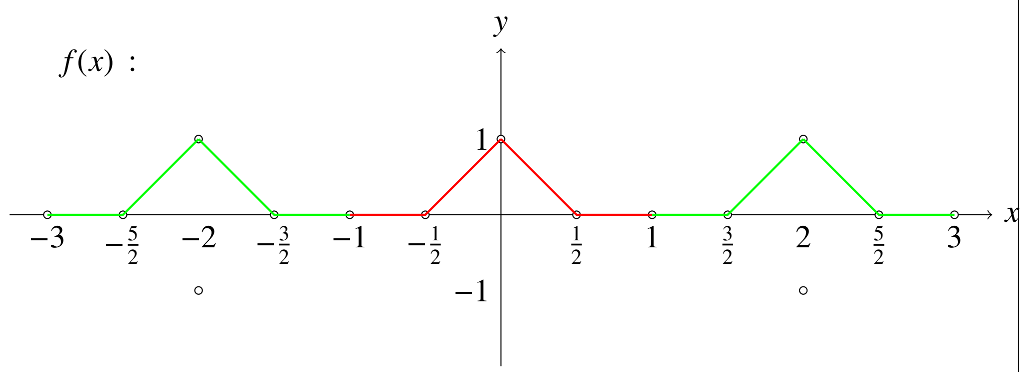
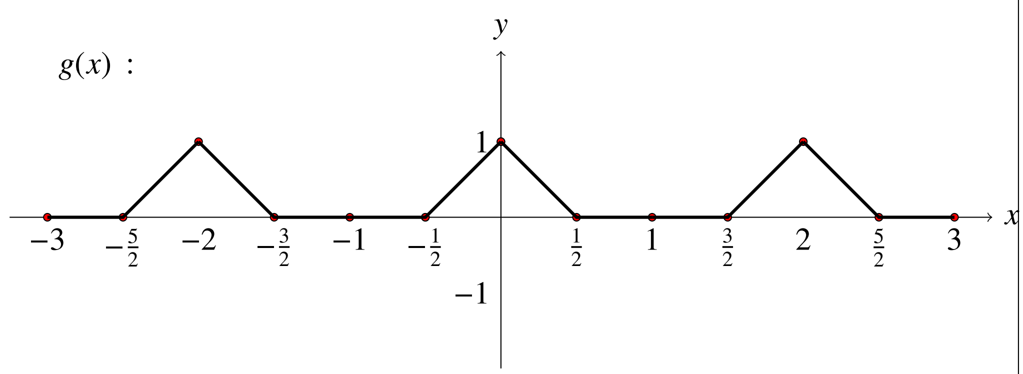
Solutions of Math 3132 Practice Questions Part 6

22. Let $f(x) = \begin{cases} 0 & \text{if } -1 < x < -\frac{1}{2} \\ 1 + 2x & \text{if } -\frac{1}{2} < x < 0 \\ 1 - 2x & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1 \end{cases}$, with $f(x+2) = f(x)$.

(a) Draw the graph of $f(x)$ and also the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges to, in the interval $-3 \le x \le 3$.

Solution:





(b) Find Fourier series of $f(x)$ and simplify as much as possible. Use it to evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$.

Solution: In fact the function can be restated as $f(x) = \begin{cases} 1 - 2x & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1 \end{cases}$

with $f(x+2) = f(x)$ and $f(-x) = f(x)$.

Since $f(x)$ is an even function so $b_n = 0$ for all $n \ge 1$, $a_0 = \frac{2}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$ for all $n \ge 1$. But $2L = 2$ so $L = 1$. Hence

$$a_0 = \frac{2}{1}\int_0^1 f(x)dx = 2[\int_0^{\frac{1}{2}}(1-2x)dx + \int_{\frac{1}{2}}^1 0dx] = \cdots = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^{\frac{1}{2}}(1-2x)\cos n\pi x\, dx.$$

Using integration by parts gives $a_n = \frac{4}{\pi^2 n^2}(1 - \cos\frac{n\pi}{2})$. Hence the Fourier series is

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x = \frac{1}{4} + \sum_{n=1}^{\infty}[\frac{4}{\pi^2 n^2}(1 - \cos\frac{n\pi}{2})]\cos n\pi x\,.$$

But $\cos\frac{n\pi}{2} = 0$ if $n = 2k+1$ and $\cos\frac{n\pi}{2} = (-1)^k$ if $n = 2k$. So

$$\begin{aligned}
g(x) =& \frac{1}{4} + \sum_{n=1}^{\infty}[\frac{4}{\pi^2 n^2}(1 - \cos\frac{n\pi}{2})]\cos n\pi x \\
=& \frac{1}{4} + \sum_{n=1}^{\infty}[\frac{4}{\pi^2(2n-1)^2}(1-0)]\cos(2n-1)\pi x + \sum_{n=1}^{\infty}[\frac{4}{\pi^2(2n)^2}(1-(-1)^n)]\cos(2n)\pi x \\
=& \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^2}]\cos(2n-1)\pi x + \frac{1}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{n^2}(1-(-1)^n)]\cos 2n\pi x \\
=& \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^2}]\cos(2n-1)\pi x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^2}]\cos 2(2n-1)\pi x\,.
\end{aligned}$$

For $x = \frac{1}{2}$ since $g(\frac{1}{2}) = 0$ so by substitution we get

$$0 = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^2}]\cos\frac{(2n-1)\pi}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}[\frac{1}{(2n-1)^2}]\cos(2n-1)\pi\,.$$

But $\cos\frac{(2n-1)\pi}{2} = 0$ and $\cos(2n-1)\pi = -1$ so then $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.
Note that you can use other points as well. For instance if you use $x = 0$ then it will gave the same result.

23. Let $f(x) = 3x - 2$ where $0 \le x \le \frac{1}{4}$. Expand $f(x)$ to a Fourier cosine series such that the Fourier cosine series converges to $f(x)$ on the interval $0 < x < \frac{1}{4}$.

Solution: So $L = \frac{1}{4}$ and we need to find a_0 and a_n such that

$$f(x) = 3x - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 4n\pi x\,.$$

Now

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{\frac{1}{4}}\int_0^{\frac{1}{4}}(3x-2)\,dx = 8(\frac{3}{2}x^2 - 2x)\Big|_0^{\frac{1}{4}} = -\frac{13}{4}\,;$$

$$\begin{aligned}
a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx &= \frac{2}{\frac{1}{4}}\int_0^{\frac{1}{4}}(3x-2)\cos\frac{n\pi x}{\frac{1}{4}}\,dx \\
&= 8\int_0^{\frac{1}{4}}(3x-2)\cos 4n\pi x\,dx\,;
\end{aligned}$$

using integration by parts, let $u = 3x - 2$ and $dv = \cos 4n\pi x\,dx$ then

$du = 3dx$ and $v = \dfrac{1}{4n\pi} \sin 4n\pi x$. Hence

$$
\begin{aligned}
a_n &= 8 \int_0^{\frac{1}{4}} (3x - 2) \cos 4n\pi x \, dx \\
&= 8 \Big[\frac{3x - 2}{4n\pi} \sin 4n\pi x \Big|_0^{\frac{1}{4}} - \frac{3}{4n\pi} \int_0^{\frac{1}{4}} \sin 4n\pi x \, dx \Big] \\
&= 8 \Big[0 + \frac{3}{16n^2\pi^2} \cos 4n\pi x \Big|_0^{\frac{1}{4}} \Big] \\
&= \frac{3}{2n^2\pi^2} [(-1)^n - 1] .
\end{aligned}
$$

Therefore the Fourier cosine series of $f(x)$ is

$$
\begin{aligned}
f(x) = 3x - 2 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 4n\pi x \\
&= \frac{-\frac{13}{4}}{2} + \sum_{n=1}^{\infty} \frac{3}{2n^2\pi^2} [(-1)^n - 1] \cos 4n\pi x \\
&= -\frac{13}{8} - \frac{3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos 4(2n-1)\pi x , \quad 0 < x < \frac{1}{4} .
\end{aligned}
$$