Q1 [9] (a) In the third row must $a=0$ and $b\neq 0$ then it becomes $0 =$ a nonzero number, which is impossible.

(b) In the third row if $a\neq 0$, then for any number b we perform $R_3 \to \frac{1}{a}R_3$ so it becomes $[0 \quad 0 \quad 1 \quad \frac{b}{a}]$ which means $z = b/a$ and therefore the system has only one solution.

(c) In the third row if $a=0$ and $b=0$, then it becomes $[0 \quad 0 \quad 0 \quad 0]$ and the system will have infinitely many solutions.

Q2 [15] (a)

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] R_1 \to -R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to -2R_1+R_2 \\ R_3 \to -4R_1+R_4 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] R_2 \to -R_3+R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] R_2 \to -R_2 \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to 2R_2+R_1 \\ R_3 \to -6R_2+R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -8 & 6 & -5 \end{array}\right] R_3 \to -R_3 \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \begin{array}{l} R_1 \to -R_3+R_1 \\ R_2 \to R_3+R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$

(b)

$$\left[\begin{array}{ccc|ccc} -40 & 16 & 9 & 1 & 0 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right] R_1 \to 3R_2+R_1 \Rightarrow \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 3 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right] R_1 \to -R_1$$

$\Longrightarrow$

Q2 (b) $\Rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 13 & -5 & -3 & 0 & 1 & 0 \\ 5 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to -13R_1 + R_2 \\ R_3 \to -5R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & 8 & -3 & 13 & 40 & 0 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right] R_2 \to -3R_3 + R_2 \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & -1 & 0 & -2 & -5 & -3 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right] R_2 \to -R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 3 & -1 & 5 & 15 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_2 + R_1 \\ R_3 \to -3R_2 + R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 0 & -1 & -1 & 0 & -8 \end{array}\right] R_3 \to -R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 0 & 8 \end{array}\right]$$

so $B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(c) $$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ -5 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to -3R_1 + R_2 \\ R_3 \to 5R_1 + R_3 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 0 & 7 & 14 & 1 & -3 & 0 \\ 0 & -7 & -14 & 0 & 5 & 1 \end{array}\right] R_3 \to R_2 + R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 0 & 1 & 0 \\ 0 & 7 & 14 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

since the left hand side cannot become identity so the matrix C is not invertible.

Q3 first
[16]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] R_3 \to -2R_1 + R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] R_2 \to 2R_3 + R_2 \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to -R_2 + R_1 \\ R_3 \to R_2 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -1 & -2 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & 0 & -1 & -6 & 1 & 3 \end{array}\right] R_3 \to -R_3$$

$\Longrightarrow$

Q3 $\Rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -1 & -2 \\ 0 & 1 & -1 & -4 & 1 & 2 \\ 0 & 0 & 1 & 6 & -1 & -3 \end{array}\right]$ $\begin{array}{l} R_1 \to -3R_3 + R_1 \\ R_2 \to R_3 + R_2 \end{array}$ $\Rightarrow$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 2 & 7 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -3 \end{array}\right]$

So $A^{-1} = \begin{bmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{bmatrix}$

Now for part (a): $A\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow \vec{x} = A^{-1}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$ i.e $x=-7,\ y=1,\ z=4$

$(-7, 1, 4)$

For part (b): $(-3A)\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} \Rightarrow A\vec{x} = \frac{1}{-3}\begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow$

$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -27 \\ 4 \\ 12 \end{bmatrix}$

so $x = -27,\ y = 4,\ z = 0$ i.e $(-27, 4, 12)$

For part (c): $A^{-1}\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = (A^{-1})^{-1}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = A\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix} \Rightarrow$ $x = 3,\ y = -4,\ z = 7$

$(3, -4, 7)$

For part (d): $A^T\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{x} = (A^T)^{-1}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{x} = (A^{-1})^T\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 & 2 & 7 \\ 2 & 0 & -1 \\ 6 & -1 & -3 \end{bmatrix}^T\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -13 & 2 & 6 \\ 2 & 0 & -1 \\ 7 & -1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -37 \\ 6 \\ 20 \end{bmatrix} \Rightarrow$

$x = -37,\ y = 6,\ z = 20$ i.e $(-37, 6, 20)$.

Q4 (a)

[15] $\det(A) = +3 \det\begin{bmatrix} 6 & 2 \\ 8 & 9 \end{bmatrix} - 4 \det\begin{bmatrix} 5 & 2 \\ 1 & 9 \end{bmatrix} + 7 \det\begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$= 3(54-16) - 4(45-2) + 7(40-6)$

$= 3(38) - 4(43) + 7(34)$

$= 114 - 172 + 238$

$= 180$

(b) we use properties of determinant:

$$\det\begin{bmatrix} 2 & 7 & -3 & 0 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 0 & 3 \\ 4 & 15 & -6 & 0 \end{bmatrix} \; R_4 \to -2R_1 + R_4 \quad = \det\begin{bmatrix} 2 & 7 & -3 & 0 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \; R_2 \leftrightarrow -R_3$$

$$= (-1) \det\begin{bmatrix} 2 & 7 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 6 & 7 \end{bmatrix} \; \begin{matrix} R_3 \to -R_2 + R_3 \\ R_4 \to -2R_2 + R_4 \end{matrix}$$

$$= (-1) \det\begin{bmatrix} 2 & 7 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 6 & 7 \end{bmatrix} \; R_3 \leftrightarrow R_4$$

$$= (-1)(-1) \det\begin{bmatrix} 2 & 7 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$= +1\left(2(1)(6)(3)\right)$

$= 36$ (or you can expand along the first column of B)

Q4. (c)

$$\det(C) = \det\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & -4 \\ -5 & 3 & 6 \end{bmatrix} = 3\det\begin{bmatrix} -2 & -4 \\ 3 & 6 \end{bmatrix} - 1\det\begin{bmatrix} 1 & -4 \\ -5 & 6 \end{bmatrix} + 2\det\begin{bmatrix} 1 & -2 \\ -5 & 3 \end{bmatrix}$$

$$= 3(-12+12) - 1(6-20) + 2(3-10)$$

$$= 3(0) - 1(-14) + 2(-7)$$

$$= 0 + 14 + (-14)$$

$$= 0$$

Q5 Expansion along the first column for both sides gives:

[8]

$$x\begin{vmatrix} 0 & 1 \\ 3 & x \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 3 & x \end{vmatrix} = -2\begin{vmatrix} x & -1 \\ 1 & -2 \end{vmatrix}$$

$$x(0-3) + 1(2x-3) = -2(-2x+1)$$

$$-3x + 2x - 3 = +4x - 2$$

$$-5x = 1$$

$$x = -\frac{1}{5}$$

Q6 must $\det(A) = 0$ but

[8]

$$\det(A) = \det\begin{bmatrix} x & 1-x & 3 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{bmatrix} = x\begin{vmatrix} x & -1 \\ 1 & 1 \end{vmatrix} - (1-x)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & x \\ 2 & 1 \end{vmatrix}$$

$$= x(x+1) - (1-x)(1+2) + 3(1-2x)$$

$$= x^2 + x - 3x + 3x + 3 - 6x$$

$$= x^2 - 2x$$

$$= x(x-2)$$

Now when $\det(A) = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$

i.e if $x = 0$ or $x = 2$, then the matrix A is singular.

Q7
[12]

(1) $\det(AB^TC) = \det(A)\det(B^T)\det(C)$
$= \det(A)\det(B)\det(C)$
$= 3(-2)(10)$
$= -60$

(2) $\det(-2A^2B^{-1}) = (-2)^5 \det(A^2)\det(B^{-1})$
$= 32\,(\det(A))^2\left(\frac{1}{\det(B)}\right)$
$= -32\,(+3)^2\left(\frac{1}{-2}\right)$
$= 144$

(3) $\det(A^{-1}DB^{-3}D^{-1}) = \det(A^{-1})\det(D)\det(B^{-3})\det(D^{-1})$
$= \frac{1}{\det(A)} \cdot \det(D)\,(\det(B^{-1}))^3\det(D^{-1})$
$= \left(\frac{1}{\det(A)}\right)\left(\frac{1}{\det(B)}\right)^3 \cancel{(\det(D))}\cancel{\left(\frac{1}{\det(D)}\right)}$
$= \left(\frac{1}{\det(A)}\right)\left(\frac{1}{\det(B)}\right)^3$
$= \left(\frac{1}{3}\right)\left(\frac{1}{-2}\right)^3$
$= -\frac{1}{24}$

Q(8) (a) A is an elementary matrix because if we perform
[9]
$R_1 \to -2R_2 + R_1$ on $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we get $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(b) B is not an elementary matrix because two elementary row operations are needed to create B.

$\Longrightarrow$

Q8 (c) C is an elementary matrix, because if we perform $R_1 \to 2R_3 + R_1$ on $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we get C.

Q9 [8]

$(3,2)$ cofactor of $A = C_{32} = (-1)^{3+2} \det \begin{bmatrix} 5 & 1 \\ 3 & 0 \end{bmatrix} = (-1)(0-3) = +3$

$(2,3)$ cofactor of $A = C_{23} = (-1)^{2+3} \det \begin{bmatrix} 5 & -2 \\ 1 & 1 \end{bmatrix} = (-1)(5+2) = -7$

Also

$(3,2)$-cofactor of $B = C_{32} = (-1)^{3+2} \det \begin{bmatrix} 0 & 1 & -1 \\ 2 & 6 & 7 \\ 0 & 2 & 1 \end{bmatrix}$

$= (-1)\left(-2 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right)$

$= +2\,(1+2)$

$= 6$

$(2,3)$-cofactor of $B = C_{23} = (-1)^{2+3} \det \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$= (-1)\left(-1 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\right)$

$= +1\,(2+1)$

$= 3$

Total: 100 marks