CHAPTER 17 LAPLACE TRANSFORMS

The Laplace transform is one of many so-called *integral transforms* in applied mathematics. Through an improper integral, the Laplace transform creates an association between a class of functions denoted by $f(t)$ and a class of functions denoted by $F(s)$. The advantage of this association as far as our discussions are concerned is that solving a differential equation for $f(t)$ is replaced by solving an algebraic equation for $F(s)$. The fact that the Laplace transform is a linear operator (in the sense of equation 16.43) makes it particularly useful for solving the linear differential equations encountered in Chapter 16. Furthermore, you will recall that in Chapter 16 we assumed continuity of nonhomogeneous terms in linear differential equations. This was a matter of convenience rather than necessity. In Exercises 26 and 27 of Section 16.9, we hinted at the awkwardness of incorporating discontinuities into the techniques of Chapter 16. We shall give other examples of discontinuous nonhomogeneities in this chapter, and show how easily they are handled by Laplace transforms. This is perhaps the biggest advantage of Laplace transforms over the methods of Chapter 16. Discontinuous forcing functions in vibrating mass-spring systems and driving voltages in RCL-circuits are easily handled by Laplace transforms.

17.1 The Laplace Transform and its Inverse

Definition 17.1 When f is a function of t, its **Laplace transform** denoted by $F = \mathcal{L}\{f\}$ is a function with values defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{17.1}$$

provided the improper integral converges.

For our purposes, s is a real variable, in which case F is a real-valued function of a real variable s. The reader should be aware, however, that in advanced applications of Laplace transforms, especially for solving partial differential equations, it is necessary to take s as a complex number, in which case F is a complex-valued function of a complex variable.

We should determine properties of a function that guarantee existence of its Laplace transform. The following three examples point us in the correct direction.

Example 17.1 Find the Laplace transform of $f(t) = e^{at}$ where a is a constant.

Solution According to equation 17.1, the Laplace transform is defined by

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left\{\frac{1}{a-s}e^{(a-s)t}\right\}_0^\infty = \frac{1}{a-s}\left[\lim_{t\to\infty} e^{(a-s)t} - 1\right].$$

This limit exists, and has value 0, only when $s > a$. Hence, the Laplace transform of $f(t) = e^{at}$ is $1/(s-a)$, but only for $s > a$. We therefore write that

$$F(s) = \frac{1}{s-a}, \qquad s > a.\bullet$$

Example 17.2 Find the Laplace transform of $f(t) = t$.

Solution According to equation 17.1, the Laplace transform is defined by

$$F(s) = \int_0^\infty t\,e^{-st}\,dt.$$

Integration by parts leads to

$$F(s) = \left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_0^\infty = \lim_{t\to\infty}\left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right) + \frac{1}{s^2}.$$

This limit exists, and has value 0, only when $s > 0$. In other words, the Laplace transform of $f(t) = t$ is $F(s) = 1/s^2$, but the function is only defined for $s > 0$.•

Example 17.3 Find the Laplace transform of the discontinuous function $f(t) = \begin{cases} 2t^2, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$.

Solution According to equation 17.1, the Laplace transform is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 2t^2 e^{-st}\,dt + \int_1^\infty e^{-st}\,dt.$$

Two integrations by parts on the first integral lead to

$$F(s) = 2\left\{\left(-\frac{t^2}{s} - \frac{2t}{s^2} - \frac{2}{s^3}\right)e^{-st}\right\}_0^1 + \left\{\frac{-e^{-st}}{s}\right\}_1^\infty = -\left(\frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3}\right)e^{-s} + \frac{4}{s^3},$$

provided $s > 0$.

What have we learned from Definition 17.1 and these three examples? First, when the function $f(t)$ is discontinuous, it is necessary to subdivide the interval $0 < t < \infty$ into subintervals in which $f(t)$ is continuous. To avoid an infinite number of such subintervals, we could demand that $f(t)$ have a finite number of discontinuities. It turns out that this is not entirely necessary, although it is often the case. Instead, we demand that $f(t)$ have a finite number of discontinuities on every interval $0 \le t \le T$ of finite length. In addition, to guarantee existence of the integral of $e^{-st}f(t)$ on each finite subinterval, we demand that right- and left-hand limits of $f(t)$ exist at every discontinuity. When a function has a finite number of discontinuities on an interval and right- and left-hand limits exist at all discontinuities in the interval, the function is said to be **piecewise-continuous** on that interval. We are assuming therefore that $f(t)$ is piecewise continuous on every interval $0 \le t \le T$ of finite length.

The second thing that we saw in Examples 17.1–17.3 is that there is always a restriction on values of s. The function $F(s)$ is not defined for all s; it is defined only for s larger than some number (a in Example 17.1 and 0 in Examples 17.2 and 17.3). This is a direct result of the fact that for improper integral 17.1 to converge, the integrand must approach 0 as $t \to \infty$, and must do so sufficiently quickly. This means that $f(t)$ must not increase so rapidly that it cannot be suppressed by e^{-st} for some value of s. A sufficient restriction on the growth of $f(t)$ for large t is contained in the following definition.

Definition 17.2 A function $f(t)$ is said to be of exponential order α, written $O(e^{\alpha t})$, if there exist constants T and M such that $|f(t)| < Me^{\alpha t}$ for all $t > T$.

What this says algebraically is that for sufficiently large t ($t > T$), $|f(t)|$ must grow no faster than a constant M times $e^{\alpha t}$. Geometrically, the graph of $|f(t)|$ must be below that of $Me^{\alpha t}$ for $t > T$. It is important to realize that the exponential order of a function $f(t)$, if it has one, is concerned with function behaviour for very large t, not for small t. The absolute value $|f(t)|$ must eventually be less than $Me^{\alpha t}$, and stay less, but it need not be so for all t. For example, an exponential function $e^{\alpha t}$ is $O(e^{\alpha t})$ since M can be chosen as 2. Constant functions are of exponential order zero. The trigonometric functions $\sin at$ and $\cos at$ are $O(e^{0t})$ since both are less than $2 = 2e^{0t}$ for all t. The exponential order of t^n is discussed in the following example.

Example 17.4 Show that the function t^n, where n is a positive integer, is $O(e^{\epsilon t})$ for arbitrarily small, positive ϵ.

Solution Consider the function $f(t) = t^n e^{-\epsilon t}$ for arbitrary $\epsilon > 0$. To draw its graph we first calculate that $f'(t) = nt^{n-1}e^{-\epsilon t} - \epsilon t^n e^{-\epsilon t} = t^{n-1}e^{-\epsilon t}(n - \epsilon t)$. There is a relative maximum at $t = n/\epsilon$ and when this is combined with the fact that $\lim_{t\to\infty} t^n e^{-\epsilon t} = 0$, the graph in Figure 17.1 results. It shows that the function $t^n e^{-\epsilon t}$ is bounded by $M = (n/\epsilon)^n e^{-n}$ for all $t \ge 0$. In other words, $t^n e^{-\epsilon t} < 2M$ for all $t > 0$; that is, $t^n < 2Me^{\epsilon t}$ for $t > 0$, and t^n is $O(e^{\epsilon t})$.

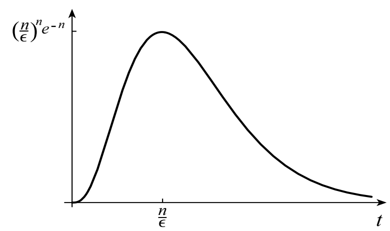


Figure 17.1

We now show that piecewise-continuous functions of exponential order always have Laplace transforms.

Theorem 17.1 If $f(t)$ is piecewise-continuous on every finite interval $0 \le t \le T$, and is of exponential order α, then its Laplace transform exists for $s > \alpha$.

Proof The improper integral in equation 17.1 can be divided into integrals over the intervals $0 \le t \le T$ and $T \le t \le \infty$, for any T,

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt.$$

Since $f(t)$ is piecewise-continuous on $0 \le t \le T$, there is no question that the first of these integrals exists. Furthermore, since $f(t)$ is $O(e^{\alpha t})$, there exist constants M and T such that $|f(t)| < Me^{\alpha t}$ for $t > T$. Hence,

$$\left|\int_T^\infty e^{-st} f(t)\,dt\right| \le \int_T^\infty e^{-st}|f(t)|\,dt < \int_T^\infty Me^{-st}e^{\alpha t}\,dt = \int_T^\infty Me^{(\alpha - s)t}\,dt$$
$$= \left\{\frac{M}{\alpha - s}e^{(\alpha - s)t}\right\}_T^\infty = \frac{M}{s - \alpha}e^{(\alpha - s)T},$$

provided $s > \alpha$. In other words, the improper integral over the interval $T \le t \le \infty$ converges when $s > \alpha$. Thus, the Laplace transform of $f(t)$ is defined for $s > \alpha$.

Theorem 17.1 provides sufficient conditions for existence of Laplace transforms. Functions that are not piecewise continuous or not of exponential order may or may not have transforms. For example, the function $f(t) = 1/\sqrt{t}$ is not piecewise continuous due to the infinite discontinuity at $t = 0$. It does, however, have a Laplace transform (see Exercise 34).

In calculating Laplace transforms of known functions by means of Definition 17.1, it is not necessary to determine whether the function is of exponential order prior to use of the integral; evaluation of the integral will yield the interval on which the transform is defined. When using techniques other than the defining integral to find Laplace transforms, however, it may be necessary to know that the function is of exponential order and piecewise-continuous on every finite interval. We shall develop other techniques in the next section. In this section we concentrate on the integral definition for Laplace transforms.

Example 17.5 Find the Laplace transform for $f(t) = t^n$, where n is a positive integer.

Solution Integration by parts gives

$$F(s) = \int_0^\infty t^n e^{-st}\,dt = \left\{\frac{t^n e^{-st}}{-s}\right\}_0^\infty - \int_0^\infty -\frac{n}{s}t^{n-1}e^{-st}\,dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt,$$

provided $s > 0$. A second integration by parts yields

$$F(s) = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt = \frac{n}{s}\left\{\frac{t^{n-1}e^{-st}}{-s}\right\}_0^\infty - \frac{n}{s}\int_0^\infty -\frac{n-1}{s}t^{n-2}e^{-st}\,dt$$
$$= \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}\,dt.$$

Further intergations by parts lead to

$$F(s) = \frac{n(n-1)(n-2)\cdots(1)}{s^n}\int_0^\infty e^{-st}\,dt = \frac{n!}{s^n}\left\{\frac{e^{-st}}{-s}\right\}_0^\infty = \frac{n!}{s^{n+1}},$$

provided again that $s > 0$. This is consistent with Theorem 17.1 and Example 17.4. According to Example 17.4, t^n is $O(e^{\epsilon t})$ for arbitrarily small, positive ϵ, and therefore its Laplace transform should exist for $s > \epsilon$ for arbitrarily small $\epsilon > 0$. This is tantamount to $s > 0$.•

Example 17.6 Find the Laplace transform for $f(t) = \sin at$, where a is a nonzero constant.

Solution According to equation 17.1,

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt.$$

Integration by parts with $u = \sin at$, $du = a\cos at\,dt$, $dv = e^{-st}\,dt$, and $v = -(1/s)e^{-st}$, gives

$$F(s) = \left\{-\frac{1}{s}e^{-st}\sin at\right\}_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}a\cos at\,dt = \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt,$$

provided $s > 0$. A second integration by parts with $u = \cos at$, $du = -a\sin at\,dt$, $dv = e^{-st}\,dt$, and $v = -(1/s)e^{-st}$, yields

$$F(s) = \frac{a}{s}\left\{-\frac{1}{s}e^{-st}\cos at\right\}_0^\infty - \frac{a}{s}\int_0^\infty -\frac{1}{s}e^{-st}(-a\sin at)\,dt = \frac{a}{s^2} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin at\,dt,$$

provided once again that $s > 0$. We can therefore write that

$$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2}F(s),$$

and when this equation is solved for $F(s)$, the result is $F(s) = a/(s^2 + a^2)$. An alternative derivation using complex exponentials is suggested in Exercise 33.•

The following table contains Laplace transforms of functions that occur very frequently in differential equations. They can be verified with equation 17.1.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$	e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$\cos at$	$\dfrac{s}{s^2+a^2}$
$t\sin at$	$\dfrac{2as}{(s^2+a^2)^2}$	$t\cos at$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
$\sin at + at\,\cos at$	$\dfrac{2as^2}{(s^2+a^2)^2}$	$\sin at - at\,\cos at$	$\dfrac{2a^3}{(s^2+a^2)^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$\cosh at$	$\dfrac{s}{s^2-a^2}$

Table 17.1

We have included transforms of the hyperbolic sine and cosine functions. We make no use of them here for the sake of those who have not studied hyperbolic functions. Those familiar with these functions will be able to provide simpler solutions to some of the examples and exercises.

The Inverse Laplace Transform

Definition 17.3 When F is the Laplace transform of f, we call f the inverse Laplace transform of F, and write

$$f = \mathcal{L}^{-1}\{F\}. \tag{17.2}$$

For instance, Table 17.1 yields $\mathcal{L}^{-1}\{1/(s+2)\} = e^{-2t}$ and $\mathcal{L}^{-1}\{s/(s^2+3)\} = \cos\sqrt{3}t$. The Laplace transform $F(s)$ of a function $f(t)$ is unique, every function has exactly one Laplace transform. On the other hand, many functions have the same transform. For example, the functions

$$f(t) = t^2 \qquad \text{and} \qquad g(t) = \begin{cases} 0, & t = 1 \\ t^2, & t \neq 1, 2 \\ 0, & t = 2 \end{cases},$$

which are identical except for their values at $t = 1$ and $t = 2$ both have the same transform $2/s^3$. The fact that $F(s) = 2/s^3$ follows from Table 17.1; $G(s) = 2/s^3$ follows from integration. What we are saying is that the inverse transform $f = \mathcal{L}^{-1}\{F\}$ in Definition 17.3 is not an inverse in the true sense of inverse; there are many possibilities for f for given F. In advanced work, a formula for calculating inverse transforms is derived, and this formula always yields a continuous function $f(t)$, when this is possible. In the event that this is not possible, the formula gives a piecewise-continuous function whose value is the average of right- and left-limits at discontinuities, namely $\lim_{\alpha\to 0}[f(t+\alpha) + f(t-\alpha)]/2$. The importance of this formula is that it defines $f = \mathcal{L}^{-1}\{F\}$ in a unique way. Other functions which have the same transform F differ from f only in their values at isolated points; they cannot differ from f over an entire interval $a \leq t \leq b$. When f is a continuous function with transform F, there cannot be another continuous function with the same transform. We adopt the procedure of always choosing a continuous function $\mathcal{L}^{-1}\{F\}$ for given F, or when this is not possible, a piecewise-continuous function.

According to the following theorem, the Laplace transform is a linear operator in the sense of equation 16.43; this is a direct result of the fact that integration is a linear operation.

Theorem 17.2 The Laplace transform and its inverse are linear operators; that is, for arbitrary functions f and g with transforms F and G, and an arbitrary constant c,

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}, \qquad \mathcal{L}\{c\,f\} = c\,[\mathcal{L}\{f\}], \tag{17.3a}$$

$$\mathcal{L}^{-1}\{F+G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\}, \qquad \mathcal{L}^{-1}\{c\,F\} = c\,[\mathcal{L}^{-1}\{F\}]. \tag{17.3b}$$

For instance, using linearity and Table 17.1,

$$\mathcal{L}\{2e^{-t} + 3\sin 4t\} = 2\mathcal{L}\{e^{-t}\} + 3\mathcal{L}\{\sin 4t\} = \frac{2}{s+1} + 3\left(\frac{4}{s^2+16}\right),$$

and

$$\mathcal{L}^{-1}\left\{\frac{2}{s^4} - \frac{4s}{s^2+5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = 2\left(\frac{t^3}{6}\right) - 4\cos\sqrt{5}t.$$

EXERCISES 17.1

In Exercises 1–10 use linearity and Table 17.1 to find the Laplace transform of the function.

1. $f(t) = t^3 - 2t^2 + 1$

2. $f(t) = t + e^t$

3. $f(t) = e^{4t}$

4. $f(t) = e^{-2t} + 2e^t$

5. $f(t) = \sin 4t + 3\cos 4t$

6. $f(t) = \cos 2t - 3\sin 4t$

7. $f(t) = t\,\cos 2t$

8. $f(t) = 3t\,\sin 4t$

9. $f(t) = 5t\,\cos t - 2t\,\sin t$

10. $f(t) = 3t\,\sin t - \cos t$

In Exercises 11–20 use linearity and Table 17.1 to find the inverse Laplace transform of the function.

11. $F(s) = \dfrac{1}{s^3}$

12. $F(s) = \dfrac{2}{s} - \dfrac{3}{s^4}$

13. $F(s) = \dfrac{1}{s+5} + \dfrac{4}{s^2}$

14. $F(s) = \dfrac{3}{s-1}$

15. $F(s) = \dfrac{s}{s^2+4} - \dfrac{3}{s^2+4}$

16. $F(s) = \dfrac{2s}{s^2+2} - \dfrac{5}{s^2+9}$

17. $F(s) = \dfrac{2s}{(s^2+2)^2}$

18. $F(s) = \dfrac{s^2}{(s^2+9)^2}$

19. $F(s) = \dfrac{3s - s^2}{(s^2+4)^2}$

20. $F(s) = \dfrac{s^2-2}{(s^2+3)^2}$

In Exercises 21–32 use Definition 17.1 to find the Laplace transform of the function. Treat a and b as constants.

21. $f(t) = \begin{cases} 0, & 0 < t < 3 \\ 1, & t > 3 \end{cases}$

22. $f(t) = \begin{cases} 1, & 0 < t < 4 \\ 2, & t > 4 \end{cases}$

23. $f(t) = \begin{cases} t, & 0 < t < 2 \\ 2, & t > 2 \end{cases}$

24. $f(t) = \begin{cases} t^2, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

25. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t^2, & t > 1 \end{cases}$

26. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ (t-1)^2, & t > 1 \end{cases}$

27. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

28. $f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

29. $f(t) = \begin{cases} 2t, & 0 < t < 1 \\ t, & t > 1 \end{cases}$

30. $f(t) = \begin{cases} 1 + t^2, & 0 < t < 1 \\ 2t, & t > 1 \end{cases}$

31. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & t > a \end{cases}$

32. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$

33. Use the expressions $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and Example 17.1 to find Laplace transforms for $\sin at$ and $\cos at$.

34. Show that the Laplace transform of $1/\sqrt{t}$ is $\sqrt{\pi/s}$. Hint: Set $u = \sqrt{t}$ in the definition of the Laplace transform of $1/\sqrt{t}$ in terms of a definite integral and use the fact that $\int_0^\infty e^{-u^2}du = \sqrt{\pi}/2$.

35. Are bounded functions (functions that satisfy $|f(t)| < M$ for all $t > 0$) of exponential order?

ANSWERS

1. $(s^3 - 4s + 6)/s^4$ **2.** $(s^2 + s - 1)/(s^3 - s^2)$ **3.** $1/(s - 4)$ **4.** $(3s + 3)/(s^2 + s - 2)$
5. $(3s + 4)/(s^2 + 16)$ **6.** $(s^3 - 12s^2 + 16s - 48)/(s^4 + 20s^2 + 64)$ **7.** $(s^2 - 4)/(s^2 + 4)^2$
8. $24s/(s^2 + 16)^2$ **9.** $(5s^2 - 4s - 5)/(s^2 + 1)^2$ **10.** $(5s - s^3)/(s^2 + 1)^2$ **11.** $t^2/2$
12. $2 - t^3/2$ **13.** $4t + e^{-5t}$ **14.** $3e^t$ **15.** $\cos 2t - (3/2)\sin 2t$ **16.** $2\cos\sqrt{2}t - (5/3)\sin 3t$
17. $(t/\sqrt{2})\sin\sqrt{2}t$ **18.** $(3t\cos 3t + \sin 3t)/6$ **19.** $(3t\sin 2t - 2t\cos 2t - \sin 2t)/4$
20. $(15t\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t)/18$ **21.** $(1/s)e^{-3s}$ **22.** $(1 + e^{-4s})/s$ **23.** $(1 - e^{-2s})/s^2$
24. $2/s^3 - e^{-s}(2 + 2s + s^2)/s^3$ **25.** $e^{-s}(s^2 + 2s + 2)/s^3$ **26.** $(2/s^3)e^{-s}$ **27.** $(e^{-s} - e^{-2s})/s$
28. $(1 - 2e^{-s} + e^{-2s})/s^2$ **29.** $2/s^2 - e^{-s}(s + 1)/s^2$ **30.** $1/s + 2(1 - e^{-s})/s^3$ **31.** $(1/s)e^{-as}$
32. $(e^{-as} - e^{-bs})/s$ **35.** Yes

17.2 Algebraic Properties of The Laplace Transform and its Inverse

Seldom is it necessary to evaluate the improper integral in Definition 17.1 to find the Laplace transform for a function; other techniques prove more efficient. Keep in mind that our intention is to use Laplace transforms to provide another method for solving linear differential equations. With this in mind, note how each of the algebraic properties of the Laplace transform uncovered in this section is directed toward the functions so prevalent in solving linear differential equations, namely, t^n, e^{at}, $\sin at$, $\cos at$, and sums and products of these functions. In Section 17.3, we derive formulas for taking Laplace transforms of derivatives of functions and use these formulas to solve differential equations.

One of two *shifting* properties is contained in the following theorem.

Theorem 17.3 When F is the Laplace transform of f:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \tag{17.4a}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t). \tag{17.4b}$$

Proof By Definition 17.1,

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}e^{-st}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt.$$

But this is equation 17.1 with s replaced by $s-a$; that is,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Equation 17.4b is 17.4a written in terms of inverse transforms rather than transforms.

The notation in equation 17.4 is not quite as described earlier. The Laplace transform and its inverse operate on functions, not on function values as is suggested by 17.4. These equations would be more properly stated in the form

$$\mathcal{L}\{e^{at}f\}(s) = F(s-a), \tag{17.4c}$$

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}f(t). \tag{17.4d}$$

We feel that the shifting property is more clearly conveyed for most readers by 17.4a,b, and we apologize to readers who are offended by the notation. It is convenient to repeat this practice in describing other properties of the Laplace transform, but we shall attempt to minimize its use.

Theorem 17.3 states that multiplication by an exponential e^{at} in the t-domain is equivalent to a translation or shift by a in the s-domain. It provides a quick way to find the Laplace transform of any function multiplied by an exponential, provided the Laplace transform of the function is known. For example, since $\mathcal{L}\{\cos 2t\} = s/(s^2+4)$, 17.4a implies that

$$\mathcal{L}\{e^{3t}\cos 2t\} = \frac{s-3}{(s-3)^2+4}.$$

Example 17.7 Find the Laplace transform for $f(t) = t^2e^{-5t}$.

Solution Since $\mathcal{L}\{t^2\} = 2/s^3$, property 17.4a gives $\mathcal{L}\{t^2e^{-5t}\} = \dfrac{2}{(s+5)^3}$.●

Example 17.8 Find the inverse Laplace transform for $F(s) = 1/(s^2-6s+14)$.

Solution First, by completing the square on the quadratic, we can express $F(s)$ in the form

$$F(s) = \frac{1}{(s-3)^2+5}.$$

We can now use property 17.4b to find the inverse transform of $f(t)$,

$$f(t) = e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\} = e^{3t}\frac{1}{\sqrt{5}}\sin\sqrt{5}t.\bullet$$

The second shifting property of the Laplace transform involves shifts in the t-domain rather than the s-domain. Such shifts are conveniently described by Heaviside's unit step functions (see Section 2.4). The fundamental unit step function is defined by

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}. \tag{17.5}$$

Its graph is shown in Figure 17.2; there is a discontinuity of magnitude unity at $t = 0$.

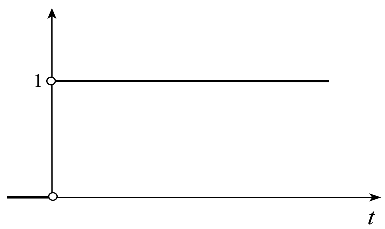


Figure 17.2

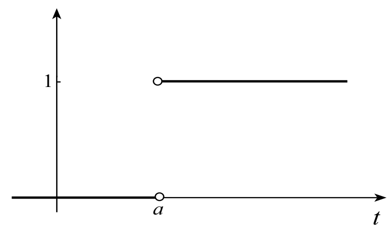


Figure 17.3

When the discontinuity occurs at $t = a$, the function is denoted by

$$h(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}. \tag{17.6}$$

Its graph is shown in Figure 17.3.

Heaviside unit step functions provide compact descriptions to functions with discontinuities. One of the most important is shown in Figure 17.4. It is called a *pulse* function. It can be expressed algebraically in the form $h(t-a) - h(t-b)$. In the event that the height of the nonzero portion is c rather than unity (Figure 17.5), we obtain $c[h(t-a) - h(t-b)]$.

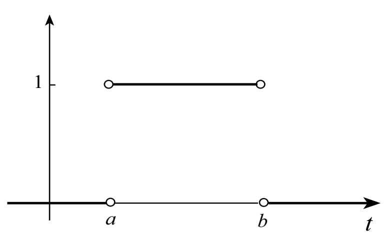


Figure 17.4

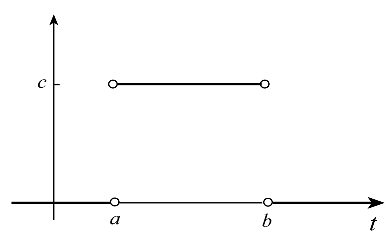


Figure 17.5

Pulse functions can be combined algebraically to produce step functions. The function in Figure 17.6 is the sum of two pulse functions,

$$4[h(t) - h(t-3)] + 2[h(t-3) - h(t-6)] = 4h(t) - 2h(t-3) - 2h(t-6).$$

The function in Figure 17.7 is the sum of three pulses,

$$\begin{aligned} 3[h(t-a) - h(t-b)] + 4[h(t-b) - h(t-c)] + h(t-c) \\ = 3h(t-a) + h(t-b) - 3h(t-c). \end{aligned}$$

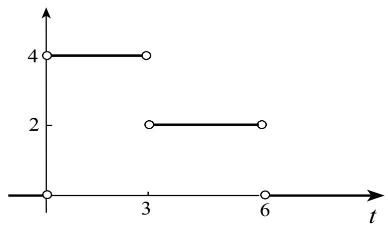


Figure 17.6

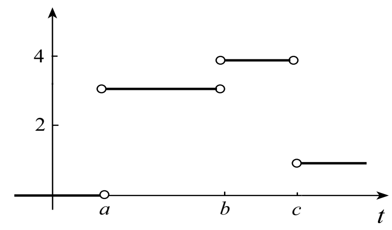


Figure 17.7

A convenient representation for the function in Figure 17.8 is $t^2[h(t) - h(t-a)]$, and for the function in Figure 17.9, $[2 - (t-a)/(b-a)][h(t-a) - h(t-b)]$.

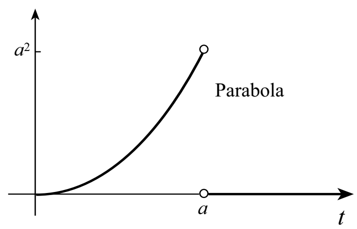


Figure 17.8

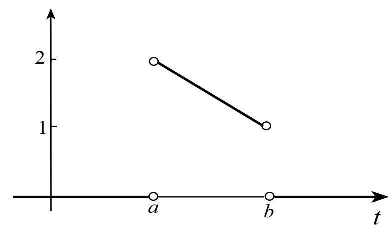


Figure 17.9

What these examples illustrate is that to "turn a function on" over the interval $a < t < b$, multiply it by $h(t-a) - h(t-b)$. It will be zero for $t < a$ and $t > b$. To turn a function on for $t > a$, multiply it by $h(t-a)$. The parabola in Figure 17.10 has equation $a^2 + (t-a)^2$ for $t > a$. To turn it on, we multiply by $h(t-a)$; that is, the function can be expressed in the form $[a^2 + (t-a)^2]h(t-a)$. For the function in Figure 17.11, we turn on the straight line $y = a - a(t-a)/(b-a)$ for $a < t < b$, and then the horizontal line $y = c$ for $t > b$,

$$[a - a(t-a)/(b-a)][h(t-a) - h(t-b)] + c\,h(t-b).$$

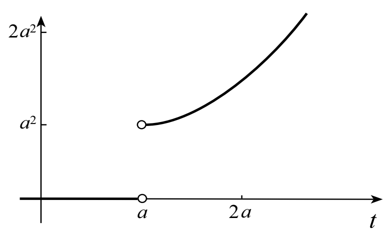


Figure 17.10

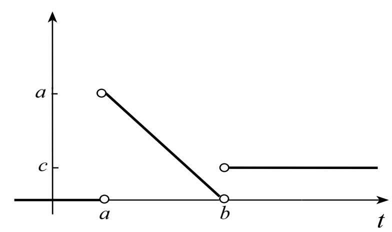


Figure 17.11

The Laplace transform of the Heaviside unit step function is

$$\mathcal{L}\{h(t-a)\} = \int_0^\infty e^{-st}h(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_a^\infty = \frac{e^{-as}}{s}, \tag{17.7}$$

provided $s > 0$. To find the Laplace transform of a function that is the product of a Heaviside function and another function we use the following theorem.

Theorem 17.4 When $f(t)$ has a Laplace transform,

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}. \tag{17.8a}$$

Proof If the graph of $f(t)$ is as shown in Figure 17.12a, the graph of $f(t)h(t-a)$ is shown in Figure 17.12b. It is that of $f(t)$ turned on for $t > a$. According to Definition 17.1,

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^\infty e^{-st} f(t)h(t-a)\,dt = \int_a^\infty e^{-st} f(t)\,dt.$$

If we change variables of integration with $u = t - a$, then

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^\infty e^{-s(u+a)} f(u+a)\,du$$
$$= e^{-as} \int_0^\infty e^{-su} f(u+a)\,du = e^{-as}\mathcal{L}\{f(t+a)\}.$$

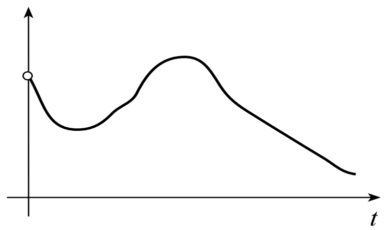


Figure 17.12a

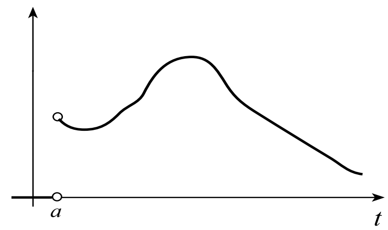


Figure 17.12b

In Section 17.1 we used integration to find the Laplace transform of discontinuous functions. Property 17.8a provides a convenient alternative, provided the discontinuous function can be expressed in terms of Heaviside functions. We illustrate in the following examples.

Example 17.9 Find the Laplace transform for the function $f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ (t-2)^2, & t > 2 \end{cases}$, shown in Figure 17.13.

Solution Since $f(t)$ can be expressed in the form $f(t) = (t-2)^2 h(t-2)$, except for its value at $t = 2$, equation 17.8a gives

$$F(s) = \mathcal{L}\{(t-2)^2 h(t-2)\} = e^{-2s}\mathcal{L}\{t^2\} = \frac{2e^{-2s}}{s^3}.\bullet$$

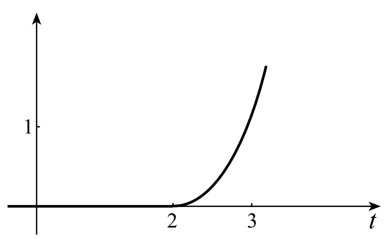


Figure 17.13

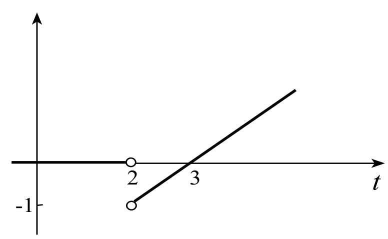


Figure 17.14

Example 17.10 Find the Laplace transform for the function $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ t-3, & t > 2 \end{cases}$, shown in Figure 17.14.

Solution Since $f(t)$ can be expressed in the form $f(t) = (t-3)h(t-2)$, its Laplace transform is

$$F(s) = \mathcal{L}\{(t-3)h(t-2)\} = e^{-2s}\mathcal{L}\{t-1\} = e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right).\bullet$$

Example 17.11 Find the Laplace transform for the function in Figure 17.15.

Solution The function is continuous, but because it is defined differently on the intervals $0 \le t \le 1$, $1 < t \le 2$, and $t > 2$, it can be represented most efficiently in terms of Heaviside functions (except for its values at $t = 1$ and $t = 2$),

$$\begin{aligned} f(t) &= 3(t-1)[h(t-1) - h(t-2)] + 3h(t-2) \\ &= 3(t-1)\,h(t-1) + (6-3t)\,h(t-2). \end{aligned}$$

We can now use equation 17.8a to find its Laplace transform,

$$F(s) = e^{-s}\mathcal{L}\{3t\} + e^{-2s}\mathcal{L}\{6 - 3(t+2)\} = \frac{3e^{-s}}{s^2} - e^{-2s}\left(\frac{3}{s^2}\right). \bullet$$

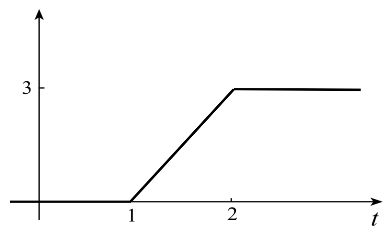


Figure 17.15

Example 17.12 Find the Laplace transform for $e^{-3t}\sin 2t\,h(t-1)$.

Solution Using property 17.8a,

$$\begin{aligned} \mathcal{L}\{e^{-3t}\sin 2t\,h(t-1)\} &= e^{-s}\mathcal{L}\{e^{-3(t+1)}\sin 2(t+1)\} \\ &= e^{-s}\mathcal{L}\{e^{-3}e^{-3t}\sin 2(t+1)\} \\ &= e^{-s-3}\mathcal{L}\{e^{-3t}\sin 2(t+1)\}. \end{aligned}$$

Since

$$\mathcal{L}\{\sin 2(t+1)\} = \mathcal{L}\{\cos 2\,\sin 2t + \sin 2\,\cos 2t\} = \frac{(\cos 2)2}{s^2+4} + \frac{(\sin 2)s}{s^2+4},$$

we can use property 17.4a to write

$$\mathcal{L}\{e^{-3t}\sin 2(t+1)\} = \frac{(\cos 2)2}{(s+3)^2+4} + \frac{(\sin 2)(s+3)}{(s+3)^2+4}.$$

Consequently,

$$\mathcal{L}\{e^{-3t}\sin 2t\,h(t-1)\} = e^{-s-3}\left[\frac{2\cos 2}{(s+3)^2+4} + \frac{(\sin 2)(s+3)}{(s+3)^2+4}\right]. \bullet$$

The equivalent of property 17.8a in terms of inverse transforms is equally as important as 17.8a itself, and it is from the inverse statement that it gets its name the *second shifting property* of Laplace transforms. We state it as a corollary to Theorem 17.4.

Corollary 17.4.1 If $f = \mathcal{L}^{-1}\{F\}$, then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)h(t-a). \tag{17.8b}$$

The graph of $f(t-a)h(t-a)$ is that of $f(t)$ (Figure 17.16a) shifted a units to the right and turned on for $t > a$ (Figure 17.16b).

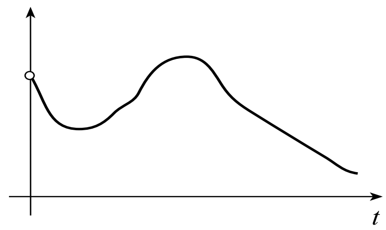


Figure 17.16a

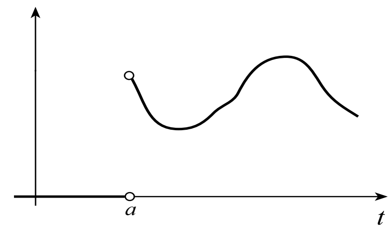


Figure 17.16b

Thus, to find the inverse transform of a function in the form $e^{-as}F(s)$, we find the inverse transform of $F(s)$, translate it a units to the right, and turn it on for $t > a$. For example, since $\mathcal{L}^{-1}\{2/s^3\} = t^2$, it follows that

$$\mathcal{L}^{-1}\left\{\frac{2e^{-4s}}{s^3}\right\} = (t-4)^2 h(t-4).$$

A graph of $(t-4)^2 h(t-4)$, except for its value at $t = 4$, is shown in Figure 17.17.

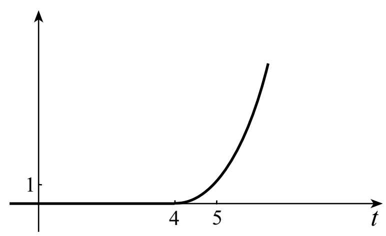


Figure 17.17

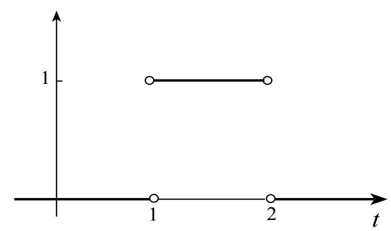


Figure 17.18

Example 17.13 Find the inverse transform for $F(s) = \dfrac{e^{-s} - e^{-2s}}{s}$.

Solution Since $\mathcal{L}^{-1}\{1/s\} = 1$, property 17.8b gives

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} = h(t-1) - h(t-2).$$

This also follows from equation 17.7. The function is shown in Figure 17.18.•

Finding inverse transforms is often a matter of finding the partial fraction decomposition of a rational function, together with the above properties and a set of tables. We illustrate in the following example.

Example 17.14 Find inverse Laplace transforms for the following functions:

$$\text{(a) } F(s) = \frac{s^2 - 9s + 9}{s^3(s^2+9)} \quad \text{(b) } F(s) = \frac{e^{-s}}{s^2 - s} \quad \text{(c) } F(s) = \frac{1}{s^2(s^2-4)}$$

Solution (a) The partial fraction decomposition of $F(s)$ is

$$F(s) = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s^2+9}.$$

Table 17.1 therefore gives $f(t) = \dfrac{t^2}{2} - t + \dfrac{1}{3}\sin 3t$.

(b) Partial fractions give $\dfrac{1}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$, and therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = e^t - 1.$$

Property 17.8b now gives $\mathcal{L}^{-1}\{F(s)\} = (e^{t-1} - 1)h(t-1)$.

(c) Partial fractions give $\dfrac{1}{s^2(s^2-4)} = \dfrac{1/16}{s-2} - \dfrac{1/16}{s+2} - \dfrac{1/4}{s^2}$, and therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-4)}\right\} = \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{t}{4}.\bullet$$

Periodic Functions

Periodic functions play a fundamental role in many applications of differential equations. The integration in Definition 17.1 for a function f with period p, can be replaced by an integral over the interval $0 \le t \le p$,

$$F(s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt. \tag{17.9}$$

To verify this, we first write the integral in equation 17.1 as an infinite series of integrals

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \sum_{n=0}^\infty \int_{np}^{(n+1)p} e^{-st} f(t)\,dt.$$

If we change variables of integration with $u = t - np$, then

$$\begin{aligned} F(s) &= \sum_{n=0}^\infty \int_0^p e^{-s(u+np)} f(u+np)\,du = \sum_{n=0}^\infty e^{-nps}\int_0^p e^{-su} f(u)\,du \\ &= \left(\int_0^p e^{-su} f(u)\,du\right)\left(\sum_{n=0}^\infty e^{-nps}\right). \end{aligned}$$

Since the series is geometric with common ratio e^{-ps},

$$F(s) = \int_0^p e^{-su} f(u)\,du \left[\frac{1}{1-e^{-ps}}\right] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt.$$

For example, the function in Figure 17.19 has period 2, and therefore

$$F(s) = \frac{1}{1-e^{-2s}}\int_0^2 (1-t)e^{-st}\,dt.$$

Integration by parts gives

$$F(s) = \frac{1}{1-e^{-2s}}\left\{\frac{(t-1)}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right\}_0^2 = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}.$$

We can avoid integration by parts by interpreting the integral over the interval $0 \le t \le 2$ as the Laplace transform of the function in Figure 17.20. Its Laplace transform is

$$\begin{aligned} \mathcal{L}\{(1-t)[h(t) - h(t-2)]\} &= \mathcal{L}\{(1-t)\,h(t)\} + \mathcal{L}\{(t-1)\,h(t-2)\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\mathcal{L}\{t+1\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right). \end{aligned}$$

Hence,

$$F(s) = \frac{1}{1-e^{-2s}}\left[\frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] = \frac{1+e^{-2s}}{s(1-e^{-2s})} - \frac{1}{s^2}.$$

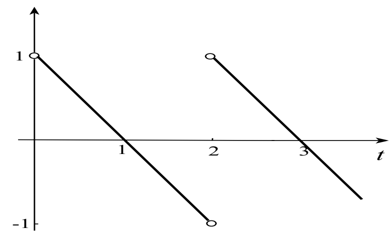


Figure 17.19

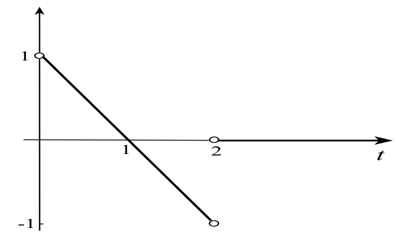


Figure 17.20

Example 17.15 Find the Laplace transform for $|\sin 2t|$.

Solution Since $|\sin 2t|$ has period $\pi/2$ (see Figure 17.21), formula 17.9 gives

$$\mathcal{L}\{|\sin 2t|\} = \frac{1}{1-e^{-\pi s/2}}\int_0^{\pi/2} e^{-st}\sin 2t\,dt.$$

To avoid integrations by parts, we write that

$$\begin{aligned}\mathcal{L}\{|\sin 2t|\} &= \frac{1}{1-e^{-\pi s/2}}\mathcal{L}\{\sin 2t[h(t)-h(t-\pi/2)]\}\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\mathcal{L}\{h(t)\sin 2t\} - \mathcal{L}\{\sin 2t\,h(t-\pi/2)\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} - e^{-\pi s/2}\mathcal{L}\{\sin 2(t+\pi/2)\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} + e^{-\pi s/2}\mathcal{L}\{\sin 2t\}\right]\\ &= \frac{1}{1-e^{-\pi s/2}}\left[\frac{2}{s^2+4} + \frac{2e^{-\pi s/2}}{s^2+4}\right]\\ &= \frac{2(1+e^{-\pi s/2})}{(s^2+4)(1-e^{-\pi s/2})}.\bullet\end{aligned}$$

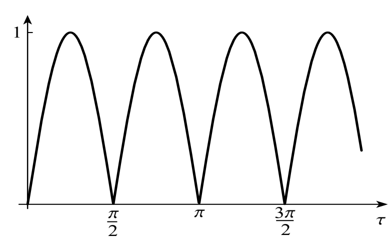


Figure 17.21

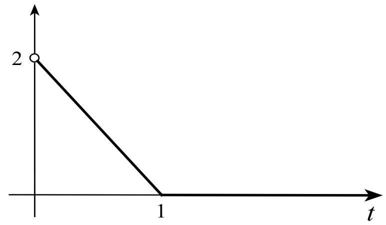
There are other algebraic properties of the Laplace transform and its inverse, but the ones discussed here suffice for our purposes.

EXERCISES 17.2

In Exercises 1–12 represent the functions in Exercises 21–32 of Section 17.1 in terms of Heaviside unit step functions. Find the Laplace transform of each function.

In Exercises 13–20 represent the function algebraically in terms of Heaviside unit step functions. Find the Laplace transform of each function.

13.



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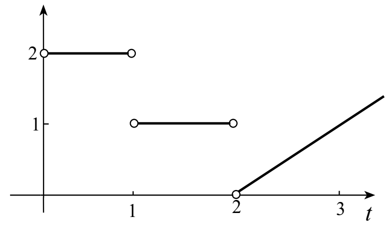

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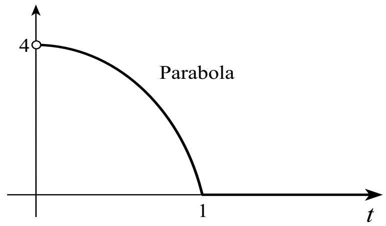

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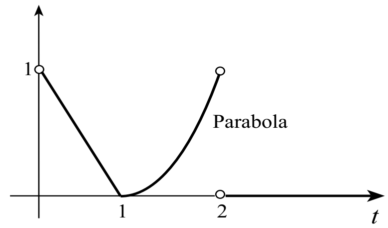

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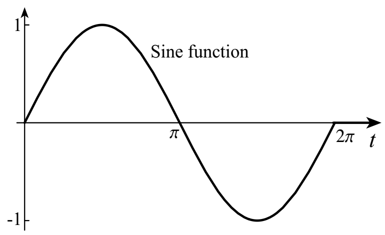

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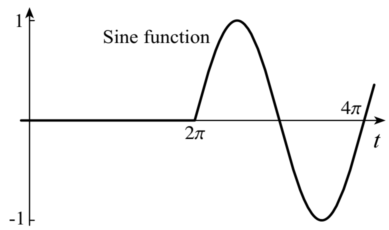

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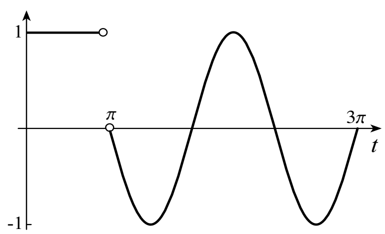

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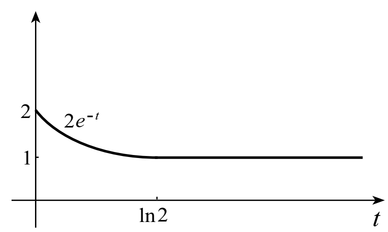


In Exercises 21–30 use property 17.4a to find the Laplace transform for the function.

21. $f(t) = t^3 e^{-5t}$

22. $f(t) = t^2 e^{3t}$

23. $f(t) = 4te^{-t} - 2e^{-3t}$

24. $f(t) = 5e^{at} - 5e^{-at}$

25. $f(t) = e^t \sin 2t + e^{-t} \cos t$

26. $f(t) = 2e^{-3t} \sin 3t + 4e^{3t} \cos 3t$

27. $f(t) = te^t \cos 2t$

28. $f(t) = te^{-2t} \sin t$

29. $f(t) = 2e^t(\cos t + \sin t)$

30. $f(t) = (t-1)e^{2-3t} \sin 4t$

In Exercises 31–40 use property 17.8a to find the Laplace transform of the function.

31. $f(t) = (t-2)^2 h(t-2)$

32. $f(t) = \sin 3(t-4)\, h(t-4)$

33. $f(t) = t\, h(t-1)$

34. $f(t) = (t+5)\, h(t-3)$

35. $f(t) = (t^2+2)\, h(t-1)$

36. $f(t) = \cos t\, h(t-\pi)$

37. $f(t) = \cos t\, h(t-2)$

38. $f(t) = e^t h(t-4)$

39. $f(t) = t^2 e^t\, h(t-3)$

40. $f(t) = e^t \cos 2t\, h(t-1)$

In Exercises 41–45 find the Laplace transform of the periodic function.

41. $f(t) = t, \quad 0 < t < a, \quad f(t+a) = f(t)$

42. $f(t) = \begin{cases} 1, & 0 < t < a \\ -1, & a < t < 2a \end{cases} \qquad f(t+2a) = f(t)$

43. $f(t) = |\sin at|$

44. $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \qquad f(t + 2a) = f(t)$

45. $f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \end{cases} \qquad f(t + 2a) = f(t)$

Find the inverse Laplace transform in Exercises 46–62.

46. $F(s) = \dfrac{1}{s^2 - 2s + 5}$

47. $F(s) = \dfrac{s}{s^2 + 4s + 1}$

48. $F(s) = \dfrac{e^{-2s}}{s^2}$

49. $F(s) = \dfrac{e^{-3s}}{s^2 + 1}$

50. $F(s) = \dfrac{se^{-5s}}{s^2 + 2}$

51. $F(s) = \dfrac{se^{-s}}{(s^2 + 4)^2}$

52. $F(s) = \dfrac{1}{4s^2 - 6s - 5}$

53. $F(s) = \dfrac{s}{s^2 - 3s + 2}$

54. $F(s) = \dfrac{4s + 1}{(s^2 + s)(4s^2 - 1)}$

55. $F(s) = \dfrac{e^{-3s}}{s + 5}$

56. $F(s) = \dfrac{e^{-2s}}{s^2 + 3s + 2}$

57. $F(s) = \dfrac{1}{s^3 + 1}$

58. $F(s) = \dfrac{5s - 2}{3s^2 + 4s + 8}$

59. $F(s) = \dfrac{e^{-s}(1 - e^{-s})}{s(s^2 + 1)}$

60. $F(s) = \dfrac{s}{(s + 1)^5}$

61. $F(s) = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$

62. $F(s) = \dfrac{s^2}{(s^2 - 4)^2}$

63. If $F(s) = \mathcal{L}\{f(t)\}$ for $s > \alpha$, for what values of s is $F(s - a)$ the Laplace transform of $e^{at}f(t)$?

64. Find the Laplace transform of the function

$$f(t) = \begin{cases} t^2/4, & 0 \le t < 1 \\ -(t^2 - 4t + 2)/4, & 1 \le t < 3 \\ (t - 4)^2/4, & 3 \le t \le 4 \end{cases} \qquad f(t + 4) = f(t).$$

65. Verify the *change of scale* property: If $F(s) = \mathcal{L}\{f(t)\}$ for $s > \alpha$, then for $a > 0$,

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad s > \alpha a.$$

ANSWERS

1. $h(t - 3)$; $(1/s)e^{-3s}$ **2.** $h(t) + h(t - 4)$; $(1 + e^{-4s})/s$ **3.** $t\,h(t) + (2 - t)\,h(t - 2)$; $(1 - e^{-2s})/s^2$
4. $t^2\,[h(t) - h(t - 1)]$; $2/s^3 - e^{-s}(2 + 2s + s^2)/s^3$ **5.** $t^2\,h(t - 1)$; $e^{-s}(2 + 2s + s^2)/s^3$
6. $(t - 1)^2 h(t - 1)$; $(2/s^3)e^{-s}$ **7.** $h(t - 1) - h(t - 2)$; $(e^{-s} - e^{-2s})/s$
8. $t\,h(t) - (2 - 2t)\,h(t - 1) + (t - 2)\,h(t - 2)$; $(1 - 2e^{-s} + e^{-2s})/s^2$
9. $2t\,h(t) - t\,h(t - 1)$; $2/s^2 - e^{-s}(s + 1)/s^2$
10. $(t^2 + 1)\,h(t) + (-t^2 + 2t + 1)\,h(t - 1)$; $1/s + 2(1 - e^{-s})/s^3$ **11.** $h(t - a)$; $(1/s)e^{-as}$
12. $h(t - a) - h(t - b)$; $(e^{-as} - e^{-bs})/s$
13. $2(1 - t)[h(t) - h(t - 1)]$, $t \neq 1$; $2(s - 1)/s^2 + (2/s^2)e^{-s}$
14. $2\,h(t) - h(t - 1) + (t - 3)\,h(t - 2)$; $(2 - e^{-s})/s + e^{-2s}(1 - s)/s^2$
15. $4(1 - t^2)[h(t) - h(t - 1)]$, $t \neq 1$; $4(s^2 - 2)/s^3 + 8e^{-s}(s + 1)/s^3$
16. $(1 - t)\,h(t) + (t^2 + 3t - 2)\,h(t - 1) - (t - 1)^2\,h(t - 2)$, $t \neq 1$; $(s - 1)/s^2 + e^{-s}(s + 2)/s^3 - e^{-2s}(s^2 + 2s + 2)/s^3$ **17.** $\sin t\,[h(t) - h(t - 2\pi)]$, $t \neq 0, 2\pi$; $(1 - e^{-2\pi s})/(s^2 + 1)$

18. $h(t-2\pi)\sin t,\ t \neq 2\pi;\ e^{-2\pi s}/(s^2+1)$
19. $h(t)+(-1+\sin t)\,h(t-\pi);\ (1-e^{-\pi s})/s - e^{-\pi s}/(s^2+1)$
20. $2e^{-t}\,h(t)+(1-2e^{-t})\,h(t-\ln 2),\ t\neq \ln 2;\ 2/(s+1)+e^{-s\ln 2}/(s^2+s)$ **21.** $6/(s+5)^4$
22. $2/(s-3)^3$ **23.** $4/(s+1)^2-2/(s+3)$ **24.** $10a/(s^2-a^2)$
25. $2/(s^2-2s+5)+(s+1)/(s^2+2s+2)$
26. $6/(s^2+6s+18)+4(s-3)/(s^2-6s+18)$ **27.** $(s^2-2s-3)/(s^2-2s+5)^2$
28. $(2s+4)/(s^2+4s+5)^2$ **29.** $2s/(s^2-2s+2)$ **30.** $-4e^2(s^2+4s+19)/(s^2+6s+25)^2$
31. $(2/s^3)e^{-2s}$ **32.** $3e^{-4s}/(s^2+9)$ **33.** $(s+1)e^{-s}/s^2$ **34.** $(8s+1)e^{-3s}/s^2$
35. $(3s^2+2s+2)e^{-s}/s^3$ **36.** $-se^{-\pi s}/(s^2+1)$ **37.** $(s\cos 2-\sin 2)e^{-2s}/(s^2+1)$
38. $e^{4-4s}/(s-1)$ **39.** $(9s^2-12s+5)e^{3-3s}/(s-1)^3$ **40.** $[(s-1)\cos 2-2\sin 2]e^{1-s}/(s^2-2s+5)$
41. $[1-e^{-as}(1+as)]/[s^2(1-e^{-as})]$ **42.** $(1-e^{-as})/[s(1+e^{-as})]$
43. $a(1+e^{-\pi s/a})/[(s^2+a^2)(1-e^{-\pi s/a})]$ **44.** $(1-e^{-as})/[s^2(1+e^{-as})]$
45. $1/[s(1+e^{-as})]$ **46.** $(1/2)e^t\sin 2t$ **47.** $(1/2-1/\sqrt{3})e^{(\sqrt{3}-2)t}+(1/2+1/\sqrt{3})e^{-(\sqrt{3}+2)t}$
48. $(t-2)\,h(t-2)$ **49.** $\sin(t-3)\,h(t-3)$ **50.** $\cos\sqrt{2}(t-5)\,h(t-5)$
51. $(1/4)(t-1)\sin 2(t-1)\,h(t-1)$ **52.** $(\sqrt{29}/58)[e^{(3+\sqrt{29})t/4}-e^{(3-\sqrt{29})t/4}]$ **53.** $2e^{2t}-e^t$
54. $e^{-t}-1+e^{t/2}-e^{-t/2}$ **55.** $e^{-5(t-3)}\,h(t-3)$ **56.** $[e^{-(t-2)}-e^{-2(t-2)}]h(t-2)$
57. $(1/3)e^{-t}+(1/3)e^{t/2}[\sqrt{3}\sin(\sqrt{3}t/2)-\cos(\sqrt{3}t/2)]$
58. $(1/3)e^{-2t/3}[5\cos(2\sqrt{5}t/3)-(8/\sqrt{5})\sin(2\sqrt{5}t/3)]$
59. $[1-\cos(t-1)]h(t-1)-[1-\cos(t-2)]h(t-2)$
60. $t^3e^{-t}(4-t)/24$ **61.** $(1/3)e^{-t}(\sin t+\sin 2t)$ **62.** $e^{2t}(2t+1)/8+e^{-2t}(2t-1)/8$
63. $s>\alpha+a$ **64.** $(1-e^{-s})^2/[2s^3(1+e^{-2s})]$

17.3 Laplace Transforms and Differential Equations

The Laplace transform is a powerful technique for solving ordinary and partial differential equations. It replaces differentiations with algebraic operations. The following theorem and its corollary simplify this process.

Theorem 17.5 Suppose f is continuous for $t \geq 0$ with a piecewise-continuous first derivative on every finite interval $0 \leq t \leq T$. If f is $O(e^{\alpha t})$, then $\mathcal{L}\{f'\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{17.10}$$

(A more precise representation of the left side of this equation is $\mathcal{L}\{f'\}(s)$.)

Proof If t_j, $j = 1, \ldots, n$ denote the discontinuities of f' in $0 \leq t \leq T$, then

$$\int_0^T e^{-st} f'(t)\, dt = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} e^{-st} f'(t)\, dt,$$

where $t_0 = 0$ and $t_{n+1} = T$. Since f' is continuous on each subinterval, we may integrate by parts on these subintervals,

$$\int_0^T e^{-st} f'(t)\, dt = \sum_{j=0}^{n} \left[\left\{ e^{-st} f(t) \right\}_{t_j}^{t_{j+1}} + s \int_{t_j}^{t_{j+1}} e^{-st} f(t)\, dt \right].$$

Because f is continuous, $f(t_j+) = f(t_j-)$, $j = 1, \ldots, n$, and therefore

$$\int_0^T e^{-st} f'(t)\, dt = -f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\, dt.$$

Thus,

$$\begin{aligned}
\mathcal{L}\{f'\} &= \int_0^\infty e^{-st} f'(t)\, dt = \lim_{T \to \infty} \int_0^T e^{-st} f'(t)\, dt \\
&= \lim_{T \to \infty} \left[-f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\, dt \right] \\
&= sF(s) - f(0) + \lim_{T \to \infty} e^{-sT} f(T),
\end{aligned}$$

provided the limit on the right exists. Since f is $O(e^{\alpha t})$, there exists M and $\overline{T}$ such that for $t > \overline{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \overline{T}$,

$$e^{-sT} |f(T)| < e^{-sT} M e^{\alpha T} = M e^{(\alpha - s)T}$$

which approaches 0 as $T \to \infty$ (provided $s > \alpha$). Consequently,

$$\mathcal{L}\{f'\} = sF(s) - f(0).$$

This result is easily extended to second and higher order derivatives. For extensions when f is only piecewise-continuous, see Exercise 45.

Corollary 17.5.1 Suppose f and f' are continuous for $t \geq 0$, and f'' is piecewise-continuous on every finite interval $0 \leq t \leq T$. If f and f' are $O(e^{\alpha t})$, then $\mathcal{L}\{f''\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f''\} = s^2 F(s) - sf(0) - f'(0). \tag{17.11}$$

Proof Since f' is continuous, f'' is piecewise-continuous, and f' is $O(e^{\alpha t})$, equation 17.10 gives

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0).$$

We can apply equation 17.10 once again to obtain

$$\mathcal{L}\{f''\} = s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0).$$

The extension to n^{th}-order derivatives is contained in the next corollary.

Corollary 17.5.2 Suppose f and its first $n-1$ derivatives are continuous for $t \geq 0$, and $f^{(n)}(t)$ is piecewise-continuous on every finite interval $0 \leq t \leq T$. If f and its first $n-1$ derivatives are $O(e^{\alpha t})$, then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > \alpha$, and

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{17.12}$$

We now show how to use Laplace transforms to solve ordinary differential equations, beginning with the initial-value problem in the following example.

Example 17.16 Solve the initial-value problem

$$y'' - 2y' + y = 2e^t, \quad y(0) = y'(0) = 0.$$

Solution First we assume that the solution of the problem is a function satisfying the conditions of Corollary 17.5.1 . We can then take Laplace transforms of both sides of the differential equation,

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2\mathcal{L}\{e^t\}.$$

Properties 17.10 and 17.11 yield

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + Y(s) = \frac{2}{s-1}.$$

We now substitute from the initial conditions $y(0) = y'(0) = 0$,

$$s^2Y(s) - 2sY(s) + Y(s) = \frac{2}{s-1},$$

and solve this equation for $Y(s)$,

$$Y(s) = \frac{2}{(s-1)^3}.$$

The required function $y(t)$ can now be obtained by taking the inverse transform of $Y(s)$,

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} \qquad \text{(by linearity)} \\
&= 2e^t\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \qquad \text{(by 17.4b)} \\
&= 2e^t\left(\frac{t^2}{2}\right) \qquad \text{(from Table 17.1)} \\
&= t^2e^t. \bullet
\end{aligned}$$

This example is typical of Laplace transforms at work on differential equations. We begin by assuming that the solution of the problem satisfies whatever conditions are necessary to apply the transform to the differential equation. In the case of Example 17.16, this meant assuming that $y(t)$ satisfies the conditions of Corollary 17.5.1. In actual fact we need only assume that $y(t)$ and $y'(t)$ are of exponential order. Since the nonhomogeneity $2e^t$ is continuous, our theory in Chapter 16 indicates that the solution has a continuous second derivative. In applying the Laplace transform to a third-order differential equation, we would assume that the solution satisfies the conditions of Corollary 17.5.2 for $n = 3$. The Laplace transform reduces the differential equation in $y(t)$ to an algebraic equation in its transform $Y(s)$. Notice how initial conditions for the solution of the differential equation are incorporated by the Laplace transform at a very early stage, unlike the techniques of Chapter 16 where they are used to determine arbitrary constants in a general solution. The algebraic equation is solved for $Y(s)$ and the inverse transform then yields a function $y(t)$. That $y(t)$ is a solution of the initial-value problem is easily verified by direct substitution into the differential equation and initial conditions. We omit this verification, although the problem is not truly solved until this action has been taken.

Example 17.17 Solve the initial-value problem

$$y'' + 4y = 3\cos 2t, \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution Assuming that the solution satisfies the conditions of Corollary 17.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions

$$[s^2Y - s(1) - 0] + 4Y = \frac{3s}{s^2+4}.$$

The solution of this equation for $Y(s)$ is

$$Y(s) = \frac{3s}{(s^2+4)^2} + \frac{s}{s^2+4},$$

and Table 17.1 gives

$$y(t) = 3\left(\frac{t}{4}\sin 2t\right) + \cos 2t. \bullet$$

Although Laplace transforms are particularly adept at handling initial conditions, they also provide general solutions to linear differential equations, as shown in the next example.

Example 17.18 Find a general solution of the differential equation $y'' + 2y' - 3y = t^2$.

Solution We denote initial values of the solution and its first derivative by $y(0) = A$ and $y'(0) = B$. If we assume that the solution of the problem satisfies the conditions of Corollary 17.5.1, and take Laplace transforms of both sides of the differential equation,

$$[s^2Y - s(A) - B] + 2[sY - A] - 3Y = \frac{2}{s^3}.$$

The solution of this equation for $Y(s)$ is

$$Y(s) = \frac{2}{s^3(s^2+2s-3)} + \frac{As + (B+2A)}{s^2+2s-3}.$$

The partial fraction decomposition of the first term is

$$\frac{2}{s^3(s^2+2s-3)} = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{1/2}{s-1} + \frac{1/54}{s+3}.$$

Hence,

$$Y(s) = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{1/2}{s-1} + \frac{1/54}{s+3} + \frac{As + (B+2A)}{(s-1)(s+3)}.$$

If we are not concerned with preserving the fact that A and B represent initial values for $y(t)$ and its first derivative, we can write that $Y(s)$ is of the form

$$Y(s) = \frac{-2/3}{s^3} - \frac{4/9}{s^2} - \frac{14/27}{s} + \frac{C}{s-1} + \frac{D}{s+3},$$

where C and D are constants. Inverse transforms now give the general solution

$$y(t) = -\frac{t^2}{3} - \frac{4t}{9} - \frac{14}{27} + Ce^t + De^{-3t}.\bullet$$

Example 17.19 A 2-kg mass is suspended from a spring with constant 128 N/m. It is pulled 4 cm above its equilibrium position and released. An external force $3\sin\omega t$ N acts vertically on the mass during its motion. If damping is negligible, find the position of the mass as a function of time.

Solution The initial-value problem describing oscillations of the mass is

$$2\frac{d^2x}{dt^2} + 128x = 3\sin\omega t, \quad x(0) = 1/25, \quad x'(0) = 0.$$

If we take Laplace transforms of both sides of the differential equation,

$$2[s^2X - s/25] + 128X = \frac{3\omega}{s^2+\omega^2} \implies X(s) = \frac{3\omega}{2(s^2+64)(s^2+\omega^2)} + \frac{s}{25(s^2+64)}.$$

When $\omega \neq 8$, partial fractions on the first term on the left leads to

$$X(s) = \frac{3\omega}{2(64-\omega^2)(s^2+\omega^2)} - \frac{3\omega}{2(64-\omega^2)(s^2+64)} + \frac{s}{25(s^2+64)}.$$

Hence, displacement in the absence of resonance is

$$x(t) = \frac{3}{2(64-\omega^2)}\sin\omega t - \frac{3\omega}{16(64-\omega^2)}\sin 8t + \frac{1}{25}\cos 8t.$$

When $\omega = 8$, the Laplace transform $X(s)$ takes the form

$$X(s) = \frac{12}{(s^2+64)^2} + \frac{s}{25(s^2+64)},$$

in which case Table 17.1 gives the resonant solution

$$x(t) = \frac{12}{2(8)^3}(\sin 8t - 8t\,\cos 8t) + \frac{1}{25}\cos 8t = \frac{3}{256}\sin 8t - \frac{3t}{32}\cos 8t + \frac{1}{25}\cos 8t.\bullet$$

Convolutions

It is often necessary in applications to find the inverse transform of the product of two functions FG when inverse transforms f and g of F and G are known. We shall see shortly that the inverse of FG is what is called the *convolution* of f and g.

Definition 17.4 The **convolution** of two functions f and g is defined as

$$f * g = \int_0^t f(u)g(t-u)\,du. \tag{17.13}$$

The following properties of convolutions are easily verified using Definition 17.4:

$$f * g = g * f, \tag{17.14a}$$
$$f * (kg) = (kf) * g = k(f * g),\ k \text{ a constant} \tag{17.14b}$$
$$(f * g) * h = f * (g * h), \tag{17.14c}$$
$$f * (g + h) = f * g + f * h. \tag{17.14d}$$

Example 17.20 Find the convolution of $f(t) = \sin t$ and $g(t) = \cos 4t$.

Solution According to equation 17.13,

$$f * g = \int_0^t \sin u\,\cos 4(t-u)\,du.$$

With the trigonometric identity $\sin A \cos B = (1/2)[\sin(A+B) + \sin(A-B)]$, we obtain

$$\begin{aligned} f * g &= \frac{1}{2}\int_0^t [\sin(4t-3u) + \sin(5u-4t)]\, du \\ &= \frac{1}{2}\left\{\frac{1}{3}\cos(4t-3u) - \frac{1}{5}\cos(5u-4t)\right\}_0^t \\ &= \frac{1}{15}(\cos t - \cos 4t). \bullet \end{aligned}$$

The importance of convolutions lies in the following theorem.

Theorem 17.6 If f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval $0 \le t \le T$, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}, \quad s > \alpha. \tag{17.15a}$$

Proof If $F = \mathcal{L}\{f\}$ and $G = \mathcal{L}\{g\}$, then

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)\, du \int_0^\infty e^{-s\tau} g(\tau)\, d\tau = \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(u) g(\tau)\, d\tau\, du.$$

Suppose we change variables of integration in the inner integral with respect to τ by setting $t = u + \tau$. Then

$$F(s)G(s) = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du = \lim_{T\to\infty} \int_0^T \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du.$$

We would like to interchange orders of integration, but to do so requires that the inner integral converge uniformly with respect to u. To verify that this is indeed the case we note that since f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval $0 \le t \le T$, there exists a constant M such that for all $t \ge 0$, $|f(t)| < Me^{\alpha t}$ and $|g(t)| < Me^{\alpha t}$. For each $u \ge 0$, we therefore have $|e^{-st} f(u) g(t-u)| < M^2 e^{-st} e^{\alpha u} e^{\alpha(t-u)} = M^2 e^{-t(s-\alpha)}$. Thus,

$$\begin{aligned} \left|\int_u^\infty e^{-st} f(u) g(t-u)\, dt\right| &< M^2 \int_u^\infty e^{-t(s-\alpha)}\, dt = M^2 \left\{\frac{e^{-t(s-\alpha)}}{\alpha - s}\right\}_u^\infty \\ &= \frac{M^2 e^{-u(s-\alpha)}}{s-\alpha} < \frac{M^2}{s-\alpha}, \end{aligned}$$

provided $s > \alpha$, and the improper integral is uniformly convergent with respect to u. The order of integration in the expression for $F(s)G(s)$ may therefore be interchanged (Figure 17.22), and we obtain

$$\begin{aligned} F(s)G(s) = \lim_{T\to\infty} \biggl[&\int_0^T e^{-st} \int_0^t f(u) g(t-u)\, du dt \\ &+ \int_T^\infty e^{-st} \int_0^T f(u) g(t-u)\, du dt \biggr]. \end{aligned}$$

Since

$$\begin{aligned} \left|\int_T^\infty e^{-st} \int_0^T f(u) g(t-u)\, du dt\right| &< \int_T^\infty \int_0^T M^2 e^{-t(s-\alpha)} du\, dt \\ &= M^2 T \left\{\frac{e^{-t(s-\alpha)}}{\alpha - s}\right\}_T^\infty = \frac{M^2 T e^{-T(s-\alpha)}}{s-\alpha} \end{aligned}$$

provided $s > \alpha$, it follows that

$$\lim_{T\to\infty}\int_T^{\infty} e^{-st}\int_0^T f(u)g(t-u)\,du\,dt = 0.$$

Thus,

$$F(s)G(s) = \lim_{T\to\infty}\int_0^T e^{-st}\int_0^t f(u)g(t-u)\,dudt = \lim_{T\to\infty}\int_0^T e^{-st} f*g\,dt = \mathcal{L}\{f*g\}.$$

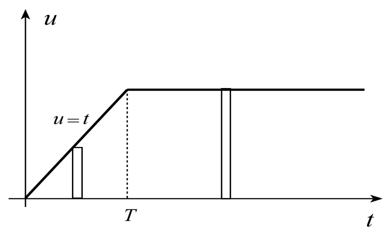


Figure 17.22

More important in practice is the inverse of property 17.15a.

Corollary 17.6.1 If $\mathcal{L}^{-1}\{F\} = f$ and $\mathcal{L}^{-1}\{G\} = g$, where f and g are $O(e^{\alpha t})$ and piecewise-continuous on every finite interval, then

$$\mathcal{L}^{-1}\{FG\} = f*g. \tag{17.15b}$$

The following example illustrates this corollary.

Example 17.21 Find the inverse transform of $F(s) = \dfrac{2}{s^2(s^2+4)}$.

Solution Since $\mathcal{L}^{-1}\{2/(s^2+4)\} = \sin 2t$ and $\mathcal{L}^{-1}\{1/s^2\} = t$, convolution propery 17.15b gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+4)}\right\} &= \int_0^t u\sin 2(t-u)\,du\\ &= \left\{\frac{u}{2}\cos 2(t-u) + \frac{1}{4}\sin 2(t-u)\right\}_0^t = \frac{t}{2} - \frac{1}{4}\sin 2t.\bullet\end{aligned}$$

Convolutions are particularly useful when solving ordinary differential equations that contain unspecified forcing functions.

Example 17.22 Find the solution of the initial-value problem

$$y'' + 2y' + 3y = f(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where $f(t)$ is piecewise-continuous for $t \geq 0$.

Solution Assuming that the solution satisfies the conditions of Corollary 17.5.1, we take Laplace transforms of both sides of the differential equation,

$$[s^2Y - s] + 2[sY - 1] + 3Y = F(s),$$

and solve for Y,

$$Y(s) = \frac{F(s)}{s^2+2s+3} + \frac{s+2}{s^2+2s+3}.$$

To find the inverse transform of this function, we first note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t.$$

Convolution property 17.15b on the first term of $Y(s)$ now yields

$$\begin{aligned} y(t) &= \int_0^t f(u)\frac{1}{\sqrt{2}}e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + \mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + e^{-t}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sin\sqrt{2}(t-u)\,du + e^{-t}\left(\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right). \bullet \end{aligned}$$

EXERCISES 17.3

In Exercises 1–16 use Laplace transforms to solve the initial-value problem.

1. $y'' + 3y' - 4y = t + 3, \quad y(0) = 1, \quad y'(0) = 0$

2. $y'' + 2y' - y = e^t, \quad y(0) = 1, \quad y'(0) = 2$

3. $y'' + y = 2e^{-t}, \quad y(0) = y'(0) = 0$

4. $y'' + 2y' + y = t, \quad y(0) = 0, \quad y'(0) = 1$

5. $y'' - 2y' + y = t^2e^t, \quad y(0) = 1, \quad y'(0) = 0$

6. $y'' + y = t, \quad y(0) = 1, \quad y'(0) = -2$

7. $y'' + 2y' + 5y = e^{-t}\sin t, \quad y(0) = 0, \quad y'(0) = 1$

8. $y'' + 6y' + y = \sin 3t, \quad y(0) = 2, \quad y'(0) = 1$

9. $y'' + y' - 6y = t + \cos t, \quad y(0) = 1, \quad y'(0) = -2$

10. $y'' - 4y' + 5y = te^{-3t}, \quad y(0) = -1, \quad y'(0) = 2$

11. $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1$, where $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

12. $y'' + 2y' - 4y = \cos^2 t, \quad y(0) = 0, \quad y'(0) = 0$

13. $y'' - 3y' + 2y = 8t^2 + 12e^{-t}, \quad y(0) = 0, \quad y'(0) = 2$

14. $y'' + 4y' - 2y = \sin 4t, \quad y(0) = 0, \quad y'(0) = 0$

15. $y'' + 8y' + 41y = e^{-2t}\sin t, \quad y(0) = 0, \quad y'(0) = 1$

16. $y'' + 2y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$, where $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

In Exercises 17–19 use Laplace transforms to solve the boundary-value problem.

17. $y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$

18. $y'' + 3y' - 4y = 2e^{-4t}, \quad y(0) = 1, \quad y(1) = 1$

19. $y'' + 2y' + 5y = e^{-t}\sin t, \quad y(0) = 0, \quad y(\pi/4) = 1$

In Exercises 20–23 use Laplace transforms to find an integral representation for the solution to the problem.

20. $y'' - 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 0$

21. $y'' + 4y' + 6y = f(t), \quad y(0) = 0, \quad y'(0) = 0$

22. $y'' + 16y = f(t)$

23. $y'' + 3y' + 2y = e^t f(t)$

In Exercises 24–27 use convolutions to find the inverse Laplace transform for the function.

24. $F(s) = \dfrac{1}{s(s+1)}$

25. $F(s) = \dfrac{1}{(s^2+1)(s^2+4)}$

26. $F(s) = \dfrac{s}{(s+4)(s^2-2)}$

27. $F(s) = \dfrac{s}{(s^2-4)(s^2-9)}$

In Exercises 28–33 use Laplace transforms to find a general solution of the differential equation.

28. $y'' - 2y' + 4y = t^2$

29. $y'' - 2y' + y = t^2e^t$

30. $y'' + y = f(t)$

31. $y'' + 2y' + 5y = e^{-t}\sin t$

32. $y'' + 4y' + y = t + 2$

33. $y'' - 4y = f(t)$

34. To find a general solution of $y'' + 9y = t\,\sin t$, replace $t\,\sin t$ by te^{ti}, solve the equation, and then take imaginary parts.

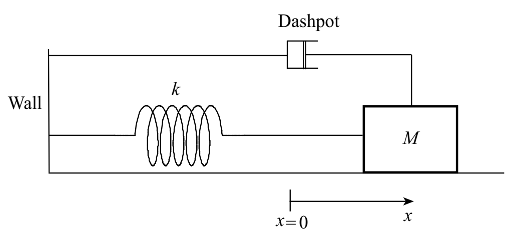
35. To find a general solution of $y'' - 2y' + 3y = t\,\cos 2t$, replace $t\,\cos 2t$ by te^{2ti}, solve the equation, and then take real parts.

Solve the problem in Exercises 36–37 .

36. $y''' - 3y'' + 3y' - y = t^2e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

37. $y''' - 3y'' + 3y' - y = t^2e^t$

One end of a spring with constant k newtons per metre is attached to a mass of M kilograms and the other end is attached to a wall (figure below).



Attached to the mass is a dashpot that provides, or represents, a resistive force on the mass directly proportional to the velocity of the mass. If all other forces are grouped into a function denoted by $f(t)$, the differential equation governing motion of the mass is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t),$$

where $\beta > 0$ is a constant. The position of M when the spring is unstretched corresponds to $x = 0$. Accompanying the differential equation will be two initial conditions $x(0) = A$ and $x'(0) = B$ representing the initial position and velocity of M. In Exercises 38–44 , solve the initial-value problem with the given information.

38. $M = 1/5, \quad \beta = 0, \quad k = 10, \quad f(t) = 0, \quad x(0) = -0.03, \quad x'(0) = 0$

39. $M = 1/5, \quad \beta = 3/2, \quad k = 10, \quad f(t) = 0, \quad x(0) = -0.03, \quad x'(0) = 0$

40. $M = 1/5, \quad \beta = 3/2, \quad k = 10, \quad f(t) = 4\sin 10t, \quad x(0) = 0, \quad x'(0) = 0$

41. $M = 2, \quad \beta = 0, \quad k = 16, \quad f(t) = 0, \quad x(0) = 0.1, \quad x'(0) = 0$

42. $M = 1/10, \quad \beta = 1/20, \quad k = 5, \quad f(t) = 0, \quad x(0) = -1/20, \quad x'(0) = 2$

43. $M = 1/10, \quad \beta = 0, \quad k = 4000, \quad f(t) = 3\cos 200t, \quad x(0) = 0, \quad x'(0) = 10$

44. $M = 1, \quad \beta = 0, \quad k = 64, \quad f(t) = 2\sin 8t, \quad x(0) = 0, \quad x'(0) = 0$

45. (a) Let f be $O(e^{\alpha t})$ and be continuous for $t \geq 0$ except for a finite discontinuity at $t = t_0 > 0$; and let f' be piecewise continuous on every finite interval $0 \leq t \leq T$. Show that

$$\mathcal{L}\{f'\} = sF(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)].$$

(b) What is the result in part (a) if $t_0 = 0$?

ANSWERS

1. $(27/80)e^{-4t} + (8/5)e^{t} - 15/16 - t/4$
2. $(1/2)e^{t} + [(1+2\sqrt{2})/4]e^{(\sqrt{2}-1)t} + [(1-2\sqrt{2})/4]e^{-(\sqrt{2}+1)t}$ **3.** $e^{-t} - \cos t + \sin t$
4. $2(1+t)e^{-t} + t - 2$ **5.** $e^{t}(1 - t + t^4/12)$ **6.** $\cos t - 3\sin t + t$
7. $(1/3)e^{-t}(\sin t + \sin 2t)$
8. $[(794\sqrt{2}+1397)/(776\sqrt{2})]e^{(-3+2\sqrt{2})t} + [(794\sqrt{2}-1397)/(776\sqrt{2})]e^{(-3-2\sqrt{2})t} - (1/194)(4\sin 3t + 9\cos 3t)$
9. $-1/36 - t/6 + (377/450)e^{-3t} + (33/100)e^{2t} + (1/50)(\sin t - 7\cos t)$
10. $(1/338)[(5+13t)e^{-3t} + e^{2t}(1364\sin t - 343\cos t)]$
11. $(1/4)(1 + 2\sin 2t - \cos 2t) + (1/4)[-1 + \cos 2(t-1)]h(t-1)$
12. $-1/8 + (1/40)(\sin 2t - 2\cos 2t) + (1/80)[(7+\sqrt{5})e^{(\sqrt{5}-1)t} + (7-\sqrt{5})e^{-(\sqrt{5}+1)t}]$
13. $14 + 12t + 4t^2 + 2e^{-t} + 8e^{2t} - 24e^{t}$
14. $-(1/290)(8\cos 4t + 9\sin 4t) + (\sqrt{6}/870)[(2\sqrt{6}+13)e^{(\sqrt{6}-2)t} + (2\sqrt{6}-13)e^{-(\sqrt{6}+2)t}]$
15. $(1/200)e^{-4t}(\cos 5t + 39\sin 5t) + (1/200)e^{-2t}(7\sin t - \cos t)$
16. $(t+2)e^{-t} + t - 2 + (2 - t - e^{1-t})h(t-1)$
17. $(1/5)(\cos 2t + 4\cos 3t + 4\sin 3t)$
18. $-(2t/5)e^{-4t} + [(5e^5 - 5e^4 - 2)/(5e^5 - 5)]e^{-4t} + [(5e^4 - 3)/(5e^5 - 5)]e^{t}$
19. $(1/3)e^{-t}\sin t + (e^{\pi/4} - \sqrt{2}/6)e^{-t}\sin 2t$
20. $(3/2)e^{t} - (1/2)e^{3t} + (1/2)\int_0^t f(u)[e^{3(t-u)} - e^{t-u}]\,du$
21. $(1/\sqrt{2})\int_0^t f(u)e^{2(u-t)}\sin\sqrt{2}(t-u)\,du$
22. $C_1\cos 4t + C_2\sin 4t + (1/4)\int_0^t f(u)\sin 4(t-u)\,du$
23. $C_1e^{-t} + C_2e^{-2t} + \int_0^t (e^{2u-t} - e^{3u-2t})f(u)\,du$
24. $1 - e^{-t}$ **25.** $(2\sin t - \sin 2t)/6$
26. $-(2/7)e^{-4t} + [(4-\sqrt{2})/28]e^{\sqrt{2}t} + [(4+\sqrt{2})/28]e^{-\sqrt{2}t}$ **27.** $(1/10)(e^{3t} + e^{-3t} - e^{2t} - e^{-2t})$
28. $t/4 + t^2/4 + e^{t}(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t)$ **29.** $(C_1 + C_2t)e^{t} + (1/12)t^4e^{t}$
30. $C_1\cos t + C_2\sin t + \int_0^t f(u)\sin(t-u)\,du$ **31.** $e^{-t}(C_1\cos 2t + C_2\sin 2t) + (1/3)e^{-t}\sin t$
32. $C_1e^{(\sqrt{3}-2)t} + C_2e^{-(\sqrt{3}+2)t} + t - 2$ **33.** $C_1e^{2t} + C_2e^{-2t} + (1/4)\int_0^t f(u)[e^{2(t-u)} - e^{2(u-t)}]\,du$
34. $(t/8)\sin t - (1/32)\cos t + C_1\cos 3t + C_2\sin 3t$
35. $e^{t}(C_1\cos\sqrt{2}t + C_2\sin\sqrt{2}t) - (t/17)\cos 2t - (4t/17)\sin 2t - (62/289)\cos 2t - (44/289)\sin 2t$
36. $e^{t}(t^5 - 30t^2 - 60t + 60)/60$ **37.** $e^{t}(C_1 + C_2t + C_3t^2 + t^5/60)$
38. $-0.03\cos 5\sqrt{2}t$ **39.** $-0.03e^{-15t/4}[\cos(5\sqrt{23}t/4) + (3/\sqrt{23})\sin(5\sqrt{23}t/4)]$
40. $-(12\cos 10t + 8\sin 10t)/65 + (4/65)e^{-15t/4}[3\cos(5\sqrt{23}t/4) + (25/\sqrt{23})\sin(5\sqrt{23}t/4)]$
41. $(1/10)\cos(2\sqrt{2}t)$ **42.** $(1/20)e^{-t/4}[(159/\sqrt{799})\sin(\sqrt{799}t/4) - \cos(\sqrt{799}t/4)]$
43. $(2+3t)(\sin 200t)/40$ **44.** $(\sin 8t - 8t\cos 8t)/64$ **45.**(b) $sF(s) - f(0+)$

17.4 Discontinuous Nonhomogeneities

Nonhomogeneities for the linear differential equations in Section 17.3 were all continuous. As a result, Laplace transforms did not prove overly advantageous compared to methods of Chapter 16. In this section we show that Laplace transforms are exceptional for handling discontinuities. We begin by illustrating the awkwardness of previous techniques on the initial-value problem

$$y'' + 2y' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where the nonhomogeneity is the discontinuous function

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}.$$

Basically what we do is solve the differential equation on the intervals $0 < t < 1$ and $t > 1$ and then match the solutions at $t = 1$. The auxiliary equation $m^2 + 2m + 1 = 0$ has double root $m = -1$. On the interval $0 < t < 1$, a particular solution of the differential equation is $y_p = t - 2$, and hence a general solution on this interval is $y_1(t) = (C_1 + C_2 t)e^{-t} + t - 2$. The initial conditions require

$$1 = y(0) = C_1 - 2, \qquad 0 = y'(0) = C_2 - C_1 + 1,$$

the solutions of which are $C_1 = 3$ and $C_2 = 2$. On the interval $0 < t < 1$, then,

$$y_1(t) = (3 + 2t)e^{-t} + t - 2.$$

For $t > 1$, the general solution of the differential equation is $y_2(t) = (D_1 + D_2 t)e^{-t}$.

In Chapter 16 we saw that the solution of a second-order, linear differential equation must be continuous and have a continuous first derivative, and this must be true even at the point of discontinuity ($t = 1$) of $f(t)$. This means that $\lim_{t\to 1^-} y_1(t) = \lim_{t\to 1^+} y_2(t)$ and $\lim_{t\to 1^-} y_1'(t) = \lim_{t\to 1^+} y_2'(t)$, and therefore

$$5e^{-1} - 1 = (D_1 + D_2)e^{-1}, \qquad -3e^{-1} + 1 = -D_1 e^{-1}.$$

These can be solved for $D_1 = 3 - e$ and $D_2 = 2$, and therefore the solution of the initial-value problem is

$$y(t) = \begin{cases} (3 + 2t)e^{-t} + t - 2, & 0 \le t \le 1 \\ (3 - e + 2t)e^{-t}, & t > 1 \end{cases}.$$

Let us now solve the problem by taking Laplace transforms of both sides of the differential equation,

$$[s^2 Y - s] + 2[sY - 1] + Y = \mathcal{L}\{f(t)\},$$

where

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\,[h(t) - h(t-1)]\} \\ &= \mathcal{L}\{t\,h(t)\} - \mathcal{L}\{[t\,h(t-1)\} \\ &= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} \\ &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).\end{aligned}$$

Thus,

$$\begin{aligned}Y(s) &= \frac{1}{(s+1)^2}\left[s + 2 + \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] \\ &= \frac{(s+1)+1}{(s+1)^2} + \frac{1}{s^2(s+1)^2} - \frac{e^{-s}}{s^2(s+1)}.\end{aligned}$$

Partial fractions on the second and third terms lead to

$$\begin{aligned}Y(s) &= \left[\frac{1}{s+1} + \frac{1}{(s+1)^2}\right] + \left[-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2}\right] + e^{-s}\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+1}\right]\\ &= -\frac{2}{s} + \frac{1}{s^2} + \frac{3}{s+1} + \frac{2}{(s+1)^2} + e^{-s}\left(\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+1}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}y(t) &= -2 + t + 3e^{-t} + 2te^{-t} + [1 - (t-1) - e^{-(t-1)}]\,h(t-1)\\ &= (3+2t)e^{-t} + t - 2 + (2 - t - e^{1-t})\,h(t-1).\end{aligned}$$

Although the Heaviside function is undefined at $t = 1$, right- and left-hand limits of this solution as $t \to 1$ are identical. So also are limits of its first derivative. In other words, if we define the solution and its first derivative at $t = 1$ in terms of limits as $t \to 1$, the solution is identical to that obtained previously, but the Heaviside representation is clearly simpler, and arriving at it with Laplace transforms was less work.

As we use Laplace transforms to solve other differential equations with discontinuous nonhomogeneities, we invite the reader to make comparisons to solutions obtained with techniques from Chapter 16.

Example 17.23 A 2-kg mass is suspended from a spring with constant 512 N/m. It is set into motion by pulling it 10 cm above its equilibrium position and then releasing it. A sinusoidal force $A\sin 8t$ acts on the mass but only for $t > 1$. Find the position of the mass as a function of time if damping is negligible.

Solution The initial-value problem for displacement is

$$2\frac{d^2x}{dt^2} + 512x = A\sin 8t\,h(t-1), \qquad x(0) = \frac{1}{10}, \qquad x'(0) = 0.$$

If we take Laplace transforms,

$$\begin{aligned}2\left(s^2X - \frac{s}{10}\right) + 512X &= Ae^{-s}\mathcal{L}\{\sin 8(t+1)\}\\ &= Ae^{-s}\mathcal{L}\{\cos 8\,\sin 8t + \sin 8\,\cos 8t\}\\ &= Ae^{-s}\left[\frac{8\cos 8}{s^2+64} + \frac{(\sin 8)s}{s^2+64}\right].\end{aligned}$$

Hence,

$$X(s) = \frac{s}{10(s^2+256)} + \frac{Ae^{-s}[8\cos 8 + (\sin 8)s]}{2(s^2+256)(s^2+64)}.$$

Partial fractions on the second term gives

$$X(s) = \frac{s}{10(s^2+256)} + \frac{Ae^{-s}}{384}\left[\frac{\cos 8 + (\sin 8)s}{s^2+64} - \frac{\cos 8 + (\sin 8)s}{s^2+256}\right],$$

and therefore

$$\begin{aligned}x(t) = \frac{1}{10}\cos 16t + \frac{A}{384}\big[&\cos 8\,\sin 8(t-1) + \sin 8\,\cos 8(t-1)\\ &- \frac{1}{2}\cos 8\,\sin 16(t-1) - \sin 8\,\cos 16(t-1)\big]\,h(t-1).\end{aligned}$$

This has been graphed in Figure 17.23 for $A = 100$. Notice the smoothness of the graph even at $t = 1$ when the force is discontinuous.•

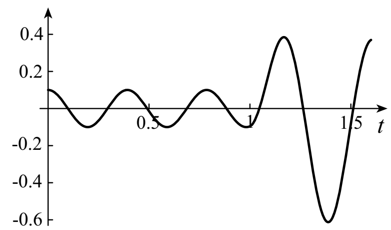


Figure 17.23

The delayed sinusoidal nonhomogeneity presented no problem in Example 17.23. When the nonhomogeneity is periodic, but not sinusoidal, additional difficulties arise. Compared to a solution by methods of Chapter 16, however, Laplace transforms are still superior. We illustrate in the following example.

Example 17.24 Solve the initial-value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where $f(t)$ is the periodic function

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t).$$

Solution When we take Laplace transforms of both sides of the differential equation,

$$\begin{aligned} s^2Y + 4Y = \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt = \frac{1}{1-e^{-2s}}\mathcal{L}\{h(t) - h(t-1)\} \\ &= \frac{1}{(1+e^{-s})(1-e^{-s})}\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) = \frac{1}{s(1+e^{-s})}. \end{aligned}$$

Thus,

$$Y(s) = \frac{1}{s(s^2+4)(1+e^{-s})}.$$

Partial fractions gives

$$\frac{1}{s(s^2+4)} = \frac{1/4}{s} - \frac{s/4}{s^2+4}.$$

Now, $1/(1+e^{-s})$ can be interpreted as the sum of a geometric series with common ratio $-e^{-s}$ so that we may write

$$\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots.$$

In other words, $Y(s)$ can be expressed as an infinite series

$$Y(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right).$$

Each term in the series has an easily calculated inverse transform,

$$y(t) = \frac{1}{4}(1 - \cos 2t) - \frac{1}{4}[1 - \cos 2(t-1)]h(t-1) + \frac{1}{4}[1 - \cos 2(t-2)]h(t-2) - \cdots.$$

In sigma notation,

$$y(t) = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n[1 - \cos 2(t-n)]\,h(t-n).$$

To evaluate $y(t)$ for any given t, it is necessary to include only those terms in the series for which $n < t$. For example, the solution at $t = 2.4$ is given by

$$y(2.4) = \frac{1}{4}[1 - \cos 2(2.4)] - \frac{1}{4}[1 - \cos 2(2.4 - 1)] + \frac{1}{4}[1 - \cos 2(2.4 - 2)] = -0.182.$$

Once again the graph of the solution in Figure 17.24 demonstrates that $y(t)$ and $y'(t)$ are continuous, even at the discontinuities $t = 1, 2, \ldots$ of $f(t)$.●

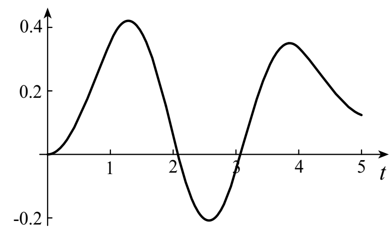


Figure 17.24

Consider using the techniques of Chapter 16 to find $y(2.4)$ in this example. It would be necessary to solve the differential equation on the intervals $0 < t < 1$, $1 < t < 2$, $2 < t < 3$, match at $t = 1$ and $t = 2$, and then find $y(2.4)$ from the solution for $2 < t < 3$. Try it. You will be convinced that Laplace transforms are superior.

Important in applications are nonhomogeneities called *unit pulses* and *unit impulses*. We discuss them in the context of the vibrating mass-spring system in Figure 17.25. When damping and surface friction are negligible, the differential equation describing the position of the mass relative to its equilibrium position is $M\,d^2x/dt^2 + kx = f(t)$ where $f(t)$ represents all forces on M other than the spring. The external force is called a **unit pulse** at time $t = t_0$ when it is of the form in Figure 17.26. It can be represented in terms of Heaviside unit step functions as

$$p(t_0, a, t) = \frac{1}{a}[h(t - t_0) - h(t - t_0 - a)]. \tag{17.16}$$

The value of t_0 identifies the time at which the pulse begins and a represents its width. The area under the graph is unity (hence the name *unit* pulse).

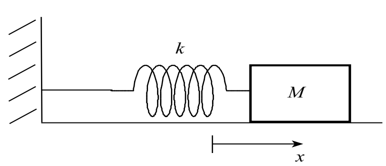


Figure 17.25

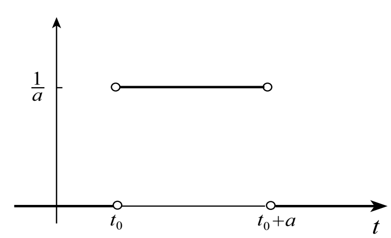


Figure 17.26

The Laplace transform of the unit pulse at $t = t_0$ is

$$\mathcal{L}\{p(t_0, a, t)\} = \frac{1}{as}\left[e^{-t_0 s} - e^{-(t_0+a)s}\right]. \tag{17.17}$$

Let us determine the reaction of the mass-spring system to a unit pulse at time $t = 0$, and this force only. We assume that the mass is motionless at its equilibrium position when this force is applied. The initial-value problem for the position of M is

$$M\frac{d^2x}{dt^2} + kx = p(0, a, t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

If we take Laplace transforms of both sides of the differential equation, and use formula 17.17 with $t_0 = 0$,

$$Ms^2X + kX = \frac{1}{as}(1 - e^{-as}) \qquad \Longrightarrow \qquad X(s) = \frac{1 - e^{-as}}{as(Ms^2 + k)}.$$

Partial fractions give

$$X(s) = \frac{1}{ka}\left(\frac{1}{s} - \frac{s}{s^2 + k/M}\right)(1 - e^{-as}),$$

from which

$$x(t) = \frac{1}{ka}\left(1 - \cos\sqrt{\frac{k}{M}}t\right) - \frac{1}{ka}\left[1 - \cos\sqrt{\frac{k}{M}}(t - a)\right]h(t - a).$$

At time $t = a$, when the unit pulse ceases, the position of the mass is given by $(1 - \cos\sqrt{k/M}a)/(ka)$ and its velocity is $[1/(a\sqrt{kM})]\sin\sqrt{k/M}a$. For most applications, a is very small; in particular, sufficiently small that $\sqrt{k/M}a < \pi/2$. In this case, the displacement of the mass from equilibrium increases for $0 < t < a$, and its velocity at time $t = a$ is positive. A graph of this function for parameter values $k = 100$, $M = 1$, and $a = 1/10$ is shown in Figure 17.27.

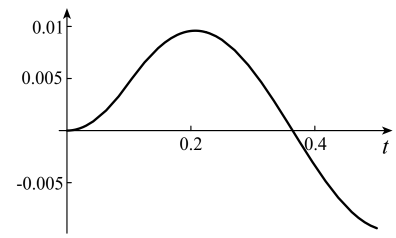


Figure 17.27

Even more important in practice is the response of a system to what is called the *unit impulse* force. It is defined to be the limit of the unit pulse $p(t_0, a, t)$ as the time interval $t_0 < t < t_0 + a$ becomes indefinitely short. As a gets smaller and smaller in Figure 17.26, the area under the curve remains unity; the force simply acts over shorter and shorter time intervals. We have shown the situation for $a = 1/10$, $1/20$, $1/40$, and $1/80$ in Figure 17.28.

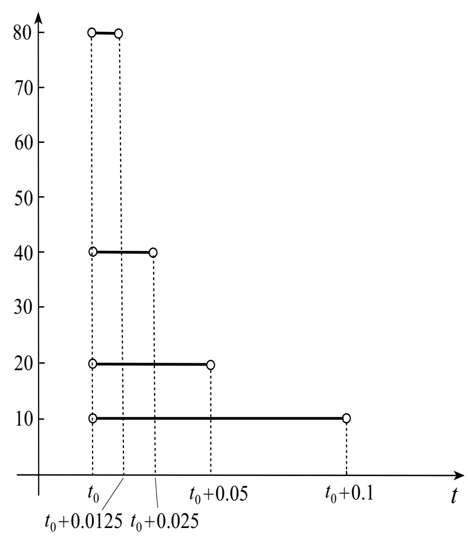


Figure 17.28

The limit of this function as $a \to 0$ is not a function in the normal sense of function. It has value 0 for all t except $t = t_0$ where its value is "infinite". Such functions are discussed in advanced mathematics; they are known as *generalized functions*. This particular one is called the **unit impulse** or the **Dirac delta** function. It is denoted by

$$\delta(t - t_0) = \lim_{a \to 0} \frac{1}{a}[h(t - t_0) - h(t - t_0 - a)]. \tag{17.18}$$

The Dirac delta function can be defined in other ways; they are essentially equivalent and lead to identical properties. Two such formulations are limits of the sequences of functions in Figures 17.29 and 17.30. In both cases, the area under each curve is unity.

Like the functions in Figure 17.28, those in Figure 17.29 are discontinuous, but they are symmetric around t_0. The functions in Figure 17.30 are continuous and symmetric around t_0.

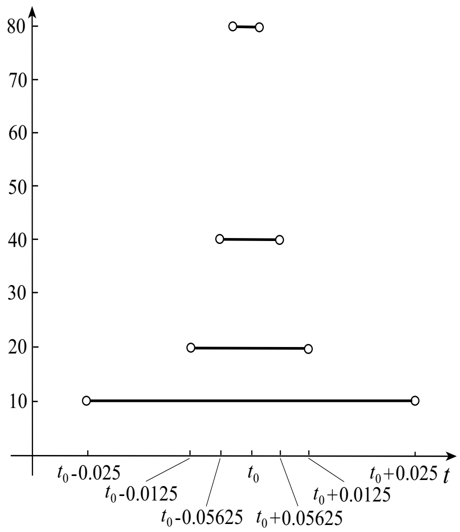


Figure 17.29

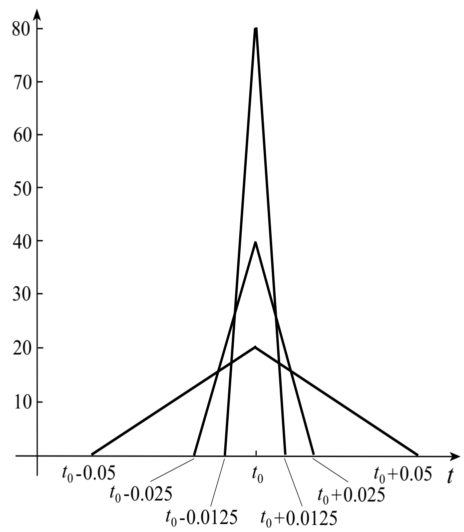


Figure 17.30

In our vibrating mass-spring system, the Dirac delta function represents a unit force instantaneously applied at time $t = t_0$. The function does not conform to the conditions of Theorem 17.1; it is of exponential order, but it is not piecewise-continuous on every finite interval. It does, however, have a Laplace transform. If we write

$$\mathcal{L}\{\delta(t-t_0)\} = \mathcal{L}\left\{\lim_{a\to 0} p(t_0, a, t)\right\},$$

it might seem natural to interchange limit and Laplace transform operations,

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0} \mathcal{L}\{p(t_0, a, t)\}.$$

Unfortunately, it not possible to justify the validity of the interchange, but it does lead to a correct formula for the Laplace transform of $\delta(t-t_0)$, and we shall therefore proceed. Substituting from equation 17.17 and using L'Hopital's rule on the limit gives

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0}\left(\frac{e^{-t_0 s} - e^{-(t_0+a)s}}{as}\right) = \lim_{a\to 0}\left(\frac{se^{-(t_0+a)s}}{s}\right) = e^{-t_0 s}. \tag{17.19}$$

Let us determine the response of the mass-spring system in Figure 17.25 to a unit impulse at time $t = 0$. To do so we solve the initial-value problem

$$M\frac{d^2x}{dt^2} + kx = \delta(t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

When we take Laplace transforms and use formula 17.19 with $t_0 = 0$,

$$Ms^2X + kX = 1 \qquad \Longrightarrow \qquad X(s) = \frac{1}{Ms^2 + k}.$$

The inverse transform is

$$x(t) = \frac{1}{M}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k/M}\right\} = \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t.$$

A graph of this function for $k = 400$ and $M = 2$ is shown in Figure 17.31.

It is straightforward to show that the same displacement results from giving the mass an initial velocity of $1/M$ and applying no impulse. In other words, the solution does not satisfy the initial condition $x'(0) = 0$. This is a result of specifying initial conditions and Dirac delta function simultaneously at $t = 0$. There would be no problem if the impulse force occured at any other time. This is illustrated in the following example.

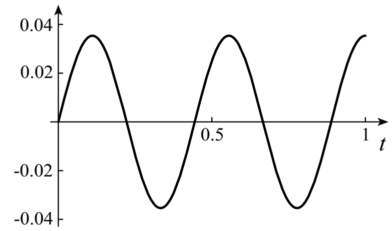


Figure 17.31

Example 17.25 A 100-gm mass is suspended from a spring with constant 50 N/m. It is set into motion by raising it 10 cm above its equilibrium position and giving it a velocity of 1 m/s downward. During the subsequent motion a damping force acts on the mass and the magnitude of this force is twice the velocity of the mass. If an impulse force of magnitude 2 N is applied vertically upward to the mass at $t = 3$ s, find the position of the mass for all time.

Solution The initial-value problem for the position of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 50x = 2\delta(t-3), \qquad x(0) = \frac{1}{10}, \qquad x'(0) = -1.$$

If we multiply the differential equation by 10, and take Laplace transforms,

$$\left(s^2X - \frac{s}{10} + 1\right) + 20\left(sX - \frac{1}{10}\right) + 500X = 20e^{-3s}.$$

Thus,

$$\begin{aligned} X(s) &= \frac{s/10 + 1}{s^2 + 20s + 500} + \frac{20e^{-3s}}{s^2 + 20s + 500} \\ &= \frac{1}{10}\left[\frac{s+10}{(s+10)^2 + 400}\right] + \frac{20e^{-3s}}{(s+10)^2 + 400}. \end{aligned}$$

The inverse transform is

$$x(t) = \frac{1}{10}e^{-10t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 400}\right\} + \mathcal{L}^{-1}\left\{\frac{20e^{-3s}}{(s+10)^2 + 400}\right\}.$$

Since $\mathcal{L}^{-1}\{20/[(s+10)^2 + 400]\} = e^{-10t}\mathcal{L}^{-1}\{20/(s^2 + 400)\} = e^{-10t}\sin 20t$, it follows that

$$x(t) = \frac{1}{10}e^{-10t}\cos 20t + e^{-10(t-3)}\sin 20(t-3)\,h(t-3).$$

It is straightforward to show that this solution satisfies the initial conditions $x(0) = 1/10$ and $x'(0) = -1$. A graph of the function is shown in Figure 17.32. Due to excessive damping, oscillations essentially disappear after 1 second, but the impulse force restores them at $t = 3$ seconds. Notice the abrupt change in slope (velocity) at $t = 3$ due to the impulse force. Damping again brings the mass essentially to rest after another second.●

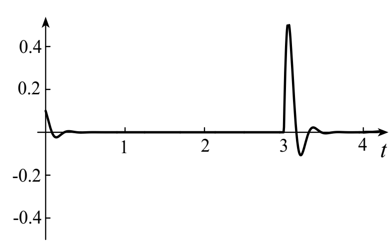


Figure 17.32

When nonhomogeneities are piecewise-continuous functions, we know that solutions are continuous and have continuous first derivatives. This example illustrates that impulse forces, not being piecewise-continuous, lead to solutions with discontinuous first derivatives at the instant of the impulse.

EXERCISES 17.4

In Exercises 1–10 solve the initial-value problem.

1. $y'' + 9y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 0, & 0 < t < 4 \\ 1, & t > 4 \end{cases}$

2. $y'' + 9y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 2, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

3. $y'' + 4y' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = -1, \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$

4. $y'' + 4y' + 4y = f(t), \quad y(0) = -1, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 2 - t, & 0 < t < 2 \\ t - 2, & t > 2 \end{cases}$

5. $y'' + 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

6. $y'' + 4y' + 3y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

7. $y'' + 2y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 3, & 0 < t < 1 \\ -3, & t > 1 \end{cases}$

8. $y'' + 2y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} 4, & 0 < t < 1 \\ -4, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

9. $y'' + 16y = f(t), \quad y(0) = 2, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$

10. $y'' + 16y = f(t), \quad y(0) = 2, \quad y'(0) = 0, \qquad \text{where } f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$

11. A 100-gm mass is suspended from a spring with constant 40 N/m. The mass is pulled 10 cm above its equilibrium position and given velocity 2 m/s downward. If a force of 100 N acts vertically upward for the first 4 seconds, find the position of the mass as a function of time. Ignore all damping.

12. Repeat Exercise 11 if the force is turned on after 4 seconds.

13. Repeat Exercise 11 if a damping force with constant $\beta = 5$ also acts on the mass.

14. Repeat Exercise 12 if a damping force with constant $\beta = 5$ also acts on the mass.

15. Repeat Exercise 11 if a damping force with constant $\beta = 1$ also acts on the mass.

16. Repeat Exercise 12 if a damping force with constant $\beta = 1$ also acts on the mass.

17. A 2-kg mass is suspended from a spring with constant 512 N/m. It is set into motion with a unit impulse force at time $t = 0$. Find the position of the mass as a function of time. Ignore all damping.

18. Repeat Exercise 17 if a damping force with constant $\beta = 80$ also acts on the mass.

19. Repeat Exercise 18 if $\beta = 8$.

20. A 2-kg mass is suspended from a spring with constant 512 N/m. It is set into motion by moving it to position x_0 and then releasing it. If a unit impulse force is applied at $t_0 > 0$, find the position of the mass for all time.

21. Repeat Exercise 20 if motion is initiated by giving the mass velocity v_0 at time $t = 0$ and position $x = 0$.

22. Repeat Exercise 20 if motion is initiated by giving the mass velocity v_0 from position x_0 at time $t = 0$.

23. A 1-kg mass is suspended from a spring with constant 100 N/m. It is subjected to a unit impulse force at $t = 0$ and again at $t = 1$. Find the position of the mass as a function of time.

24. Repeat Exercise 23 if unit impulse forces are applied one each second beginning at time $t = 0$. Express the solution in sigma notation.

25. Repeat Exercise 24 if unit impulse forces are $\pi/5$ seconds apart, the first at time $t = 0$. Is there resonance?

ANSWERS

1. $\cos 3t + (2/3)\sin 3t + (1/9)[1 - \cos 3(t-4)]\,h(t-4)$
2. $2/9 + (7/9)\cos 3t + (2/3)\sin 3t - (2/9)[1 - \cos 3(t-4)]\,h(t-4)$
3. $(1/4)(-1 + t + e^{-2t} - 3te^{-2t}) + (1/4)(2 - t - te^{2-2t})\,h(t-1)$
4. $(1/4)(3 - t - 7e^{-2t} - 13te^{-2t}) + (1/2)(t - 3 - e^{4-2t} + te^{4-2t})\,h(t-2)$
5. $(5/2)e^{-t} - (3/2)e^{-3t} + [(1/20)e^{3\pi-3t} - (1/4)e^{\pi-t} + (1/10)\sin t - (1/5)\cos t]\,h(t-\pi)$
6. $(11/4)e^{-t} - (31/20)e^{-3t} + (1/10)\sin t - (1/5)\cos t + [(1/4)e^{\pi-t} - (1/20)e^{3\pi-3t} - (1/10)\sin t +$ $(1/5)\cos t]\,h(t-\pi)$
7. $(3/10)(2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t) - (3/5)[2 - 2e^{1-t}\cos(2t-2) - e^{1-t}\sin(2t-2)]\,h(t-1)$
8. $(2/5)(2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t) + (4/5)[-2 + 2e^{1-t}\cos(2t-2) + e^{1-t}\sin(2t-2)]\,h(t-1) +$ $(2/5)[2 - 2e^{2-t}\cos(2t-4) - e^{2-t}\sin(2t-4)]\,h(t-2)$
9. $2\cos 4t + (1/64)\sum_{n=0}^{\infty}(-1)^n[4(t-n) - \sin 4(t-n)]\,h(t-n)$ $-(1/16)\sum_{n=0}^{\infty}[1 - \cos 4(t-2n-1)]\,h(t-2n-1)$
10. $2\cos 4t + (1/64)\sum_{n=0}^{\infty}(-1)^n[4(t-n) - \sin 4(t-n)]\,h(t-n)$ $+(1/64)\sum_{n=0}^{\infty}(-1)^{n+1}[4(t-n-1) - \sin 4(t-n-1)]\,h(t-n-1)$
11. $(1/10)\cos 20t - (1/10)\sin 20t + (5/2)[1 - h(t-4) - \cos 20t + \cos 20(t-4)\,h(t-4)]$
12. $(1/10)\cos 20t - (1/10)\sin 20t + (5/2)[1 - \cos 20(t-4)]\,h(t-4)$
13. $(1/30)(e^{-40t} + 2e^{-10t}) + (5/2)[1 - h(t-4)] + (5/6)(e^{-40t} - 4e^{-10t}) + (5/6)(4e^{40-10t} -$ $e^{160-40t})\,h(t-4)$
14. $(1/30)(e^{-40t} + 2e^{-10t}) + (5/6)(3 - 4e^{40-10t} + e^{160-40t})\,h(t-4)$
15. $(1/50)e^{-5t}(5\cos 5\sqrt{15}t - \sqrt{15}\sin 5\sqrt{15}t) + (5/2)[1 - h(t-4)] - (\sqrt{15}/6)e^{-5t}(\sqrt{15}\cos 5\sqrt{15}t +$ $\sin 5\sqrt{15}t) + (\sqrt{15}/6)e^{20-5t}[\sqrt{15}\cos 5\sqrt{15}(t-4) + \sin 5\sqrt{15}(t-4)]\,h(t-4)$
16. $(1/50)e^{-5t}(5\cos 5\sqrt{15}t - \sqrt{15}\sin 5\sqrt{15}t) + (5/2)h(t-4) - (\sqrt{15}/6)e^{20-5t}[\sqrt{15}\cos 5\sqrt{15}(t-4)$ $+ \sin 5\sqrt{15}(t-4)]\,h(t-4)$
17. $(1/32)\sin 16t$ **18.** $(1/48)(e^{-8t} - e^{-32t})$ **19.** $(\sqrt{7}/84)e^{-2t}\sin 6\sqrt{7}t$
20. $x_0\cos 16t + (1/32)\sin 16(t-t_0)\,h(t-t_0)$
21. $(v_0/16)\sin 16t + (1/32)\sin 16(t-t_0)\,h(t-t_0)$
22. $x_0\cos 16t + (v_0/16)\sin 16t + (1/32)\sin 16(t-t_0)\,h(t-t_0)$
23. $(1/10)\sin 10t + (1/10)\sin 10(t-1)\,h(t-1)$
24. $(1/10)\sum_{n=0}^{\infty}\sin 10(t-n)\,h(t-n)$ **25.** $(1/10)\sum_{n=0}^{\infty}\sin 10(t-n\pi/5)\,h(t-n\pi/5)$ Yes

17.5 Deflections of Beams

An important application of differential equations in structural engineering is to determine the shape of a horizontal beam when it is subjected to various loads. By analyzing internal forces and moments, it can be shown that the shape $y(x)$ of a uniform beam with constant cross section (Figure 17.33) is governed by the equation

$$\frac{d^4y}{dx^4} = \frac{F(x)}{EI} \tag{17.20}$$

where E is a constant called Young's modulus of elasticity (depending on the material of the beam), and I is also a constant (the moment of inertia of the cross section of the beam). Quantity $F(x)$ is the load placed on the beam; it is the vertical force per unit length in the x-direction, placed at position x, including the weight of the beam itself. For example, if a beam has mass 100 kg and length 10 metres (Figure 17.34), then the load due to its weight is a constant $F(x) = -9.81(100/10) = -98.1$ N/m at every point of the beam.

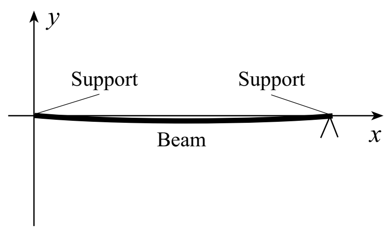


Figure 17.33

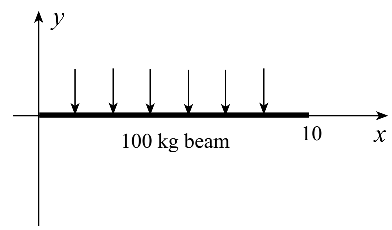


Figure 17.34

Suppose a block with mass 40 kg, uniform in cross section, and length 4 metres is centred on the beam in Figure 17.34 (see Figure 17.35). It adds an additional load of 9.81(10)=98.1 N/m over the interval $3 < x < 7$. The total load can be represented in terms of Heaviside unit step functions as

$$F(x) = -98.1 - 98.1[h(x-3) - h(x-7)].$$

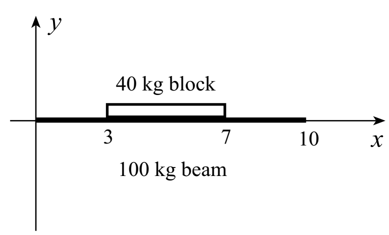


Figure 17.35

Accompanying differential equation 17.20 will be four boundary conditions defining the type of support (if any) at each end of the beam. Three types of supports are common. We discuss them at the left end of the beam, but they also occur at the right end.

1. **Simple Support**

The end of a beam is simply-supported when it cannot move vertically, but it is free to rotate. Visualize that a horizontal pin perpendicular to the xy-plane passes through a hole in the end of the beam at $x = 0$ (Figure 17.36). The pin is fixed, but the end of the beam can rotate on the pin. In this case, $y(x)$ must satisfy the **boundary conditions**

$$y(0) = y''(0) = 0. \tag{17.21a}$$

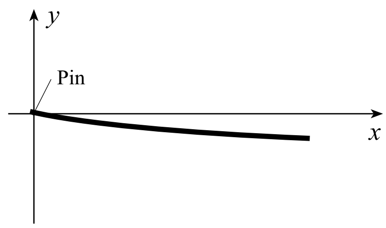


Figure 17.36

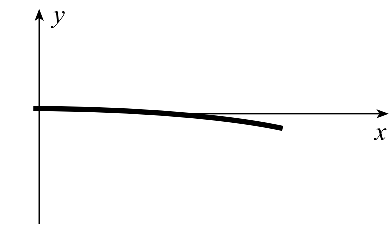


Figure 17.37

2. **Built-in End**

If the end $x = 0$ of the beam is permanently fixed in a horizontal position (Figure 17.37), $y(x)$ satisfies

$$y(0) = y'(0) = 0. \tag{17.21b}$$

3. **Free Support**

If the end $x = 0$ of the beam is not supported (Figure 17.38), $y(x)$ satisfies

$$y''(0) = y'''(0) = 0. \tag{17.21c}$$

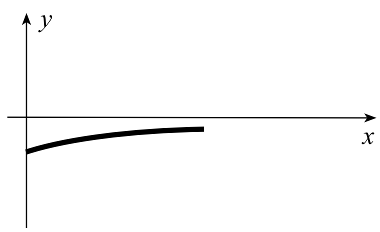


Figure 17.38

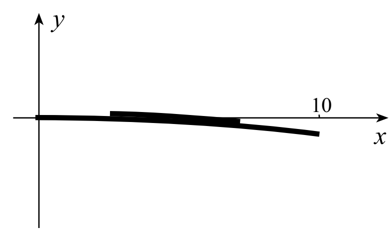


Figure 17.39

When two boundary conditions at each end of a beam accompany differential equation 17.20, we have what is called a **boundary-value problem**. For example, if the end $x = 0$ of the beam in Figure 17.35 is horizontally built-in, and the right end is free, just like a diving board (Figure 17.39), the boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}[-98.1 - 98.1\,h(x-3) + 98.1\,h(x-7)],$$
$$y(0) = y'(0) = 0, \quad y''(10) = y'''(10) = 0.$$

To solve this problem without Laplace transforms we would solve the differential equation on the intervals $0 < x < 3$, $3 < x < 7$, $7 < x < 10$, and match $y(x)$, $y'(x)$, $y''(x)$, and $y'''(x)$ at $x = 3$ and $x = 7$. Try it. Laplace transforms with respect to x are much simpler.

To have a Laplace transform with respect to x, a function must be defined for all $x > 0$ except perhaps for isolated points. Such is not the case for beam deflections; the deflection curve $y(x)$ is defined only for the length of the beam. To remedy this in the present problem (and others), we extend the beam indefinitely to the right, but assign a load of zero beyond its natural end at $x = 10$. The -98.1 term in the load is replaced by $-98.1[h(x) - h(x-10)]$. If we denote the constant $-98.1/(EI)$ by k, the boundary-value problem becomes

$$\frac{d^4y}{dx^4} = k[h(x) - h(x-3) + h(x-7) - h(x-10)],$$
$$y(0) = y'(0) = 0, \quad y''(10) = y'''(10) = 0.$$

In this way we can take transforms, and the fact that the load vanishes for $x > 10$ means that the beam is unaffected for $0 < x < 10$ due to this artificial extension.

Because we have a boundary-value problem, and not an initial-value problem, it will be necessary to specify unknown constants for $y''(0)$ and $y'''(0)$ in taking the Laplace transform of d^4y/dx^4. If we set $y''(0) = A$ and $y'''(0) = B$, and use equation 17.12 with $n = 4$,

$$s^4Y - s^3(0) - s^2(0) - As - B = \frac{k}{s}(1 - e^{-3s} + e^{-7s} - e^{-10s}).$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} + \frac{k}{s^5}(1 - e^{-3s} + e^{-7s} - e^{-10s}).$$

Inverse transforms give the deflection curve for the beam

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} + \frac{k}{24}\big[x^4 - (x-3)^4h(x-3) + (x-7)^4h(x-7) - (x-10)^4h(x-10)\big].$$

The last term contributes nothing to the curve for $0 < x < 10$, and is therefore dropped from further calculations. To find A and B we use the boundary conditions at $x = 10$. For $x > 7$,

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} + \frac{k}{24}\big[x^4 - (x-3)^4 + (x-7)^4\big],$$

and therefore

$$0 = y''(10) = A + 10B + \frac{k}{24}[12(10)^2 - 12(7)^2 + 12(3)^2],$$
$$0 = y'''(10) = B + \frac{k}{24}[24(10) - 24(7) + 24(3)].$$

These can be solved for $A = 30k$ and $B = -6k$, and hence

$$y(x) = 15kx^2 - kx^3 + \frac{k}{24}\big[x^4 - (x-3)^4h(x-3) + (x-7)^4h(x-7)\big]$$
$$= \frac{-9.81}{EI}\left\{15x^2 - x^3 + \frac{1}{24}\big[x^4 - (x-3)^4h(x-3) + (x-7)^4h(x-7)\big]\right\}.$$

A graph of this function for $EI = 10^4$ is shown in Figure 17.40. The deflection at the right end of the beam is $y(10) = -0.80$ m. As theory in Chapter 16 suggests, the function and its derivative (slope) are continuous even at $x = 3$ and $x = 7$ where the load is discontinuous. So also are $y''(x)$ and $y'''(x)$, although we cannot see this graphically.

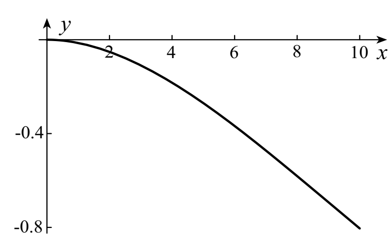


Figure 17.40

In the next example we use the Dirac delta function to represent a point load on a beam.

Example 17.26 A uniform beam of length L and mass m has both ends fixed horizontally in concrete. A force of P Newtons acting vertically downward at the centre of the beam is represented as a load $P\,\delta(x - L/2)$. Find the deflection curve of the beam.

Solution The boundary-value problem for deflections (Figure 17.41) is

$$\frac{d^4y}{dx^4} = -\frac{P}{EI}\delta(x - L/2) - \frac{mg}{EIL}[h(x) - h(x - L)],$$
$$y(0) = y'(0) = 0, \quad y(L) = y'(L) = 0,$$

where $g = 9.81$.

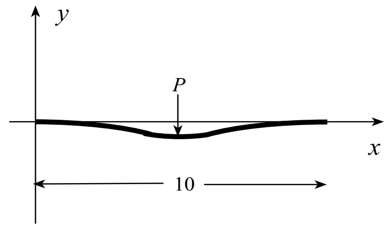


Figure 17.41

If we let $y''(0) = A$ and $y'''(0) = B$, and take Laplace transforms of both sides of the differential equation,

$$s^4Y - AS - B = -\frac{P}{EI}e^{-Ls/2} - \frac{mg}{EILs}(1 - e^{-Ls}).$$

Thus,

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} - \frac{P}{EIs^4}e^{-Ls/2} - \frac{mg}{EILs^5}(1 - e^{-Ls}),$$

and

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} - \frac{P}{6EI}(x - L/2)^3h(x - L/2) - \frac{mgx^4}{24EIL} + \frac{mg}{24EIL}(x - L)^4h(x - L).$$

The last term contributes nothing to the curve for $0 < x < L$, and is therefore dropped from the solution. For $y(L) = y'(L) = 0$,

$$0 = \frac{AL^2}{2} + \frac{BL^3}{6} - \frac{P(L/2)^3}{6EI} - \frac{mgL^3}{24EI}, \qquad 0 = AL + \frac{BL^2}{2} - \frac{P(L/2)^2}{2EI} - \frac{mgL^2}{6EI}.$$

These can be solved for $A = -\dfrac{PL}{8EI} - \dfrac{mgL}{12EI}$ and $B = \dfrac{P}{2EI} + \dfrac{mg}{2EI}$, and therefore

$$y(x) = -\frac{L(3P + 2mg)x^2}{48EI} + \frac{(P + mg)x^3}{12EI} - \frac{P}{6EI}(x - L/2)^3h(x - L/2) - \frac{mgx^4}{24EIL}.$$

The graph of this function should be symmetric about $x = L/2$. To verify this we express the function in the form

$$y(x) = \begin{cases} -\dfrac{L(3P + 2mg)x^2}{48EI} + \dfrac{(P + mg)x^3}{12EI} - \dfrac{mgx^4}{24EIL}, & 0 \le x \le L/2 \\ -\dfrac{L(3P + 2mg)x^2}{48EI} + \dfrac{(P + mg)x^3}{12EI} - \dfrac{mgx^4}{24EIL} - \dfrac{P}{6EI}(x - L/2)^3, & L/2 < x \le L \end{cases}.$$

If we replace x by $L - x$ in the second part, we obtain

$$-\frac{L(3P + 2mg)}{48EI}(L - x)^2 + \frac{(P + mg)(L - x)^3}{12EI} - \frac{mg(L - x)^4}{24EIL} - \frac{P}{6EI}(L - x - L/2)^3,$$

and this simplifies to

$$-\frac{L(3P+2mg)x^2}{48EI}+\frac{(P+mg)x^3}{12EI}-\frac{mgx^4}{24EIL},$$

thus verifying symmetry about $x = L/2$.

EXERCISES 17.5

1. (a) A uniform beam with mass m and length L is simply-supported at each end. Find the deflection curve.
(b) Is the deflection curve symmetric about $x = L/2$?

2. (a) Repeat Exercise 1 if the end at $x = 0$ is fixed horizontally and the end at $x = L$ is free.
(b) How far is the end $x = L$ from the horizontal?

3. (a) A uniform beam with mass m and length L has its end at $x = 0$ fixed horizontally and its end at $x = L$ is free. An additional mass M is distributed uniformly along the right half of the beam. Find the deflection curve.
(b) Are right- and left-hand limits of $y(x)$, $y'(x)$, $y''(x)$, and $y'''(x)$ at $x = L/2$ equal?

4. A uniform beam with mass m and length L has its left end horizontally fixed and its right end simply-supported. An additional mass M is distributed along the left third of the beam. Find the deflection curve.

5. (a) Repeat Exercise 4 if M is distributed over the middle third.
(b) Is the deflection curve symmetric about $x = L/2$?

6. (a) A uniform beam of length L has a concentrated force of P Newtons acting vertically downward at $x = L/3$. Both ends of the beam are clamped horizontally. If P is so large that the mass of the beam is negligible in comparison, find the deflection curve.
(b) Where is deflection a maximum?
(c) Compare right- and left-hand limits of $y(x)$, $y'(x)$, $y''(x)$, and $y'''(x)$ at $x = L/3$. Are they the same? Did you expect them to be the same?

7. (a) A uniform beam with mass m and length L has concentrated forces of P Newtons acting vertically downward at $x = L/3$ and $x = 2L/3$. Both ends of the beam are clamped horizontally. Find the deflection curve.
(b) What is the maximum deflection?

8. A uniform beam of length L has its left end fixed horizontally and its right end is free. A concentrated force of P Newtons acts vertically downward at $x = L/2$. If P is so large that the mass of the beam is negligible by comparison, find the deflection curve.

9. Repeat Exercise 8 if the mass of the beam is taken into account.

10. Repeat Exercise 8 if both ends of the beam are simply-supported.

11. Repeat Exercise 9 if both ends of the beam are simply-supported.

12. A uniform beam of length L and mass m is simply-supported at $x = 0$. If the right end is free, what physically should happen to the beam? Does equation 17.20 with boundary conditions 17.21a at $x = 0$ and 17.21c at $x = L$ confirm this?

13. A uniform beam extends between $x = 0$ and $x = L$ on the x-axis. Its left end is fastened in concrete in such a way that it points upwards making an angle of $\pi/10$ radians with the horizontal. A concentrated force of P Newtons acts vertically downward at $x = L/2$. If P is so large that the mass of the beam is negligible by comparison, and the right end of the beam is free, find its deflection curve.

14. A uniform beam of length L and mass m is simply-supported at both ends. What is the maximum value of m for deflections not to exceed 1% of the length of the beam?

15. A uniform beam of length L and mass m has both ends fixed horizontally. A concentrated force of P Newtons is applied vertically downward at its midpoint. What is the maximum value of P if deflections must not exceed 5% of the length of the beam?

ANSWERS

1.(a) $-[mg/(24EIL)](x^4 - 2Lx^3 + L^3x)$ (b) Yes
2.(a) $-[mg/(24EIL)](x^4 - 4Lx^3 + 6L^2x^2)$ (b) $-mgL^3/(8EI)$
3.(a) $-[Lg(2m+3M)/(8EI)]x^2 + [g(m+M)/(6EI)]x^3 - [mg/(24EIL)]x^4 - [Mg/(12EIL)]$ $(x-L/2)^4h(x-L/2)$ (b) Yes
4. $-[gL(25M+27m)/(432EI)]x^2 + [g(205M+135m)/(1296EI)]x^3 - [mg/(24EIL)]x^4$ $-[Mg/(8EIL)]x^4 + [Mg/(8EIL)](x-L/3)^4h(x-L/3)$
5.(a) $-[gL(9m+13M)/(144EI)]x^2 + [g(45m+49M)/(432EI)]x^3 - [mg/(24EIL)]x^4$ $-[Mg/(8EIL)](x-L/3)^3h(x-L/3) + [Mg/(8EIL)](x-2L/3)^3h(x-2L/3)$ (b) No
6.(a) $-[P/(6EI)](x-L/3)^3h(x-L/3) + 10Px^3/(81EI) - 2PLx^2/(27EI)$ (b) $x = 3L/7$ (c) $y'''(x)$ is not continuous at $x = L/3$
7.(a) $-[L(8P+3mg)/(72EI)]x^2 + [(2P+mg)/(12EI)]x^3 - [mg/(24EIL)]x^4 - [P/(6EI)](x-L/3)^3h(x-L/3) - [P/(6EI)](x-2L/3)^3h(x-2L/3)$ (b) $-L^3(27mg+80P)/(10368EI)$
8. $-[PL/(4EI)]x^2 + [P/(6EI)]x^3 - [P/(6EI)](x-L/2)^3h(x-L/2)$
9. $-[L(P+mg)/(4EI)]x^2 + [(P+mg)/(6EI)]x^3 - [mg/(24EIL)]x^4 - [P/(6EI)](x-L/2)^3h(x-L/2)$
10. $-[PL^2/(16EI)]x + [P/(12EI)]x^3 - [P/(6EI)](x-L/2)^3h(x-L/2)$
11. $-[L^2(3P+2mg)/(48EI)]x + [(P+mg)/(12EI)]x^3 - [mg/(24EIL)]x^4 - [P/(6EI)](x-L/2)^3h(x-L/2)$
12. Not a posible situation
13. $[\tan(\pi/10)/(EI)]x - [PL/(4EI)]x^2 + [P/(6EI)]x^3 - [P/(6EI)](x-L/2)^3h(x-L/2)$
14. $96EI/(125gL^2)$ **15.** $48EI/(5L^2) - mg/2$