$ is a unit vector in the direction of $\vec{u}$.

$$\|\vec{v}\| = \sqrt{4+9+4} = \sqrt{17}$$

so $\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{17}}(2,3,-2) = \left(\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, -\frac{2}{\sqrt{17}}\right)$ is a unit vector in the direction of $\vec{v}$.

Q7 Let the point $\vec{Q}(x, y, z, u)$ be the tail of $\vec{u}$ i.e
[8]

$$\vec{u} = \overrightarrow{QP} \quad \text{but } \overrightarrow{QP} = (1, 6, -1, -2) - (x, y, z, u)$$
$$= (1-x,\ 6-y,\ -1-z,\ -2-u)$$

so must

$$(-1, 3, -3, 4) = (1-x,\ 6-y,\ -1-z,\ -2-u)$$

i.e
$$1-x=-1 \Rightarrow x=2$$
$$6-y=3 \Rightarrow y=3$$
$$-1-z=-3 \Rightarrow z=2$$
$$-2-u=4 \Rightarrow u=-6$$

so tail is the point $Q(2, 3, 2, -6)$.

Q8
[8]

$$\text{L.H.S.} = c \odot (A \oplus B) = c \odot (A+B+I) = c(A+B+I) = cA+cB+cI$$

$$\text{R.H.S} = (c \odot A) \oplus (c \odot B) = (cA) \oplus (cB) = cA+cB+I$$

since in general $cA+cB+cI \neq cA+cB+I$
for all c in $\mathbb{R}$ (except 1)

So in general L.H.S. is not equal to R.H.S.

i.e the property fails.

Total: 100 marks