Math 3132 Practice Questions (Fall 2019)

Part 1 (Sections 14.1–14.4)

1. Let $\mathbf{F}(x,y,z) = \frac{1}{2}x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$ and $f(x,y,z) = \dfrac{-1}{x+2y+6z}$. Find values of a, b, and c such that

$$(\nabla \cdot \mathbf{F})^2\, \nabla f - (a,\, 2b,\, 3c) = \nabla \times (\nabla f - \mathbf{F})\,.$$

2. Let $\mathbf{F}$ be a vector field. If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then is $\mathbf{F} = \mathbf{0}$? If yes prove it and if no give a counter example.

3. Evaluate the line integral of $f(x,y,z) = (x^2 + \frac{y^2}{3})^2 + 60\,x^3 y - 1$ along the curve C, where C is that part of $x^2 + \frac{y^2}{3} = 1,\ z = 0$ from $(1,0,0)$ to $(0,\ \sqrt{3},\ 0)$.

4. Let C be a curve with initial and final points A and B. Also let $f(x,y,z)$ and $\mathbf{F}(x,y,z)$ be a real valued function and a vector field defined along C such that $\int_C f\,ds = 2$, $\int_C \nabla f \cdot d\mathbf{r} = 4$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = 5$. Show that

$$\int_{-C} (3\nabla f - 2\mathbf{F}) \cdot d\mathbf{r} + \int_{-C} 4f\,ds = 6,$$

where $-C$ is the same as C with the opposite direction, that is with initial and final points B and A. Explain your work.

5. (a) Is the line integral $\int y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$ independent of path in $\mathbf{R}^3$? Why?

   (b) Evaluate $\int_C y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$ where C is the curve with parametric equations

$$C: \quad x = (1-t)^2, \quad y = t^2, \quad z = t, \quad 0 \le t \le 2\,.$$

6. Evaluate $\int_C e^{x+yz^2}\,dx + z^2e^{x+yz^2}\,dy + 2yz\,e^{x+yz^2}\,dz$ where C is the curve $\frac{x^2}{16} + (y-4)^2 = 1, \quad y - z = 5$ from $(4, 4, -1)$ to $(0, 5, 0)$.

Part 2 (Sections 14.5–14.7)

7. Let $\mathbf{F}(x,y) = (\frac{2}{3}\,xy\,\sqrt{y})\,\hat{\mathbf{i}} + (\frac{3}{2}\,x^2\,\sqrt{y})\,\hat{\mathbf{j}}$. Find the work done by $\mathbf{F}$ on a particle that moves along C where C traverses once counter-clockwise around the region in the $xy-$plane bounded by the parabolas $y = x^2$, $y = (x-2)^2$ and the line $y = 0$.

8. Find the area of that part of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which is not in the third quadrant.

9. Evaluate the surface integral $\iint_S xy\,dS$ where S is that part of the paraboloid $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$ inside the sphere $x^2 + y^2 + z^2 = 3$ in the first octant.

10. Evaluate the surface integral of $f(x, y, z) = 2y^2z$ over the surface S, where S is that part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

Part 3 (Sections 14.8–14.9)

11. Let S be that part of sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = 2$. Evaluate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, where

$$\mathbf{F}(x, y, z) = y\,\hat{\mathbf{i}} - \mathbf{x}\,\hat{\mathbf{j}} + \sqrt[3]{(1 - x^2 - y^2)^2}\;\hat{\mathbf{k}},$$

and $\hat{\mathbf{n}}$ is the unit upper normal to the surface S.

12. Evaluate the surface integral $\oiint_S \left(\frac{4}{3}x^3\,\hat{\mathbf{i}} + \frac{4}{3}y^3\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{n}}\, dS$, where S is the surface enclosing the volume defined by $z = 2 - x^2 - y^2$, $z = \sqrt{x^2 + y^2}$, and $\hat{\mathbf{n}}$ is the unit outer normal to S.

Part 4 (sections 14.10 and 17.1)

13. Evaluate the line integral $\oint_C y^2\,dx + xz^3\,dy + x^3\,dz$ where C is the curve of intersection of sphere $x^2 + y^2 + z^2 = 8$ and the cone $x^2 + y^2 = z^2$ with $z \geq 0$, directed clockwise as viewed from the origin.(Do it with and also without Stokes' s Theorem.)

14. Evaluate $I = \oint_C [\,(xy + 3x^2y^2)\,\hat{\mathbf{i}} + (z + 2x^3y)\,\hat{\mathbf{j}} + (z^2 + x^2z^2)\,\hat{\mathbf{k}}\,] \cdot d\mathbf{r}$, where C is the curve $x^2 + z^2 = 1$, $x^2 + y^2 = 1$, $z = y$, directed counterclockwise as viewed from a point far up the positive z−axis.

15. Assuming that $y = \sum_{n=0}^{\infty} a_n\,(x - 1)^n$ is a solution of the differential equation

$$(x - 1)^2\,y'' - (x - 1)\,y' - (x^2 - 2x)\,y = 0,$$

find a recurrence relation for a_n and simplify it as much as possible. (Do not continue after finding the recurrence relation).

16. For the differential equation $x\,y'' + 3\,y' - xy = 0$, when you use the power series $y(x) = \sum_{n=0}^{\infty} a_n\,x^n$ to solve the differential equation, you get

$$3\,a_1 + \sum_{n=2}^{\infty} [\,n(n + 2)\,a_n + a_{n-2}\,]\,x^{n-1} = 0.$$

You do not need to prove this relation. Use it to find the solution of the differential equation. Write your solution using sigma notation and **simplify as much as possible**.

17. Use $y = \sum_{n=0}^{\infty} a_n\,x^n$ to solve the differential equation

$$x^2\,y'' + x\,y' + (x^2 - 1)\,y = 0.$$

Simplify as much as possible. Is this solution a general solution? What is the interval of convergence?

Part 5 (sections 17.2, 18.1 and 18.2)

18. Consider the differential equation

$$(\sin x)\, y'' \;+\; \frac{\sin x}{x^2+16}\, y' \;+ (x\cos x)\, y \;=\; 0\,.$$

   (a) Is $x = 0$ a singular point for the differential equation? Why?

   (b) Find all real or complex singular points of the differential equation.

   (c) What can be said about the radius of convergence of a power series solution about $x = 3$ for the differential equation?
   (You are **not** asked to solve the differential equation.)

19. Consider the differential equation

$$(x^2 - 2x + 2)\, y'' \;+\; x^2\, y' \;-\; (\sin^2 x)\, y \;=\; 0\,.$$

   (a) Find all real or complex singular points of the differential equation.

   (b) If $y(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x+1)^n$ is used to solve this differential equation, will the result be a general solution ? What can be said about the radius of convergence of this power series solution ? Justify your conclusions.(You do not need to solve the differential equation.)

20. Let $f(x) = \begin{cases} x-1 & \text{if } 0 < x < 1 \\ 0 & \text{if } \; 1 < x < 2 \end{cases}$ with $f(x+2) = f(x)$.

   (a) On the interval $-4 \le x \le 4$, draw the graph of $f(x)$; also draw the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges. Describe $g(x)$.

   (b) Find the Fourier series for the periodic function $f(x)$. Simplify your answer as much as possible.

   (c) Use part (b) to find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

21. Let $f(x) = \begin{cases} 8x + 16x^2 & \text{if } -\frac{1}{4} < x \le 0 \\ 8x - 16x^2 & \text{if } \; 0 < x \le \frac{1}{4} \end{cases}$ with $f(x + \frac{1}{2}) = f(x)$.

   (a) Draw the graph of $f(x)$ in the interval $-\frac{5}{4} \le x \le \frac{5}{4}$.

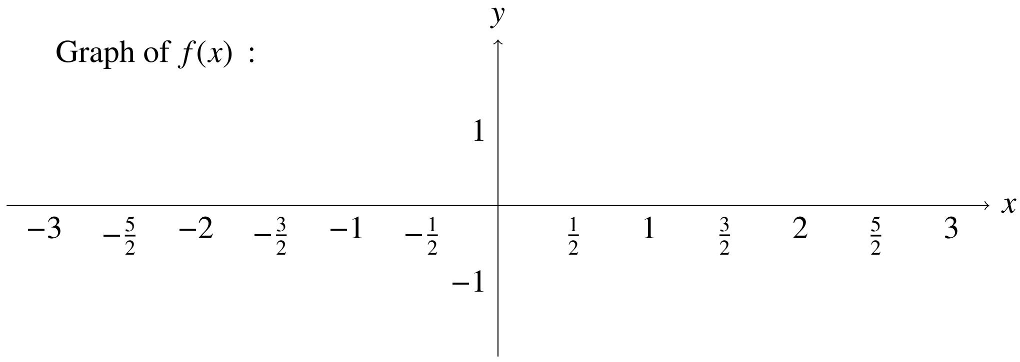
   (b) Draw the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges to, in the interval $-\frac{5}{4} \le x \le \frac{5}{4}$. Describe $g(x)$.

Part 6 (section 18.2)

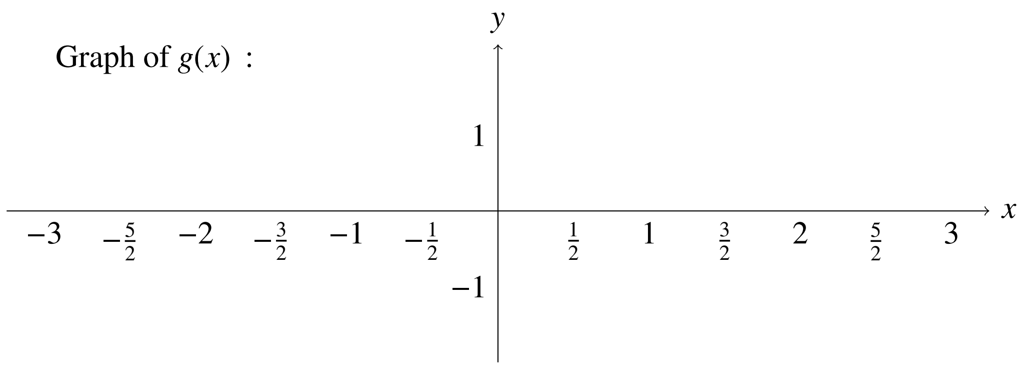
22. Let $f(x) = \begin{cases} 0 & \text{if } -1 < x < -\frac{1}{2} \\ 1+2x & \text{if } -\frac{1}{2} < x < 0 \\ 1-2x & \text{if } \; 0 < x < \frac{1}{2} \\ 0 & \text{if } \; \frac{1}{2} < x < 1 \end{cases}$, with $f(x+2) = f(x)$.

(a) Draw the graph of $f(x)$ and also the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges to, in the interval $-3 \le x \le 3$.

Graph of $f(x)$:



Graph of $g(x)$:



(b) Find Fourier series of $f(x)$ and simplify as much as possible. Use it to evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$.

23. Let $f(x) = 3x - 2$ where $0 \le x \le \dfrac{1}{4}$. Expand $f(x)$ to a Fourier cosine series such that the Fourier cosine series converges to $f(x)$ on the interval $0 < x < \dfrac{1}{4}$.

Part 7 (sections 19.1, 19.2 and 20.1–20.3)

24. Let $f(x) = 3x - 2$ where $0 \le x \le \dfrac{1}{4}$. Expand $f(x)$ in terms of the eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad 0 < x < \frac{1}{4}, \quad y(0) = 0, \quad y'(L) = 0.$$

25. Consider the Sturm-Liouville system

$$\begin{aligned} y'' + 4y' + (1 - 2\lambda)\, y &= 0, \qquad 0 < x < L, \\ y(0) &= 0, \\ y(L) &= 0. \end{aligned}$$

(a) Find the standard form of the Sturm-Liouville system.

(b) If $\lambda < -\dfrac{3}{2}$, find all eigenvalues and eigenfunctions of the Sturm-Liouville system.

26. A string with constant linear density ρ is stretched tightly between the points $x = 0$ and $x = 12$ on the x-axis. The tension in the string is a constant τ. The displacement of the string at time $t = 0$ is shown in the figure below, and from this position, it is released. The right end of the string is fixed on the x-axis, but the left end is looped around a vertical rod, and can move vertically without friction. A restoring force proportional

to displacement and also gravity are taken into account. What is the initial-value problem for displacement $y(x,t)$ of the string? Include the partial differential equation, and all boundary and initial conditions, and include intervals on which they must be satisfied.