Solutions of Math 3132 Practice Questions Part 3

11. Let S be that part of sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = 2$. Evaluate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$, where

$$\mathbf{F}(x, y, z) = y\,\hat{\mathbf{i}} - \mathbf{x}\,\hat{\mathbf{j}} + \sqrt[3]{(1 - x^2 - y^2)^2}\ \hat{\mathbf{k}},$$

and $\hat{\mathbf{n}}$ is the unit upper normal to the surface S.

Solution: We offer two solutions:

Solution 1: $z = \sqrt{5 - x^2 - y^2}$ and

$$\mathbf{n} = \nabla(z - \sqrt{5 - x^2 - y^2}) = \Big(\frac{x}{\sqrt{5 - x^2 - y^2}},\ \frac{y}{\sqrt{5 - x^2 - y^2}},\ 1\Big) \Rightarrow$$

$$|\mathbf{n}| = \sqrt{\frac{x^2}{5 - x^2 - y^2} + \frac{y^2}{5 - x^2 - y^2} + 1} = \frac{\sqrt{5}}{\sqrt{5 - x^2 - y^2}} \Rightarrow$$

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{\sqrt{5}}(x,\ y,\ \sqrt{5 - x^2 - y^2});$$

$$\mathbf{F} \cdot \hat{\mathbf{n}} = (y, -x, \sqrt[3]{(1 - x^2 - y^2)^2}) \cdot \frac{1}{\sqrt{5}}(x,\ y,\ \sqrt{5 - x^2 - y^2}) = \frac{1}{\sqrt{5}}\sqrt[3]{(1 - x^2 - y^2)^2}\ \sqrt{5 - x^2 - y^2}$$

$$dS = \sqrt{1 + \Big(-\frac{x}{\sqrt{5 - x^2 - y^2}}\Big)^2 + \Big(-\frac{y}{\sqrt{5 - x^2 - y^2}}\Big)^2}\, dA = \frac{\sqrt{5}}{\sqrt{5 - x^2 - y^2}}\, dA.$$

Therefore

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS &= \iint_{S_{xy}} \frac{1}{\sqrt{5}}\sqrt[3]{(1 - x^2 - y^2)^2}\ \sqrt{5 - x^2 - y^2}\Big(\frac{\sqrt{5}}{\sqrt{5 - x^2 - y^2}}\, dA\Big) \\
&= \iint_{S_{xy}} (1 - (x^2 + y^2))^{\frac{2}{3}}\, dA \\
&= \int_0^{2\pi}\int_0^1 r(1 - r^2)^{\frac{2}{3}}\, dr d\theta \\
&= \int_0^{2\pi} -\frac{3}{10}(1 - r^2)^{\frac{5}{3}}\Big|_0^1\, d\theta \\
&= \frac{3}{10}\int_0^{2\pi} d\theta \\
&= \frac{3\pi}{5}.
\end{aligned}$$

Solution 2: In order to use Divergence Theorem we need a closed surface. But

$$z = 2 \Rightarrow x^2 + y^2 + 2^2 = 5 \Rightarrow x^2 + y^2 = 1$$

So let S_1 be the surface $z = 2$, $x^2 + y^2 \le 1$ then $S \cup S_1$ is closed and

$$\oiint_{S \cup S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS + \iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\ dS \qquad (1)$$

Now by Divergence Theorem:

$$\begin{aligned}
\oiint_{S \cup S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\ dS &= \iiint_V \Big[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}\big(\sqrt[3]{(1 - x^2 - y^2)^2}\,\big)\Big]\ dV \\
&= \iiint_V 0\, dV = 0.
\end{aligned}$$

On S_1 , $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and

$$\mathbf{F} \cdot \hat{\mathbf{n}} = (y, -x, \sqrt[3]{(1 - x^2 - y^2)^2}\,) \cdot (0, 0, -1) = -\sqrt[3]{(1 - x^2 - y^2)^2}\,,$$

also $dS = \sqrt{1+0+0}\,dA = dA$, hence

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\ dS &= \iint_{S_{xy}} -\sqrt[3]{(1 - x^2 - y^2)^2}\ dA \\
&= -\int_0^{2\pi} \int_0^1 r(1 - r^2)^{\frac{2}{3}}\ drd\theta \\
&= \frac{1}{2} \int_0^{2\pi} \frac{3}{5}(1 - r^2)^{\frac{5}{3}}\Big|_0^1 d\theta \\
&= -\frac{3}{10} \int_0^{2\pi} d\theta \\
&= -\frac{3\pi}{5}\,.
\end{aligned}$$

Substitution in (1) gives $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = 0 - (-\frac{3\pi}{5}) = \frac{3\pi}{5}$.

12. Evaluate the surface integral $\oiint_S \left(\frac{4}{3}x^3\,\hat{\mathbf{i}} + \frac{4}{3}y^3\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{n}}\ dS$, where S is the surface enclosing the volume defined by $z = 2 - x^2 - y^2$, $z = \sqrt{x^2 + y^2}$, and $\hat{\mathbf{n}}$ is the unit outer normal to S .

Solution: First we find the intersection of the two surfaces:

$\sqrt{x^2 + y^2} = z \Rightarrow z = 2 - z^2 \Rightarrow (z-1)(z+2) = 0 \Rightarrow z = 1$, $z = -2$ NA $\Rightarrow x^2 + y^2 = 1$.

Using Divergence Theorem we get

$$\begin{aligned}
\oiint_S \left(\frac{4}{3}x^3\,\hat{\mathbf{i}} + \frac{4}{3}y^3\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{n}}\,dS &= \iiint_V \nabla \cdot \mathbf{F}\ dV \\
&= \iiint_V (4x^2 + 4y^2 - 1)\ dV \quad \text{(Using Cylindrical System)} \\
&= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r(4r^2 - 1)\ dzdrd\theta \\
&= \int_0^{2\pi} \int_0^1 (4r^3 - r)z\Big|_r^{2-r^2} drd\theta \\
&= \int_0^{2\pi} \int_0^1 (-4r^5 - 4r^4 + 9r^3 + r^2 - 2r)\ drd\theta \\
&= \int_0^{2\pi} (\frac{-4}{6}r^6 - \frac{4}{5}r^5 + \frac{9}{4}r^4 + \frac{1}{3}r^3 - r^2)\Big|_0^1 d\theta \\
&= \frac{7}{60} \int_0^{2\pi} d\theta \\
&= \frac{7\pi}{30}\,.
\end{aligned}$$