Math 3132 Practice Questions (Fall 2019)

Part 1 (sections 14.1–14.4)

1. Let $\mathbf{F}(x,y,z) = \frac{1}{2}x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$ and $f(x,y,z) = \dfrac{-1}{x+2y+6z}$. Find values of a, b, and c such that

$$(\nabla \cdot \mathbf{F})^2\,\nabla f - (a,\ 2b,\ 3c) = \nabla \times (\nabla f - \mathbf{F})\,.$$

Solution:

$$\begin{aligned}(\nabla \cdot \mathbf{F})^2\,\nabla f &= (x+2y+6z)^2\Big(\frac{1}{(x+2y+6z)^2},\ \frac{2}{(x+2y+6z)^2},\ \frac{6}{(x+2y+6z)^2}\Big)\\ &= (1,\ 2,\ 6)\,.\end{aligned}$$

$$\nabla \times (\nabla f - \mathbf{F}) = \nabla \times \nabla f - \nabla \times \mathbf{F} = \mathbf{0} - \nabla \times \mathbf{F} = -\nabla \times \mathbf{F}\,.$$

But

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}x^2 & y^2 & 3z^2 \end{vmatrix} = (0-0)\hat{\mathbf{i}} - (0-0)\hat{\mathbf{j}} + (0-0)\hat{\mathbf{k}} = \mathbf{0}\,.$$

Therefore $\nabla \times (\nabla f - \mathbf{F}) = \mathbf{0}$. Hence

$$(1,\ 2,\ 6) - (a,\ 2b,\ 3c) = \mathbf{0} \Rightarrow (a,\ 2b,\ 3c) = (1,\ 2,\ 6) \Rightarrow a = 1,\ b = 1,\ c = 2\,.$$

2. Let $\mathbf{F}$ be a vector field. If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then is $\mathbf{F} = \mathbf{0}$? If yes prove it and if no give a counter example.

Solution: The answer is no; for example for any non zero constant vector field $\mathbf{F} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + \mathbf{c}\hat{\mathbf{k}}$, $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$.
There are many other examples as well for instance for any of the following vector field we have the same result:
$\mathbf{F}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$, $\mathbf{F}(x,y,z) = 2x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$,
$\mathbf{F}(x,y,z) = (y+z)\,\hat{\mathbf{i}} + (x+z)\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$, $\mathbf{F}(x,y,z) = yz\,\hat{\mathbf{i}} + xz\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$.

3. Evaluate the line integral of $f(x,y,z) = (x^2 + \frac{y^2}{3})^2 + 60\,x^3y - 1$ along the curve C, where C is that part of $x^2 + \frac{y^2}{3} = 1,\ z = 0$ from $(1,0,0)$ to $(0,\ \sqrt{3},\ 0)$.

Solution: **(question 1)** Let $x = \cos t$ then $y = \sqrt{3}\sin t$ so

$$C: \quad x = \cos t, \quad y = \sqrt{3}\sin t, \quad z = 0, \qquad 0 \le t \le \frac{\pi}{2},$$

$$\begin{aligned} ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\,dt &= \sqrt{(-\sin t)^2 + (\sqrt{3}\cos t)^2 + 0}\,dt \\ &= \sqrt{\sin^2 t + 3\cos^2 t}\,dt \end{aligned}$$

$$= \sqrt{1 + 2\cos^2 t}\, dt\,.$$

$$\int_C f ds = \int_0^{\frac{\pi}{2}} (1^2 + 60 \cos^3 t(\sqrt{3}\sin t) - 1)\, \sqrt{1+2\cos^2 t}\, dt$$
$$= \sqrt{3}\int_0^{\frac{\pi}{2}} 60 \cos^3 t \sin t\, \sqrt{1+2\cos^2 t}\, dt\,.$$

Now let $u = \sqrt{1+2\cos^2 t}$ then $u^2 = 1 + 2\cos^2 t$ and $2udu = -4\sin t \cos t\, dt$ that is $dt = \dfrac{-u}{2\sin t\cos t}\, du$ and $\cos^2 t = \frac{1}{2}(u^2-1)$. Hence

$$\int_C f ds = \sqrt{3}\int_{t=0}^{t=\frac{\pi}{2}} (\,60 \cos^3 t \sin t)\,)\,(u)\,(\frac{-u}{2\sin t\cos t}\, du)$$
$$= \sqrt{3}\int_{t=0}^{t=\frac{\pi}{2}} -30\,(\cos^2 t) u^2\, du$$
$$= \sqrt{3}\int_{t=0}^{t=\frac{\pi}{2}} -15\,(u^2-1) u^2\, du$$
$$= 15\sqrt{3}\int_{t=0}^{t=\frac{\pi}{2}} (u^2 - u^4)\, du$$
$$= 15\sqrt{3}\,[\,\frac{1}{3}u^3 - \frac{1}{5}u^5)\,]\Big|_{t=0}^{t=\frac{\pi}{2}}$$
$$= \sqrt{3}\,[5(1+2\cos^2 t)^{\frac{3}{2}} - 3(1+2\cos^2 t)^{\frac{5}{2}}]\Big|_0^{\frac{\pi}{2}}$$
$$= \sqrt{3}\,[(5-3) - (5(3)^{\frac{3}{2}} - 3(3)^{\frac{5}{2}})]$$
$$= \sqrt{3}\,(12\sqrt{3} + 2)$$
$$= 36 + 2\sqrt{3}\,.$$

4. Let C be a curve with initial and final points A and B. Also let $f(x,y,z)$ and $\mathbf{F}(x,y,z)$ be a real valued function and a vector field defined along C such that $\int_C f\, ds = 2$, $\int_C \nabla f \cdot d\mathbf{r} = 4$ and $\int_C \mathbf{F}\cdot d\mathbf{r} = 5$. Show that

$$\int_{-C} (3\nabla f - 2\mathbf{F})\cdot d\mathbf{r} + \int_{-C} 4f\, ds = 6\,,$$

where $-C$ is the same as C with the opposite direction, that is with initial and final points B and A. Explain your work.

Solution: We know that in general $\int_{-C} f\, ds = \int_C f\, ds$ and $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$. Now

$$\int_{-C} (3\nabla f - 2\mathbf{F})\cdot d\mathbf{r} + \int_{-C} 4f\, ds$$
$$= 3\int_{-C} \nabla f\cdot d\mathbf{r} - 2\int_{-C} \mathbf{F}\cdot d\mathbf{r} + 4\int_{-C} f\, ds$$
$$= 3(-\int_C \nabla f\cdot d\mathbf{r}) - 2(-\int_C \mathbf{F}\cdot d\mathbf{r}) + 4\int_C f\, ds$$
$$= 3(-4) + 2(5) + 4(2)$$

$$= 6\,.$$

5. (a) Is the line integral $\int y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$ independent of path in $\mathbf{R}^3$? Why?

Solution:

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \\ &= (6xyz^2 - 6xyz^2)\hat{\mathbf{i}} - (3y^2z^2 - 3y^2z^2)\hat{\mathbf{j}} + (2yz^3 - 2yz^3)\hat{\mathbf{k}} \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\,;\end{aligned}$$

and since $\mathbf{R}^3$ is simply connected, so yes the line integral is independent of path.

(b) Evaluate $\int_C y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$ where C is the curve with parametric equations

$$C: \quad x = (1-t)^2, \quad y = t^2, \quad z = t, \quad 0 \le t \le 2\,.$$

Solution: Let $f(x,y,z) = xy^2z^3$ then $\mathbf{F} = \nabla f$. Now $t = 0$ corresponds to the point $(1,0,0)$ and $t = 2$ corresponds to the point $(1,4,2)$, therefore

$$\begin{aligned}\int y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz &= f(1,4,2) - f(1,0,0) \\ &= 1(4^2)(2^3) - 1(0^2)(0^3) \\ &= 128\,.\end{aligned}$$

Alternatively one could use the given parametrization of the curve C and evaluate the integral.

6. Evaluate $\int_C e^{x+yz^2}\,dx + z^2e^{x+yz^2}\,dy + 2yz\,e^{x+yz^2}\,dz$ where C is the curve $\frac{x^2}{16} + (y-4)^2 = 1, \quad y - z = 5$ from the point $(4,4,-1)$ to the point $(0,5,0)$.

Solution: Let $\mathbf{F}(x,y,z) = e^{x+yz^2}\hat{\mathbf{i}} + z^2e^{x+yz^2}\hat{\mathbf{j}} + 2yze^{x+yz^2}\hat{\mathbf{k}}$ then

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+yz^2} & z^2e^{x+yz^2} & 2yze^{x+yz^2}\hat{\mathbf{k}} \end{vmatrix} \\ &= e^{x+yz^2}(2z + 2yz^3 - 2z - 2yz^3)\hat{\mathbf{i}} - e^{x+yz^2}(2yz - 2yz)\hat{\mathbf{j}} + e^{x+yz^2}(z^2 - z^2)\hat{\mathbf{k}} \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\,;\end{aligned}$$

and since $\mathbf{R}^3$ is simply connected, so the line integral is independent of path. By inspection for $f(x,y,z) = e^{x+yz^2}$ we have $\mathbf{F} = \nabla f$. Therefore

$$\begin{aligned}\int_C e^{x+yz^2}\,dx + z^2e^{x+yz^2}\,dy + 2yze^{x+yz^2}\,dz &= f(0,5,0) - f(4,4,-1) \\ &= e^{0+0(5^2)} - e^{4+4(-1)^2} \\ &= 1 - e^8\,.\end{aligned}$$