A GENUS TWO ARITHMETIC SIEGEL-WEIL FORMULA ON $X_0(N)$

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ABSTRACT. We define a family of arithmetic zero cycles in the arithmetic Chow group of a modular curve $X_0(N)$, for $N > 3$ odd and squarefree, and identify the arithmetic degrees of these cycles as q-coefficients of the central derivative of a Siegel Eisenstein series of genus two. This parallels work of Kudla-Rapoport-Yang for Shimura curves.

CONTENTS

1. INTRODUCTION

In a series of work culminating in the book [KRY06], Kudla, Rapoport and the third named author studied certain families of arithmetic "special" cycles that live on arithmetic models of Shimura curves. Among their results are *arithmetic Siegel-Weil formulas* in genus one and two, which identify generating series of heights of arithmetic cycles with derivatives of Eisenstein series. The results of [KRY06] comprise the most fully-developed example in Kudla's program, which seeks

2010 *Mathematics Subject Classification.* 11G18, 11F46, 14G40, 14G35.
SS was partially supported by an NSERC Discovery grant. TY was partially supported by Van Vleck Research grant and Dorothy Gollmar chair fund.

to establish systematic relations between arithmetic cycles on Shimura varieties and automorphic forms; while a substantial body of work has arisen in support of these conjectures, the case of modular curves has been largely overlooked.

In this note, we fill this gap in the literature and prove the arithmetic Siegel-Weil formula for arithmetic zero cycles on $\mathcal{X}_0(N)$ for odd, squarefree N with $N > 3$. More precisely, we construct a family of arithmetic zero cycles $\widehat{\mathcal{Z}}(T, v)$, viewed in the arithmetic Chow group $\widehat{\mathrm{CH}}^2(\mathcal{X}_0(N))$. Our main result, explicated in more detail in Section 2.5 and Section 2.6 below, is the identity

$$\sum_{T \in \mathrm{Sym}_2(\mathbb{Z})^\vee} \widehat{\deg}\, \widehat{\mathcal{Z}}(T, v)\, q^T \;=\; C \cdot E'(\tau, 0, \Phi^{\mathcal{L}}); \tag{1}$$

here

- $\tau \in \mathbb{H}_2$, the Siegel upper half space of genus 2 and $v = \mathrm{Im}(\tau)$;
- $q^T = e^{2\pi i \mathrm{tr}(\tau T)}$;
- $C = \frac{1}{24} \prod_{p|N} (p + 1)$;
- $\widehat{\deg}$ is the arithmetic degree; and
- $E'(\tau, 0, \Phi^{\mathcal{L}})$ is the derivative of the Siegel Eisenstein series, evaluated at the centre of symmetry $s = 0$, associated to the quadratic lattice $\mathcal{L}$ given by

$$\mathcal{L} = \left\{ x = \begin{pmatrix} a & b/N \\ c & -a \end{pmatrix} :\, a, b, c \in \mathbb{Z} \right\}, \quad Q_{\mathcal{L}}(x) = N det(x) = -Na^2 - bc. \tag{2}$$

Our proof of the main result, which proceeds by matching the q-coefficients term-by-term, is closely modelled on the arguments in [KRY06]. For positive definite T, it turns out that the cycle $\widehat{\mathcal{Z}}(T, v)$ is concentrated in a special fibre at a prime p, and the geometric side of the theorem factors as the product of a local intersection number, determined by Gross and Keating [GK93], and a point count; we then relate this computation to the corresponding coefficient of the Eisenstein series using explicit formulas due to the third named author. An important step is to prove a local arithmetic Siegel-Weil formula (Proposition 3.3), which identifies the local intersection number with the central derivative of a local Whittaker function. For non-degenerate T of signature $(1, 1)$ or $(0, 2)$, the special cycles are purely archimedean, and the result is essentially a special case of [GS19, Theorem 5.1]. Finally, for degenerate T's, the identity can be reduced to the genus one case of the Siegel-Weil formula, as considered in [DY19].

We note that there is another quadratic lattice L that is naturally associated to $X_0(N)$, which is given by

$$L = \left\{ x = \begin{pmatrix} a & b \\ Nc & -a \end{pmatrix} :\, a, b, c \in \mathbb{Z} \right\}, \quad Q_L(x) = det(x) = -a^2 - Nbc. \tag{3}$$

The precise relation between $E'(\tau, 0, \Phi^{\mathcal{L}})$ and $E'(\tau, 0, \Phi^L)$ is given by Proposition 5.3 and (319). For the purposes of the computations in the present paper, it seems that the lattice $\mathcal{L}$ is more convenient.

2. Arithmetic special cycles on modular curves

2.1. Modular curves.

Throughout, we fix $N > 3$ odd and squarefree. Let

$$\mathcal{X} = \mathcal{X}_0(N) \tag{4}$$

denote the integral model, over $\mathrm{Spec}(\mathbb{Z})$ of the modular curve of level $\Gamma_0(N)$. More precisely, $\mathcal{X}$ denotes the Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z})$ whose S points, for a scheme S over $\mathrm{Spec}(\mathbb{Z})$, parametrize diagrams

$$\varphi\colon \mathcal{E} \to \mathcal{E}' \tag{5}$$

where $\mathcal{E}$ and $\mathcal{E}'$ are generalized elliptic curves over S, and φ is a cyclic isogeny of degree N. As usual, we let $\mathcal{Y} = \mathcal{Y}_0(N)$ denote the open modular curve $\mathcal{Y} = \mathcal{X} \setminus \{\text{cusps}\}$. The stack $\mathcal{X}$ is regular of dimension two, flat over $\operatorname{Spec}(\mathbb{Z})$, and smooth over $\operatorname{Spec}\mathbb{Z}[1/N]$; see [KM85] for details.

For later use, we recall the following description of the complex points $Y_0(N) = \mathcal{Y}(\mathbb{C})$ as an $O(1,2)$ Shimura variety. Consider the quadratic space

$$\mathcal{V} := M_2(\mathbb{Q})^{tr=0}, \qquad Q(x) = N\det(x) \tag{6}$$

of signature $(1,2)$, and let

$$\mathbb{D} = \mathbb{D}(\mathcal{V}) \;=\; \{z \in \mathcal{V} \otimes_{\mathbb{Q}} \mathbb{C} \mid \langle z,z\rangle = 0, \langle z,\overline{z}\rangle < 0\}/\mathbb{C}^\times \tag{7}$$

denote the symmetric space attached to $\mathcal{V} \otimes \mathbb{C} = M_2(\mathbb{C})^{tr=0}$; here $\langle x,y\rangle = -N\operatorname{tr}(xy)$ is the complex bilinear form with $\langle x,x\rangle = 2Q(x) = 2N\det x$. The space $\mathbb{D}$ decomposes into two connected components

$$\mathbb{D} = \mathbb{D}^+ \coprod \mathbb{D}^-, \tag{8}$$

where

$$\mathbb{D}^+ := \left\{z = \left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right) \in \mathcal{V}\otimes\mathbb{C} \mid \langle z,z\rangle = 0,\ \langle z,\overline{z}\rangle < 0,\ \operatorname{Im}(a/c) > 0\right\}/\mathbb{C}^\times. \tag{9}$$

The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathcal{V}\otimes\mathbb{C}$ by $\gamma\cdot v = \gamma v\gamma^{-1}$, which descends to an action on $\mathbb{D}$. A straightforward computation shows that $\mathbb{D}^+$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$, and the matrix $\operatorname{diag}(1,-1)$ interchanges the components.

Moreover, we have a $\mathrm{SL}_2(\mathbb{R})$-equivariant identification

$$\mathbb{H} \xrightarrow{\sim} \mathbb{D}^+, \qquad \tau \mapsto \operatorname{span}_{\mathbb{C}} z(\tau) \tag{10}$$

where

$$z(\tau) = \begin{pmatrix} \tau & -\tau^2 \\ 1 & -\tau \end{pmatrix}. \tag{11}$$

This construction gives rise to identifications

$$\mathcal{Y}(\mathbb{C}) \simeq [\Gamma_0(N)\backslash\mathbb{H}] \simeq [\Gamma_0(N)\backslash\mathbb{D}^+]. \tag{12}$$

Note here we are viewing the quotients as orbifolds.

2.2. **Special cycles.** We begin by recalling the construction of special cycles, following [DY19] or [BY09]. For a point $(\varphi\colon E \to E') \in \mathcal{X}_0(N)(S)$, for some base scheme S, let

$$\operatorname{End}(\varphi) = \{x \in \operatorname{End}(E) \mid \varphi\circ x\circ\varphi^{-1} \in \operatorname{End}(E')\} \tag{13}$$

and define

$$L_\varphi \;:=\; \left\{\alpha \in \operatorname{End}(\varphi) \mid \alpha + \alpha^\dagger = 0\right\}, \tag{14}$$

where $\alpha^\dagger$ is the Rosati dual. We may equip L_φ with the positive definite quadratic form

$$Q_\varphi(\alpha) = \deg(\alpha) = -\alpha^2. \tag{15}$$

Definition 2.1. *For $m \in \mathbb{Z}$, let $\mathcal{Z}(m)$ denote the moduli stack whose S points, for a base scheme S, are given by*

$$\mathcal{Z}(m)(S) := \{(\varphi\colon E \to E', \alpha)\} \tag{16}$$

where $\alpha \in L_\varphi$ satisfies the following conditions:

(a) $Q_\varphi(\alpha) = mN$; and

(b) the composition $\alpha\circ\varphi^{-1}$ is an isogeny from E' to E.

Remark 2.2. (i) More generally, in [DY19, BY09] one finds a definition for cycles $\mathcal{Z}(m,\mu_r)$ parametrized by $m \in \frac{1}{4N}\mathbb{Z}$ and $r \in \mathbb{Z}/2N\mathbb{Z}$ satisfying $r^2 \equiv -4Nm \pmod{4N}$; for $m \in \mathbb{Z}$, the moduli problem for $\mathcal{Z}(m)$ described above coincides with $\mathcal{Z}(m,\mu_0)$ in the notation of loc. cit.

(ii) If $\alpha \in L_\varphi$ satisfies condition (b) above, then $Q_\varphi(\alpha)$ is divisible by N. Moreover, if one omits this condition, the resulting moduli problem does not define a divisor on $\mathcal{X}_0(N)$.

There is a natural forgetful map $\mathcal{Z}(m) \to \mathcal{X}_0(N)$, which allows us to view $\mathcal{Z}(m)$ as a cycle on $\mathcal{X}_0(N)$ (which, abusing notation, we denote by the same symbol).

Lemma 2.3. *The cycle $\mathcal{Z}(m)$ is a divisor on $\mathcal{X}_0(N)$. Moreover, it is equal to the flat closure of its generic fibre $Z(m) = \mathcal{Z}(m)_{/\mathbb{Q}}$.*

Proof. We begin by showing that $\mathcal{Z}(m)$ is a divisor. This is clear on the generic fibre, so it will suffice to verify this claim in an étale neighbourhood of a point in characteristic p. To this end, let $z \in \mathcal{Z}(m)(\bar{\mathbb{F}}_p)$, lying above the point $x \in \mathcal{X}_0(N)(\bar{\mathbb{F}}_p)$, and let R_z (resp. R_x) denote the complete étale local rings of z and x respectively. Thus we have a surjection

$$R_x \to R_z = R_x/J \tag{17}$$

for some ideal J that we would like to show is principal. By Nakayama's lemma, it will suffice to show that $I = J/mJ$ is principal, where m is the maximal ideal of R_x. Note that $I^2 = 0$, so that the surjection

$$A := R_x/mJ \to R_z \tag{18}$$

is a square-zero infinitesimal extension.

Let $\widetilde{\varphi}\colon \mathcal{E} \to \mathcal{E}'$ denote the universal diagram over R_x, and $\widetilde{\varphi}_A\colon \mathcal{E}_A \to \mathcal{E}'_A$ (resp. $\widetilde{\varphi}_z\colon \mathcal{E}_{R_z} \to \mathcal{E}'_{R_z}$) its base change to A (resp. R_z).

By assumption, we have an action $\alpha_z \in \mathrm{End}(\widetilde{\varphi}_z)$ such that

$$\delta := \alpha_z \circ \widetilde{\varphi}_z^{-1} \tag{19}$$

is an isogeny $\mathcal{E}'_{R_z} \to \mathcal{E}_{R_z}$. Moreover, the ideal $I \subset A$ is, by definition, the largest ideal such that δ deforms to a homomorphism in $\mathrm{Hom}(\mathcal{E}'_{A/I}, \mathcal{E}_{A/I})$.

Now we apply Grothendieck-Messing theory. Consider the Hodge exact sequences

$$0 \to \mathcal{F}(\mathcal{E}'_A) \to H^1_{dR}(\mathcal{E}'_A) \to \mathrm{Lie}(\mathcal{E}'_A) \to 0 \qquad \text{and} \qquad 0 \to \mathcal{F}(\mathcal{E}_A) \to H^1_{dR}(\mathcal{E}_A) \to \mathrm{Lie}(\mathcal{E}_A) \to 0 \tag{20}$$

for $\mathcal{E}'_A$ and $\mathcal{E}_A$ respectively. Since the map $A \to R_z$ is a square-zero thickening, the homomorphism $\delta\colon \mathcal{E}'_{R_z} \to \mathcal{E}_{R_z}$ induces a canonical A-linear map

$$H^1_{dR}(\mathcal{E}'_A) \to H^1_{dR}(\mathcal{E}_A); \tag{21}$$

composing this with the maps in the Hodge exact sequences above, we obtain a map

$$\widehat{\delta}\colon \mathcal{F}(\mathcal{E}'_A) \to \mathrm{Lie}(\mathcal{E}_A). \tag{22}$$

Then Grothendieck-Messing theory implies that for any intermediate ring B with

$$A \to B \to R_z \tag{23}$$

the homomorphism δ lifts to an element of $\mathrm{Hom}(\mathcal{E}'_B, \mathcal{E}_B)$ if and only if $\widehat{\delta} \otimes B = 0$. This immediately implies that the ideal I is given by the vanishing locus of $\widehat{\delta}$, and since $\mathcal{F}(\mathcal{E}'_A)$ and $\mathrm{Lie}(\mathcal{E}_A)$ are free A-modules of rank 1, it follows that I is principal, as required.

It remains to show that $\mathcal{Z}(m)$ is horizontal, i.e. does not contain any irreducible components in finite characteristic. Let p be a prime, and suppose $z \in \mathcal{Z}(m)(\bar{\mathbb{F}}_p)$ corresponds to $(\varphi\colon E \to E', \alpha)$

as before. If p splits in $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{-mN})$, then E and E' are necessarily ordinary; if p is non-split, then E and E' are both supersingular. However, every irreducible component of $\mathcal{X}_0(N)(\bar{\mathbb{F}}_p)$ contains both ordinary and supersingular points, hence such a component cannot be contained in the support of $\mathcal{Z}(m)$. □

We will also require a description of the complex points $\mathcal{Z}(m)(\mathbb{C})$. Consider the quadratic lattice

$$\mathcal{L} := \left\{ \begin{pmatrix} a & b/N \\ c & -a \end{pmatrix} \Big| a, b, c \in \mathbb{Z} \right\} \subset \mathcal{V} \tag{24}$$

where $\mathcal{V} = M_2(\mathbb{Q})^{tr=0}$, with quadratic form $Q(x) = N \det x$.

Now given $x = \left(\begin{smallmatrix} a & b/N \\ c & -a \end{smallmatrix}\right) \in \mathcal{L}$ with $x \neq 0$, let

$$\mathbb{D}_x^+ \ := \ \{ z \in \mathbb{D}^+ \mid \langle z, x \rangle = 0 \}. \tag{25}$$

If $Q(x) \leq 0$, then $\mathbb{D}_x^+$ is empty. Otherwise, $\mathbb{D}_x^+ = \{z(\tau_x)\}$ consists of exactly one point, where $\tau_x \in \mathbb{H}$ is the root of the equation

$$c\tau^2 + 2a\tau - b/N \tag{26}$$

with positive imaginary part.

Lemma 2.4. *We have identifications*

$$\begin{array}{ccc} \mathcal{Z}(m)(\mathbb{C}) & \xrightarrow{\sim} & \left[\Gamma_0(N) \backslash \coprod_{\substack{x \in \mathcal{L} \\ Q(x) = m}} \mathbb{D}_x^+ \right] \\ \downarrow & & \downarrow \\ \mathcal{Y}_0(N)(\mathbb{C}) & \xrightarrow{\sim} & \left[\Gamma_0(N) \backslash \mathbb{H} \right] \end{array} \tag{27}$$

Proof. Given $x = \left(\begin{smallmatrix} a & b/N \\ c & -a \end{smallmatrix}\right) \in \mathcal{L}$, let $\tau = \tau_x$ as above. Then the corresponding point of $\mathcal{Y}_0(N)(\mathbb{C})$ is the diagram

$$\varphi_\tau \colon E_\tau = \mathbb{C}/\Lambda_\tau \ \xrightarrow{\times N} \ E'_\tau = \mathbb{C}/\Lambda_{N\tau} \tag{28}$$

where φ_τ is multiplication by N, and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ for $\tau \in \mathbb{H}$. Define an endomorphism $\alpha \in \mathrm{End}(\varphi_\tau)$ by

$$\alpha(z + \Lambda_\tau) = z(Na + Nc\tau) + \Lambda_\tau. \tag{29}$$

It is straightforward to check that α satisfies the conditions in Definition 2.1, and every such endomorphism arises in this way. □

Next, we turn our attention to 0-cycles. Consider the vector space $\mathrm{Sym}_2(\mathbb{Q})$ equipped with bilinear form

$$(x, y) \mapsto tr(xy).$$

With respect to this form, the dual of the lattice $\mathrm{Sym}_2(\mathbb{Z})$ is given by

$$\mathrm{Sym}_2(\mathbb{Z})^\vee = \left\{ \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) | a, b, c \in \mathbb{Z} \right\}.$$

Now suppose

$$T = \begin{pmatrix} t_{11} & * \\ * & t_{22} \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})^\vee \tag{30}$$

and assume for the moment that $\det T \neq 0$.

Definition 2.5. *Let $\mathcal{Z}(T)$ denote the moduli stack whose S points parametrize tuples*

$$\mathcal{Z}(T)(S) \ := \ \{(\varphi\colon E \to E', \alpha_1, \alpha_2)\} \tag{31}$$

where

(a) $\alpha_i \in L_\varphi$ for $i = 1, 2$;
(b) the moment matrix

$$T_\varphi(\alpha_1, \alpha_2) = (\frac{1}{2}(\alpha_i, \alpha_j)) = \begin{pmatrix} -\alpha_1^2 & -\frac{1}{2}(\alpha_1\alpha_2 + \alpha_2\alpha_1) \\ -\frac{1}{2}(\alpha_1\alpha_2 + \alpha_2\alpha_1) & -\alpha_2^2 \end{pmatrix}$$

satisfies

$$T_\varphi(\alpha_1, \alpha_2) = NT,$$

(c) and finally, that $\alpha_i \circ \varphi^{-1} \in \mathrm{Hom}(E', E)$ for $i = 1, 2$.

To describe the geometric points of this stack, we first recall the following representability criterion. Working more generally for the moment, let $V/\mathbb{Q}_\ell$ be any quadratic space of dimension $m = 3$, with corresponding bilinear form $(\cdot,\cdot)$. Fix a basis $\{v_1, v_2, v_3\}$ and let $A \in \mathrm{Sym}_2(\mathbb{Q}_\ell)$ denote the moment matrix, i.e. $A_{ij} = \frac{1}{2}(v_i, v_j)$. We define the *determinant* of V to be the class $\det(V) := \det(A) \in \mathbb{Q}_\ell^\times/\mathbb{Q}_\ell^{\times,2}$, which is independent of the choice of basis. Define the character

$$\chi_{V,\ell}(x) = (x, (-1)^{m(m-1)/2} \det V)_\ell = (x, -\det V)_\ell. \tag{32}$$

For a given choice of determinant $\det(V) \in \mathbb{Q}_\ell^\times/\mathbb{Q}_\ell^{\times,2}$, there are exactly two non-isomorphic quadratic spaces $V_\ell^\pm$, distinguished by their Hasse invariants.

Now if $T \in \mathrm{Sym}_2(\mathbb{Q}_\ell)$ is non-degenerate, we have that T is represented by V_ℓ if and only if

$$\varepsilon_\ell(V) = \varepsilon_\ell(T)\, \chi_{V.\ell}(\det T) \tag{33}$$

[Kud97, Prop. 1.3]; here $\varepsilon(V)$ is the Hasse invariant of V, cf. [Lam05, p.118], and $\varepsilon(T)$ is the Hasse invariant of the quadratic space $\mathbb{Q}_\ell^2$ whose quadratic form is given by T in the standard basis.

Returning to the quadratic space $\mathcal{V}$ as in (6), define the *difference set* $\mathrm{Diff}(T, \mathcal{V})$ to be

$$\mathrm{Diff}(T, \mathcal{V}) = \{\ell \text{ finite prime} \mid T \text{ is not represented by } \mathcal{V}_\ell\}; \tag{34}$$

one can check that $\epsilon_\ell(\mathcal{V}) = (-1, -N)_\ell$, so that using (33), we have

$$\mathrm{Diff}(T, \mathcal{V}) = \{\ell \mid \varepsilon_\ell(T) = -(-\det(T), -N)_\ell\}. \tag{35}$$

Note that when $T > 0$, we must have $\mathrm{Diff}(T, \mathcal{V}) \neq \emptyset$; to see this, recall the product formulas

$$\prod_{v \leq \infty} \varepsilon_v(T) = 1 = \prod_{v \leq \infty} (-\det(T), -N)_v \tag{36}$$

for the Hasse invariants and the Hilbert symbols. On the other hand, if $T > 0$ then $\varepsilon_\infty(T) = 1$ and $(-\det T, -N)_\infty = -1$. Hence there must be at least one finite place ℓ for which $\varepsilon_\ell(T) = -(-\det(T), -N)_v$; in other words, $\mathrm{Diff}(T, \mathcal{V}) \neq \emptyset$ in this case.

Now suppose that $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ is non-degenerate. The following lemma determines the support of $\mathcal{Z}(T)$ in terms of $\mathrm{Diff}(T, \mathcal{V})$.

Lemma 2.6. *Suppose T is non-degenerate.*

(1) If T is not positive definite, or if $T > 0$ and $\#\mathrm{Diff}(T, \mathcal{V}) \geq 2$, then $\mathcal{Z}(T) = \emptyset$.
(2) If $T > 0$ and $\mathrm{Diff}(T, \mathcal{V}) = \{p\}$, then $\mathcal{Z}(T)$ is supported in the supersingular locus in the special fibre $\mathcal{X}_0(N)_{/\mathbb{F}_p}$. In particular, $\mathcal{Z}(T) \to \mathcal{X}_0(N)$ determines a zero cycle.

Proof. Suppose we have a geometric point $z \in \mathcal{Z}(T)(\mathbb{F})$ for an algebraically closed field $\mathbb{F}$, corresponding to a tuple $\varphi\colon E \to E'$. Let $V_\varphi = \mathrm{End}(\varphi)^{tr=0} \otimes_{\mathbb{Z}} \mathbb{Q}$, equipped with the quadratic form $x \mapsto N^{-1} \deg x$. By assumption, we have that V_φ represents T. However, V_φ is positive definite, so if T is not positive definite, we obtain a contradiction; thus $\mathcal{Z}(T) = \emptyset$ if T is not positive definite. t f Assuming that $T > 0$ and continuing, we note that $\dim V_\varphi \geq 3$, so the characteristic of $\mathbb{F}$ is nonzero and E, E' are supersingular elliptic curves. Setting $p = \mathrm{char}(\mathbb{F})$, we may identify $V_\varphi = B^{tr=0}$, where B is the rational quaternion algebra ramified at p and ∞, equipped with the quadratic form $N^{-1}\mathrm{nm}$, where $\mathrm{nm}(x)$ is the reduced norm. It follows that for $\ell \neq p$, the space $V_{\varphi,\ell}$ is isometric to $\mathcal{V}_\ell$, and so $\mathrm{Diff}(T, \mathcal{V}) \subset \{p\}$.

Finally, note that $\det(V_{\varphi,p}) = \det(\mathcal{V}_p)$ and $\epsilon_p(V_{\varphi,p}) = -\epsilon_p(\mathcal{V}_p)$. Hence, applying the representability criterion (33), we conclude that $p \in \mathrm{Diff}(T, \mathcal{V})$. As we have already seen that any geometric point of $\mathcal{Z}(T)$ is in the supersingular locus, we have concluded the proof of the lemma. □

2.3. **Classes in arithmetic Chow groups.** Let

$$\widehat{\mathrm{CH}}^\bullet_{\mathbb{R}}(\mathcal{X}) = \bigoplus_{n=0}^{2} \widehat{\mathrm{CH}}^n_{\mathbb{R}}(\mathcal{X}) \tag{37}$$

denote the arithmetic Chow ring, as originally constructed by Gillet and Soulé, see e.g. [Sou92]. A discussion extending the construction to the case of Deligne-Mumford stacks of dimension two can be found in [KRY06, §2].

Roughly, a class in $\widehat{\mathrm{CH}}^n_{\mathbb{R}}(\mathcal{X})$ is represented by an *arithmetic cycle* $(\mathcal{Z}, g)$, where $\mathcal{Z}$ is a codimension n cycle on $\mathcal{X}$, with $\mathbb{R}$-coefficients, and g_Z is a Green current for $\mathcal{Z}(\mathbb{C})$, i.e. g_Z is a current on $\mathcal{X}(\mathbb{C})$ of degree $(n-1, n-1)$ for which there exists a smooth form ω such that

$$\mathrm{dd^c} g_Z + \delta_{\mathcal{Z}(\mathbb{C})} = [\omega], \tag{38}$$

holds; here $[\omega]$ is the current defined by integration against ω. The rational arithmetic cycles are those of the form $\widehat{\mathrm{div}}(f) = (\mathrm{div} f, i_*[-\log|f|^2])$, where $f \in \mathbb{Q}(W)^\times$ for a codimension $n-1$ integral subscheme $\iota\colon W \hookrightarrow \mathcal{X}$, together with classes of the form $(0, \partial\eta + \overline{\partial}\eta')$; by definition, the arithmetic Chow group $\widehat{\mathrm{CH}}^n_{\mathbb{R}}(\mathcal{X})$ is the quotient of the space of arithmetic cycles by the $\mathbb{R}$-subspace spanned by the rational cycles.

Note that if $n = 2$, for dimension reasons $\mathcal{Z}(\mathbb{C}) = 0$ and Green's equation (38) is devoid of content; in particular, there is nothing linking $\mathcal{Z}$ and g in this case.

We will also require a generalization of these Chow groups due to Kühn and Burgos-Kramer-Kühn. For a cusp P of $X_0(N)$, let $\mathcal{P}$ denote its flat closure in $\mathcal{X}_0(N)$, and let $\mathcal{S} = \sum \mathcal{P}$ denote the cuspidal divisor. In [BGKK07], the authors present an abstract framework that allows for more general kinds of Green objects in the theory of arithmetic Chow groups. One of the examples they present utilizes "pre-log-log forms" relative to a fixed normal crossing divisor. In our case, we take this fixed divisor to be the cuspidal divisor $\mathcal{S}$, and denote by

$$\widehat{\mathrm{CH}}^\bullet_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) = \bigoplus_{n} \widehat{\mathrm{CH}}^n_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \tag{39}$$

the corresponding Chow ring; here the notation '$\mathcal{D}_{\mathrm{pre}}$' indicates the use of the Gillet complex of pre-log-log forms in the construction, as defined in [BGKK07, §7]. In particular, there is a natural map

$$\widehat{\mathrm{CH}}^n(\mathcal{X}) \to \widehat{\mathrm{CH}}^n(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}), \tag{40}$$

an intersection product

$$\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \times \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \to \widehat{\mathrm{CH}}^2_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \qquad (\widehat{Z}_1, \widehat{Z}_2) \mapsto \widehat{Z}_1 \cdot \widehat{Z}_2, \tag{41}$$

and a degree map

$$\widehat{\deg} \colon \widehat{\mathrm{CH}}^2_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \to \mathbb{R}. \tag{42}$$

For convenience, we will often abbreviate

$$\langle \widehat{\mathcal{Z}}_1, \widehat{\mathcal{Z}}_2 \rangle = \widehat{\deg}\left(\widehat{\mathcal{Z}}_1 \cdot \widehat{\mathcal{Z}}_2 \right), \qquad \widehat{\mathcal{Z}}_1, \widehat{\mathcal{Z}}_2 \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}) \tag{43}$$

for the intersection pairing, which recovers the intersection pairing defined in [Küh01]. We will not require precise formulas for any of the aforementioned structures; the interested reader may consult [BGKK07, §7] for a complete discussion.

Remark 2.7. Strictly speaking, the constructions in [BGKK07] and [Küh01] only apply to schemes, and not Deligne-Mumford stacks. In our case, this technicality is essentially immaterial. Indeed, we have identifications

$$\mathcal{X}(\mathbb{C}) \simeq [\Gamma_0(N)\backslash \mathbb{H}^*] \simeq [\{\pm 1\}\backslash X_0(N)] \tag{44}$$

as orbifolds, where $\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps}\}$, the space $X_0(N)$ is the Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$, and $\{\pm 1\}$ acts trivially on $X_0(N)$. Hence the analytic constructions of [BGKK07] can be carried out on $X_0(N)$; the only difference is that in numerical formulas, we include a factor of $1/2$ to reflect the presence of the group $\{\pm 1\}$.

2.4. **Arithmetic special divisors.** Here we recall the special divisors defined in [DY19], which include certain additional boundary components. For $v \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}$, let

$$g(m, v) = \begin{cases} \frac{\sqrt{N}}{2\pi\sqrt{v}} \beta_{3/2}(-4\pi m v), & \text{if } m < 0 \text{ and } -Nm \text{ is a square} \\ \frac{\sqrt{N}}{2\pi\sqrt{v}}, & \text{if } m = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{45}$$

where, for $s \in \mathbb{R}$,

$$\beta_s(r) = \int_1^\infty e^{-rt} t^{-s} dt. \tag{46}$$

The modified special cycle is defined to be

$$\mathcal{Z}^*(m, v) := \mathcal{Z}(m) + g(m, v)\mathcal{S}. \tag{47}$$

where $\mathcal{S} = \sum_{\mathcal{P} \text{ cusps}} \mathcal{P}$ is the cuspidal divisor. Note that by definition, $\mathcal{Z}(m) = 0$ when $m \leq 0$.

We now describe Green functions for these cycles, following [Kud97]. First, for $x \in \mathcal{V}$ and $z \in \mathbb{D}$, let

$$R(x, z) := -\frac{|\langle x, \zeta \rangle|^2}{\langle \zeta, \overline{\zeta} \rangle} \tag{48}$$

where $\zeta \in z$ is any non-zero vector; note that the definition is clearly independent of ζ. For $v \in \mathbb{R}_{>0}$, and $m \neq 0$, we define *Kudla's Green function*

$$\xi(m, v) := \sum_{\substack{x \in \mathcal{L} \\ Q(x) = m}} \int_1^\infty e^{-2\pi R(x,z) v t} \frac{dt}{t}, \tag{49}$$

which is a smooth function on $\mathcal{Y}_0(N)(\mathbb{C}) - \mathcal{Z}(m)(\mathbb{C})$. The behaviour of these functions near the cusps was determined by Du and Yang. In particular, [DY19, Theorem 5.1] implies that $\xi(m,v)$ is a Green function for $\mathcal{Z}^*(m,v)$, and so we obtain a class

$$\widehat{\mathcal{Z}}(m,v) \ := (\mathcal{Z}^*(m,v),\, \xi(m,v)) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) \tag{50}$$

The case $m=0$ requires a bit more notation. First, let $\varphi_{\mathrm{un}}\colon \mathcal{E}_{\mathrm{un}} \to \mathcal{E}'_{\mathrm{un}}$, denote the universal diagram over $\mathcal{X} = \mathcal{X}_0(N)$, where $\pi\colon \mathcal{E}_{\mathrm{un}} \to \mathcal{X}$ is a generalized elliptic curve, and let

$$\omega_N = \pi_*\left(\Omega^1_{\mathcal{E}_{\mathrm{un}}/\mathcal{X}}\right) \tag{51}$$

denote the Hodge bundle; sections of this bundle correspond to modular forms of weight one. This bundle can be metrized on $Y_0(N) = \mathcal{Y}_0(N)(\mathbb{C})$ as follows: at a point $\tau \in \mathbb{H}^{\pm} \simeq \mathbb{D}$, corresponding to the diagram $\varphi_\tau\colon E_\tau \to E'_\tau$ as in (28), the metric $\|\cdot\|_\tau$ at τ is determined by the formula

$$\|dz\|^2_\tau = (2\sqrt{\pi}e^{-\gamma/2})\, v, \qquad \tau = u+iv \tag{52}$$

where dz is the coordinate differential with respect to the uniformization $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, and $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant. This metric is logarithmically singular at the cusps; this metrized line bundle induces a class in the generalized arithmetic Chow group, which we also denote by

$$\widehat{\omega}_N \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}). \tag{53}$$

Following [DY19], we will need a slight modification to this bundle. Recall that for a prime $p|N$, the fibre $\mathcal{X}_p$ at p decomposes into two irreducible components $\mathcal{X}_p^0$ and $\mathcal{X}_p^\infty$, where $\mathcal{X}_p^0$ (resp. $\mathcal{X}_p^\infty$) is the component that contains the reduction of the cusp $\mathcal{P}_0$ (resp. $\mathcal{P}_\infty$) mod p. We then define

$$\widehat{\omega} := -2\widehat{\omega}_N - \sum_{p|N} \widehat{\mathcal{X}}_p^0 + (0, \log N), \tag{54}$$

where $\widehat{\mathcal{X}}_p^0 = (\mathcal{X}_p^0, 0) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ is the corresponding class.

Remark 2.8. The class $\widehat{\omega}$ can be identified as (the class of) the dual of the Hodge bundle of the product $\mathcal{E}_{un} \times \mathcal{E}'_{un}$ over $\mathcal{X}_0(N)$ with an appropriately scaled metric; see [How20, §2.2] for details.

Next, define

$$\xi(0,v) \ = \ \sum_{\substack{x\in\mathcal{L}\\ Q(x)=0\\ x\neq 0}} \int_1^\infty e^{-2\pi v R(x,z)t}\, \frac{dt}{t} \tag{55}$$

which is a smooth function on $Y_0(N)$. As shown in [DY19], it is a Green function for the divisor $g(0,v)S$, with $\log-\log$ singularities at the boundaries, which therefore determines a class

$$(g(0,v)\mathcal{S}, \xi(0,v)) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}). \tag{56}$$

It turns out that the linear combination $\widehat{\omega} + (g(0,v)\mathcal{S}, \xi(0,v))$, in fact lands in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$, the usual Gillet-Soulé Chow group, see [DY19, Proposition 6.6].

Finally, we set

$$\widehat{\mathcal{Z}}(0,v) \ := \ \widehat{\omega} + (g(0,v)\mathcal{S}, \xi(0,v)) - (0, \log v) \ \in \ \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) \tag{57}$$

2.5. **Arithmetic special cycles in codimension two.** In this section, we will attach an arithmetic cycle $\widehat{\mathcal{Z}}(T,v)$ for every $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ and $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$.

We begin with the case that T is positive definite. As in Lemma 2.6, if the cycle $\mathcal{Z}(T)$ is non-zero, it is supported in the fibre $\mathcal{X}_p$ for a prime p. We may then define

$$\widehat{\mathcal{Z}}(T,v) \ := \ (\mathcal{Z}(T),0) \ \in \ \widehat{\mathrm{CH}}^2_{\mathbb{R}}(\mathcal{X}); \tag{58}$$

note that this class is independent of v.

Next, suppose T is non-degenerate and of signature $(1,1)$ or $(0,2)$, so that $\mathcal{Z}(T) = \emptyset$. In this case, we define a purely archimedean class via the construction of [GS19]. We briefly recall the construction in the case at hand. Suppose $x = (x_1,x_2) \in \mathcal{V}_{\mathbb{R}}^{\oplus 2}$ is a linearly independent pair of vectors, and consider the Schwartz form

$$\nu(x) \in S(\mathcal{V}_{\mathbb{R}}^2) \otimes A^{1,1}(\mathbb{H}) \tag{59}$$

defined in [GS19]. Explicitly, we have

$$\nu(x) \ := \ \nu^o(x) e^{-2\pi(Q(x_1)+Q(x_2))} \tag{60}$$

where

$$\nu^o(x) = \psi^o(x)\, d\mu \tag{61}$$

with

$$\psi^o(x,\tau) = \left(-\frac{1}{\pi} + 2\sum_{i=1}^{2} (R(x_i,\tau) + 2Q(x_i)) \right) e^{-2\pi(R(x_1,\tau)+R(x_2,\tau))}, \qquad x=(x_1,x_2) \tag{62}$$

and

$$d\mu = \frac{du\wedge dv}{v^2} \tag{63}$$

for $\tau = u+iv \in \mathbb{H}$; here we are abusing notation and writing $R(x,\tau) = R(x,[z_\tau])$ under the identification $\mathbb{H}^\pm \simeq \mathbb{D}$ as in (10).

Next, we set

$$\Psi(x,\tau) = \int_1^\infty \psi^o(\sqrt{t}x,\tau)\,\frac{dt}{t}. \tag{64}$$

If the components of x span a space of signature $(1,1)$ or $(0,2)$, then $\Psi(x,\cdot)$ is a smooth function on $\mathbb{H}$.

Finally, for $T \in \mathrm{Sym}_2(\mathbb{Q})$ of signature $(1,1)$ or $(0,2)$, and $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, we choose $a \in \mathrm{GL}_2(\mathbb{R})$ with $v = a\cdot {}^t a$ and set

$$\Psi(T,v,\tau) = \sum_{\substack{x\in\mathcal{L}^2 \\ T(x)=T}} \Psi(xa,\tau) \tag{65}$$

and

$$\Xi(T,v) := \Psi(T,v,\tau) d\mu(\tau). \tag{66}$$

The sum converges absolutely to a $\Gamma_0(N)$-invariant function on $\mathbb{H}$, and $\Xi(T,v)$ defines a smooth form on $Y_0(N)$.

Lemma 2.9. *The form $\Xi(T,v)$ is absolutely integrable on $X_0(N)$, i.e.*

$$\int_{X_0(N)} |\Psi(T,v,\tau)| d\mu(\tau) < \infty.$$

Proof. We first claim that it suffices to prove that for any $x = (x_1, x_2) \in \Omega(T)$, the form $\Psi(xa, \tau)d\mu(\tau)$ is absolutely integrable on $\mathbb{H}$; indeed, if this is the case, we would have

$$\int_{X_0(N)} \Psi(T, v, \tau)d\mu(\tau) = \int_{\Gamma_0(N)\backslash\mathbb{H}} \left(\sum_{\substack{x\in L^2 \\ T(x)=T}} \Psi(xa, \tau) \right) d\mu(\tau) = \sum_{\substack{x\in\Omega(T) \\ \text{mod } \Gamma_0(N)}} \int_{\mathbb{H}} \Psi(xa, \tau)\, d\mu(\tau), \quad (67)$$

where we use the fact that the stabilizer of x in $\Gamma_0(N)$ is $\{\pm 1\}$, which acts trivially on $\mathbb{H}$; since there are finitely many $\Gamma_0(N)$ orbits, the lemma would follow by Fubini.

To show the integrability of $\Psi(xa, \tau)$, fix a matrix $k \in SO(2)$ such that

$$xak = (x_1, x_2)ak = (y_1, y_2) \quad (68)$$

for a tuple $y = (y_1, y_2) \in V_{\mathbb{R}}^2$ such that

$$T(y) = \begin{pmatrix} \delta_1 & \\ & \delta_2 \end{pmatrix}, \quad (69)$$

with $\delta_1, \delta_2 \in \mathbb{R}^\times$ and $\delta_1 < 0$. It follows from [GS19] that $\Psi(xa, \tau) = \Psi(y, \tau)$. Furthermore, we have

$$\Psi(\gamma y, \tau) = \Psi(y, \gamma \cdot \tau) \quad (70)$$

for $\gamma \in \mathrm{GL}_2(\mathbb{R})$, acting diagonally on $V_{\mathbb{R}}^2$. Hence, in the integral over $\mathbb{H}$, we may act by an appropriate choice of γ and assume without loss of generality that

$$y_1 = |\delta_1|^{\frac{1}{2}} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad y_2 = |\delta_2|^{\frac{1}{2}} \begin{pmatrix} & 1 \\ \pm 1 & \end{pmatrix} \quad (71)$$

where the sign in the matrix defining y_2 is $-$ if T is of signature $(1, 1)$, and $+$ if T is of signature $(0, 2)$.

First, suppose T has signature $(0, 2)$. In this case, we have

$$R_1 \ := \ R(y_1, \tau) = \frac{|\delta_1|}{4v^2} \cdot |\tau|^2, \qquad R_2 := R(y_2, \tau) = \frac{|\delta_2|}{4v^2} \cdot |1 - \tau^2|^2. \quad (72)$$

Setting $c = \frac{\pi}{2} \min(|\delta_1|, |\delta_2|)$ and writing $\tau = u + iv$, we then have

$$\begin{aligned} 2\pi(R_1 + R_2) &\geq \frac{c}{v^2} \left(|\tau|^2 + |1 - \tau^2|^2 \right) \\ &= \frac{c}{v^2} \left(u^2 + v^2 + (1 - u^2)^2 + 2v^2(1 + u^2) + v^4 \right) \\ &= \frac{c}{v^2} \left(v^2 + (1/2 - u^2)^2 + 3/4 + 2v^2(1 + u^2) + v^4 \right) \\ &> c \left(3/4 \cdot v^{-2} + v^2 + 2u^2 \right). \end{aligned}$$

This estimate easily implies that the integrals

$$\int_{\mathbb{H}} \int_1^\infty e^{-2\pi t(R_1+R_2)} \frac{dt}{t} d\mu(\tau) < \int_{\mathbb{H}} \frac{1}{2\pi(R_1 + R_2)} e^{-2\pi(R_1+R_2)} d\mu(\tau) \quad (73)$$

and

$$\int_{\mathbb{H}} \int_1^\infty (tR_i) \cdot e^{-2\pi t(R_1+R_2)} \frac{dt}{t} d\mu(\tau) = \int_{\mathbb{H}} \frac{R_i}{2\pi(R_1 + R_2)} e^{-2\pi(R_1+R_2)} d\mu(\tau) \quad (74)$$

are finite. So the integral (67) is absolutely convergent, which in turn implies the statement of the lemma. The case of signature $(1, 1)$ is analogous. $\square$

The preceding lemma implies that $\Xi(T,v)$ defines a current on $X_0(N)$ so in this case we obtain a class

$$\widehat{\mathcal{Z}}(T,v) = (0, \Xi(T,v)) \in \widehat{\mathrm{CH}}^2_{\mathbb{R}}(\mathcal{X}). \tag{75}$$

Next, we suppose T has rank 1. We begin with the following simple observation:

Lemma 2.10. *Suppose* $\operatorname{rank}(T) = 1$. *Then there exists* $t \in \mathbb{Z}$ *and an element* $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ *such that*

$$T = {}^t\gamma \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right) \gamma. \tag{76}$$

Moreover, the integer t *is uniquely determined by* T, *and is invariant upon replacing* T *by* ${}^t\sigma T \sigma$ *with* $\sigma \in \mathrm{GL}_2(\mathbb{Z})$.

Proof. For the existence, write $T = \left(\begin{smallmatrix} t_1 & m/2 \\ m/2 & t_2 \end{smallmatrix}\right)$, so that $m^2 = 4t_1t_2$. If $t_1 = 0$, then we may take $\gamma = Id$ and $t = t_2$. Otherwise, write $-m/2t_1 = a/c$ where a and c are relatively prime, and choose b, d such that $ad - bc = 1$; then $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfies ${}^t\gamma T \gamma = \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)$ for some t, as required.

To show that t is uniquely determined, suppose that we have $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ with

$${}^t\gamma \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right) \gamma = \left(\begin{smallmatrix} 0 & \\ & t' \end{smallmatrix}\right) \tag{77}$$

Then γ is of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, hence $d = \pm 1$, which in turn implies that $t' = t$. □

Following [KRY06], we make the following definition.

Definition 2.11. *Suppose* T *is of rank 1, and let* $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$. *Let* t *be as in Lemma 2.10. Define*

$$\widehat{\mathcal{Z}}(T,v) \ := \ \widehat{\mathcal{Z}}(t, v_0) \cdot \widehat{\omega} - \left(0, \log\left(\frac{\det v}{v_0}\right) \delta_{\mathcal{Z}^*(t,v_0)(\mathbb{C})}\right) \in \widehat{\mathrm{CH}}^2(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}), \tag{78}$$

where

$$v_0 := t^{-1}\mathrm{tr}(Tv) \in \mathbb{R}_{>0}, \tag{79}$$

the cycle $\mathcal{Z}^*(t, v_0)(\mathbb{C})$ *is the complex points of the modified cycle* (47), *and* $\widehat{\omega}$ *is the class defined in* (54).

We observe two invariance properties, which both follow immediately from definitions: the first is the identity

$$\widehat{\mathcal{Z}}({}^t\gamma T \gamma, v) = \widehat{\mathcal{Z}}(T, \gamma v {}^t\gamma), \qquad \gamma \in \mathrm{GL}_2(\mathbb{Z}). \tag{80}$$

For the second invariance property, suppose $\theta = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$; then

$$\widehat{\mathcal{Z}}\left(\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right), \theta v {}^t\theta\right) = \widehat{\mathcal{Z}}\left(\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right), v\right) \tag{81}$$

for any $t \in \mathbb{Z}$ and $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$.

It remains to define the constant term.

Definition 2.12. *When* $T = 0$, *we set*

$$\widehat{\mathcal{Z}}(0,v) = \widehat{\omega} \cdot \widehat{\omega} \ + \ (0, \log\det v \cdot [\Omega]) \ \in \ \widehat{\mathrm{CH}}^2(\mathcal{X}, \mathcal{D}_{\mathrm{pre}}), \tag{82}$$

where $\Omega = \frac{dx \wedge dy}{2\pi y^2}$, *and* $[\Omega]$ *is the current given by integration against* Ω.

2.6. **The main theorem.** We begin by briefly reviewing the theory of Siegel Eisenstein series, mostly to fix notation; for details, see [KRY06, §8], for example. Let

$$\mathrm{Sp}_r = \left\{ g \in \mathrm{GL}_{2r} \mid {}^t g \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix} g = \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix} \right\} \tag{83}$$

viewed as an algebraic group over $\mathbb{Q}$.

For a place $v \leq \infty$, let $G_{r,v}$ denote the metaplectic cover of $\mathrm{Sp}_r(\mathbb{Q}_v)$; as a set, we have $G_{r,v} = \mathrm{Sp}_r(\mathbb{Q}_v) \times \mathbb{C}^1$ with multiplication given by the normalized Leray cocycle (see [KRY06, §8.5]). Similarly, $G_{r,\mathbb{A}} = \mathrm{Mp}_r(\mathbb{A})$ denote the metaplectic cover of $\mathrm{Sp}_r(\mathbb{A})$, which can be realized as a restricted direct product of the $G_{r,v}$. There is a unique splitting $\mathrm{Sp}_r(\mathbb{Q}) \to G_{r,\mathbb{A}}$ whose image we denote by $G_{r,\mathbb{Q}}$. Let $P_r = M_r N_r$ denote the standard Siegel parabolic of Sp_r; here

$$M_r = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r \right\} \qquad \text{and} \qquad N_r = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \;\middle|\; b \in \mathrm{Sym}_r \right\}. \tag{84}$$

We denote by $\tilde{P}_{r,\mathbb{A}}, \tilde{M}_{r,\mathbb{A}}$ and $\tilde{N}_{r,\mathbb{A}}$, the inverse images of $P_r(\mathbb{A})$, $M_r(\mathbb{A})$, and $N_r(\mathbb{A})$ under the covering map $G_{r,\mathbb{A}} \to \mathrm{Sp}_r(\mathbb{A})$.

Given the quadratic space $\mathcal{V}$ as in (6), we may define a character $\chi_{\mathcal{V}}$ on $\tilde{P}_{\mathbb{A}}$ by the formula

$$\chi_{\mathcal{V}}\left([m(a)n(b), \epsilon]\right) = \varepsilon\left(\det(a), -\det \mathcal{V}\right)_{\mathbb{A}} = \varepsilon\left(\det a, -N\right)_{\mathbb{A}}, \tag{85}$$

where $(\cdot,\cdot)_{\mathbb{A}}$ is the global Hilbert symbol and $\varepsilon \in \mathbb{C}^1$. Similarly, for $s \in \mathbb{C}$, we define a character $|\cdot|^s$ on $\widetilde{P}_{\mathbb{A}}$ by

$$|p|^s = |\det(a)|^s_{\mathbb{A}}, \qquad p = [m(a)n(b), \epsilon]. \tag{86}$$

Consider the degenerate principal series representation

$$I_r(s, \chi_{\mathcal{V}}) := \{\Phi\colon G_{r,\mathbb{A}} \to \mathbb{C} \text{ smooth} \mid \Phi(pg, s) = \chi_{\mathcal{V}}(p)|p|^{s+\rho}\Phi(g,s) \text{ for all } g \in G_{r,\mathbb{A}}, p \in \tilde{P}_{r,\mathbb{A}}\}, \tag{87}$$

with $G_{r,\mathbb{A}}$ acting by right translation; here $\rho = \frac{r+1}{2}$. This representation factors as a restriced direct product

$$I_r(s, \chi_{\mathcal{V}}) = \bigotimes_{v \leq \infty}{}' I_{r,v}(s, \chi_{\mathcal{V},v}). \tag{88}$$

where

$$I_{r,v}(s, \chi_{\mathcal{V},v}) := \{\Phi\colon G_{r,v} \to \mathbb{C} \text{ smooth} \mid \Phi(pg, s) = \chi_{\mathcal{V},v}(p)|p|_v^{s+\rho}\Phi(g,s) \text{ for all } g \in G_{r,v}, p \in \tilde{P}_{r,v}\},$$

is the local analogue of $I_r(s, \chi_{\mathcal{V}})$.

For each finite prime p, let $K_p = \mathrm{Sp}_r(\mathbb{Z}_p)$ and let $\tilde{K}_p$ denote its inverse image in $G_{r,p}$. Similarly, let $K_\infty \simeq U(r)$ denote the standard maximal compact subgroup of $\mathrm{Sp}_r(\mathbb{R})$, let $\tilde{K}_\infty$ denote its inverse image in $G_{r,\infty}$, and set $\tilde{K} \subset G_{r,\mathbb{A}}$ to be the inverse image of $K_\infty \prod_p K_p$.

We say that a section $\Phi \in I_r(s, \chi_{\mathcal{V}})$ (resp. $\Phi_v \in I_{r,v}(s, \chi_{\mathcal{V},v})$) is *standard* if its restriction to $\tilde{K}$ (resp. $\tilde{K}_v$) is independent of s. Note that by the Cartan decomposition, a standard section is determined by its value at any fixed $s \in \mathbb{C}$.

In this paper, we will primarily be concerned with following standard sections for $r = 1, 2$.

Definition 2.13. *Let $\mathcal{L}$ be the lattice described in (24). We define a standard section*

$$\Phi_r^{\mathcal{L}}(s) = \otimes_{v \leq \infty} \Phi_{r,v}^{\mathcal{L}}(s) \in I_r(s, \chi_{\mathcal{V}}) \tag{89}$$

as follows:

- *At the place* ∞*, we set* $\Phi^{\mathcal{L}}_{r,\infty}(s) = \Phi^{3/2}_{r,\infty}(s)$*, the standard section of scalar* $\tilde{K}_\infty$*-type* $\det^{3/2}$*. More precisely, there exists a character* $\xi_{\frac{1}{2}}$ *of* $\tilde{K}_\infty$ *whose square descends to the determinant map on* $U(r)$*, cf.* [KRY06, §8.5.6]*. Then* $\Phi^{\mathcal{L}}_{r,\infty}(s)$ *is the standard section determined by the relation* $\Phi^{\mathcal{L}}_{r,\infty}(k,s) = \xi_{\frac{1}{2}}(k)^3$ *for all* $k \in \tilde{K}_\infty$*.*
- *For* $v < \infty$*, let* $\Phi^{\mathcal{L}}_{r,v}(g,s)$ *be the local standard section attached to the lattice* $\mathcal{L}_v^{\oplus r}$ *via the Rallis map; more precisely,* $\Phi^{\mathcal{L}}_{r,v}$ *is determined by the relation*

$$\Phi^{\mathcal{L}}_{r,v}(g,s_0) = \left(\omega_{\mathcal{V},v}(g)\varphi^{\mathcal{L}}_v\right)(0), \qquad s_0 = \frac{3-r-1}{2} \tag{90}$$

  where $\omega_{\mathcal{V},v}$ *is the Weil representation acting on* $S(\mathcal{V}_v^r)$*, and* $\varphi^{\mathcal{L}}_v$ *is the characteristic function of* $\mathcal{L}_v^r$*.*

Our primary interest is in the case $r = 2$, though we will have to consider the case $r = 1$ as well. For $Re(s) > 3/2$ and $g \in G_{2,\mathbb{A}}$, let

$$E(g,s,\Phi^{\mathcal{L}}_2) \;=\; \sum_{\gamma \in P_{2,\mathbb{Q}}\backslash G_{2,\mathbb{Q}}} \Phi^{\mathcal{L}}_2(\gamma g, s) \tag{91}$$

denote the corresponding Eisenstein series; by the general theory of Eisenstein series, this series admits a meromorphic continuation to $s \in \mathbb{C}$. Moreover, $E(g,s,\Phi^{\mathcal{L}}_2)$ is *incoherent* in the sense of [Kud97], and hence $E(g,0,\Phi^{\mathcal{L}}_2) = 0$.

It will be convenient to introduce "classical" coordinates as follows. Let

$$\mathbb{H}_2 = \{\tau = u + iv \in \mathrm{Sym}_2(\mathbb{C}) \mid v > 0\} \tag{92}$$

denote the Siegel upper half-space; for $\tau = u + iv \in \mathbb{H}_2$, choose any matrix $a \in \mathrm{GL}_2(\mathbb{R})$ with $\det(a) > 0$ and $v = a \cdot {}^t a$, and write

$$g_{\tau,\infty} = [n(u)m(a), 1] \in G_{2,\infty}, \qquad g_\tau = (g_{\tau,\infty}, 1, \dots) \in G_{2,\mathbb{A}}. \tag{93}$$

Define

$$E(\tau,s,\Phi^{\mathcal{L}}_2) = \det(v)^{-3/4} E(g_\tau, s, \Phi^{\mathcal{L}}_2), \tag{94}$$

which is a (non-holomorphic) Siegel modular form of scalar weight 3/2. We write its q-expansion as

$$E(\tau,s,\Phi^{\mathcal{L}}_2) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} C_T(v,s,\Phi^{\mathcal{L}}_2)\, q^T \tag{95}$$

with $q^T = e^{2\pi i tr(Tv)}$.

Theorem 2.14. *For every* $T \in \mathrm{Sym}_2(\mathbb{Q})$*, we have*

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(T,v) = C \frac{d}{ds} C_T(v,s,\Phi^{\mathcal{L}}_2)\Big|_{s=0}. \tag{96}$$

Here $C = \frac{\prod_{p|N}(p+1)}{24}$ *is the constant given in the introduction. In particular, we have an identity of* q*-expansions*

$$\sum_T \widehat{\deg}\, \widehat{\mathcal{Z}}(T,v) q^T = C\, E'(\tau,0,\Phi^{\mathcal{L}}_2). \tag{97}$$

This theorem will be proved in Section 4.

3. Local special cycles and Whittaker functionals

In this section, we study local analogues of the special cycles, defined in terms of deformations of p-divisible groups. A result of Gross-Keating computes the degrees of these cycles, which we relate to derivatives of Whittaker functionals.

3.1. Degrees of local special cycles.

Fix a prime p, let $\mathbb{F} = \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, $W = W(\mathbb{F})$ the ring of Witt vectors and $W_{\mathbb{Q}} = W \otimes_{\mathbb{Z}} \mathbb{Q}$ its field of fractions. Denote by Nilp_p the category of local W-algebras such that p is nilpotent.

Let $\mathbb{X}$ denote the (unique, up to isomorphism) supersingular p-divisible group of height 2 and dimension 1 over $\mathbb{F}$. Then $\mathrm{End}(\mathbb{X})$ is the maximal order in the division quaternion algebra over $\mathbb{Q}_p$; we denote the main involution by $x \mapsto x^\iota$, and the reduced norm by $\mathrm{nm}(x)$.

Fix a uniformizer ϖ. Let $\varphi \in \mathrm{End}(\mathbb{X})$ be an isogeny of degree N. By composing with an automorphism of $\mathbb{X}$, we may assume without loss of generality that

$$\varphi = \begin{cases} id, & p \nmid N \\ \varpi, & p | N. \end{cases} \tag{98}$$

With this setup in place, we recall the relevant Rapoport-Zink space: let $\mathcal{M} = \mathcal{M}_{\Gamma_0(N)}$ denote the moduli space (over Nilp_p) of diagrams $(\tilde{\varphi} \colon X \to X')$ where X and X' are deformations of $\mathbb{X}$ and $\tilde{\varphi}$ is an isogeny lifting $\varphi \in \mathrm{End}(\mathbb{X})$. We denote by $\mathcal{M}_0$ the moduli space parametrizing deformations of $\mathbb{X}$ alone, which is isomorphic to $\mathrm{Spf}(W[\![t]\!])$; note that for $p \nmid N$, we have $\varphi = id$, and so $\mathcal{M}_{\Gamma_0(N)}$ is isomorphic to $\mathcal{M}_0$.

Definition 3.1 (Local special cycles)**.** *(i) For $y \in \mathrm{End}(\mathbb{X})^{tr=0}$, let $Z(y)$ denote the moduli problem (over Nilp_p) parametrizing tuples $\{(\phi \colon X_1 \to X_2, \delta)\}$ where*

- *X_1 and X_2 are deformations of $\mathbb{X}$;*
- *ϕ is an isogeny lifting φ;*
- *$\delta \in \mathrm{Hom}(X_2, X_1)$ is an isogeny lifting $y \circ \varphi^{-1}$.*

Note that if the last condition holds, then $\delta \circ \phi$ is an element of $\mathrm{End}(\phi)$ lifting y.

(ii) If $y = (y_1, y_2)$ is a linearly independent pair of elements in $\mathrm{End}(\mathbb{X})^{tr=0}$, then we set

$$Z(y) = Z(y_1) \times_{\mathcal{M}} Z(y_2) \tag{99}$$

to be the intersection.

For a pair of vectors $f_1, f_2 \in \mathrm{End}(\mathbb{X})$, let $\langle f_1, f_2 \rangle = f_1 \cdot f_2^\iota + f_2 \cdot f_1^\iota$ denote the bilinear form attached to the quadratic form $Q(f) = \mathrm{nm}(f)$. We set

$$T(f_1, f_2) \ := \ \frac{1}{2} \begin{pmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle \\ \langle f_1, f_2 \rangle & \langle f_2, f_2 \rangle \end{pmatrix}. \tag{100}$$

Proposition 3.2 (Gross-Keating)**.** *Suppose $y_1, y_2 \in \mathrm{End}(\mathbb{X})^{tr=0}$ is a linearly independent pair of vectors, and let $\delta_i = y_i \circ \varphi^{-1}$ for $i = 1, 2$. Let*

$$T = T(\delta_1, \delta_2) \tag{101}$$

and let $0 \leq a_1 \leq a_2 \leq a_3$ denote the Gross-Keating invariants of the matrix $\mathrm{diag}(N, T)$. Then

$$\nu_p(T) := \deg Z(y) \tag{102}$$

only depends on T, and is given by the following explicit formulas.

If $a_1 \equiv a_2 \pmod 2$*, then*

$$\nu_p(T) = \sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-2)/2}(a_1+1)(2a_1+a_2+a_3-4i)p^i \tag{103}$$

$$+\frac{a_1+1}{2}(a_3-a_2+1)p^{(a_1+a_2)/2} \tag{104}$$

and if $a_1 \not\equiv a_2 \pmod 2$*, then*

$$\nu_p(T) = \sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-1)/2}(a_1+1)(2a_1+a_2+a_3-4i)p^i. \tag{105}$$

Proof. Given any $f \in \mathrm{End}(\mathbb{X})$, let $Z(f)_{GK}$ denote the locus on $\mathcal{M}_0 \times \mathcal{M}_0$ on which f deforms to an isogeny $f\colon X_1 \to X_2$ where X_1 and X_2 are deformations of $\mathbb{X}$. Then the deformation locus $Z(y)$ coincides with the triple intersection $Z(\varphi)_{GK} \cdot Z(\delta_1)_{GK} \cdot Z(\delta_2)_{GK}$ on $\mathcal{M}_0 \times \mathcal{M}_0$.

Note that for $i = 1, 2$,

$$\langle \delta_i, \varphi\rangle = \delta_i \cdot \varphi^\iota + \varphi \cdot (\delta_i)^\iota = (y_1\varphi^{-1}) \cdot (-\varphi) + \varphi(-\varphi^{-1} \cdot y_i^\iota) = -tr(y_i) = 0 \tag{106}$$

Therefore, the restriction of the quadratic form $Q(x) = \deg(x)$ to $\mathrm{span}\{\varphi, \delta_1, \delta_2\}$ is represented by the matrix $\mathrm{diag}(N, T)$ with respect to the basis $\{\varphi, \delta_1, \delta_2\}$. The desired formula is then [GK93, Proposition 5.4]. □

3.2. **Local Whittaker functionals and special cycles.** The main point of this section is to express the local intersection number, defined in the previous section, in terms of Whittaker functionals. Recall that for a standard section $\Phi_p \in I_{2,p}(s, \chi_{\mathcal{V},p})$ and $T \in \mathrm{Sym}_2(\mathbb{Q}_p)$, the Whittaker functional is defined to be

$$W_{T,p}(g, s, \Phi_p) := \int_{\mathrm{Sym}_2(\mathbb{Q}_p)} \Phi(w_p^{-1}n(b)g, s)\, \psi_p(-tr(Tb))\, db \tag{107}$$

where

- $w_p = \left[\left(\begin{smallmatrix} & -1_2 \\ 1_2 & \end{smallmatrix}\right), 1\right] \in G_p$;
- for $b \in \mathrm{Sym}_2(\mathbb{Q}_p)$, we write $n(b) = \left[\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right), 1\right]$;
- $\psi_p\colon \mathbb{Q}_p \to \mathbb{C}$ is the standard additive character that is trivial on $\mathbb{Z}_p$
- and db is the additive Haar measure on $\mathrm{Sym}_2(\mathbb{Q}_p)$ that is self-dual with respect to the pairing $(b_1, b_2) \mapsto \psi_p(tr(b_1b_2))$.

In addition to the section $\Phi_p^{\mathcal{L}} = \Phi_{2,p}^{\mathcal{L}}$ as in Definition 2.13, we also need the following auxilliary section. Let

$$V_p^{ra} := (B_p)^{tr=0} \tag{108}$$

denote the space of traceless elements of the division quaternion algebra B_p over $\mathbb{Q}_p$. We equip V_p^{ra} with the quadratic form $Q^{ra}(x) = \mathrm{nm}(x)$, the reduced norm of x. Note that the quadratic character $\chi_p' := \chi_{V^{ra},p}$ associated to V_p^{ra} is given by

$$\chi_p'(x) = (x, -1)_p. \tag{109}$$

Let $L_p^{ra} = \mathcal{O}_p \cap V_p^{ra}$, where $\mathcal{O}_p$ is the maximal order; finally, we define the local section

$$\Phi_p^{ra}(s) \in I_{2,p}(s, \chi') \tag{110}$$

to be the standard section attached to $(L_p^{ra})^{\oplus 2}$. The main result of this section is:

Proposition 3.3. *Suppose $y_1, y_2 \in \mathrm{End}(\mathbb{X})^{tr=0}$ are linearly independent vectors, and let $\delta_i = y_i \circ \varphi^{-1}$ and*

$$T := T(\delta_1, \delta_2) = N^{-1} T(y_1, y_2) \tag{111}$$

as in (100). Then

$$\deg\, Z(y) \cdot \log p = 2 c_p \left(\frac{\gamma_p(V_p^{ra})}{\gamma_p(\mathcal{V})} \right)^2 \frac{W'_{T,p}(e, 0, \Phi_p^{\mathcal{L}})}{W_{NT,p}(e, 0, \Phi_p^{ra})} \tag{112}$$

where

$$c_p = \left(\frac{1}{p-1} \right) \times \begin{cases} p+1, & \text{if } p|N \\ 1, & \text{if } p \nmid N. \end{cases} \tag{113}$$

and $\gamma_p(V_p^{ra})$ and $\gamma_p(\mathcal{V})$ are the local Weil indices, cf. e.g. [KRY06, §8.5.3].

Remark 3.4. If p is odd, it follows from [RR93, Appendix A] that $\gamma_p(V_p^{ra})^2 = 1$ and $\gamma_p(\mathcal{V}_p)^2 = (-1, p)_p$.

Proof for $p \nmid N$. First, we observe the following general fact: for a lattice L over $\mathbb{Z}_\ell$, let $\Phi_\ell^L \in I_\ell(s, \chi_L)$ denote the section corresponding to (the characteristic function of) the lattice $L^{\oplus 2}$. Then, for $T \in \mathrm{Sym}_2(\mathbb{Z}_\ell)$, we have

$$W_{T,\ell}(e, s, \Phi_\ell^L) = \gamma(V_\ell)^2 \cdot [L^\vee : L]^{-1} \cdot |2|_\ell^{\frac{1}{2}} \cdot \alpha_\ell(X, T, L)|_{X = p^{-s}} \tag{114}$$

where $V = L_{\mathbb{Q}_\ell}$, and $L^\vee$ is the dual lattice, cf. [KRY06, Lemma 5.7.1]. Here $\alpha_\ell(X, T, L)$ is the representation density polynomial, as in [Yan98, Yan04].

Now we return our situation, and consider the case $p \nmid N$. In this case, the result is contained in [Yan04]. Indeed, note that if $0 \le a_1 \le a_2 \le a_3$ are the Gross-Keating invariants of $\mathrm{diag}(N, NT)$, which are also the Gross-Keating invariants of $\mathrm{diag}(1, T)$, then $a_1 = 0$, cf. [Yan04, Appendix B].

Since N is a p-adic unit, carefully tracing through the contructions in [Yan04] shows that

$$\alpha_p(X, T, \mathcal{L}) = \alpha_p(X, NT, L_p^0) \tag{115}$$

where

$$L_p^0 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z}_p \right\}, \qquad Q(x) = \det(x).$$

Thus, [Yan04, Proposition 5.7] gives

$$\begin{aligned} \alpha'_p(1, T, \mathcal{L}_p)|_{X=1} &= -(1 - p^{-2}) \begin{cases} \sum_{0 \le i \le \frac{a_2 - 1}{2}} (a_2 + a_3 - 4i) p^i, & \text{if } a_2 \equiv 0 \pmod 2, \\ \sum_{0 \le i \le \frac{a_2}{2} - 1} (a_2 + a_3 - 4i) p^i - \frac{a_2 - a_3 + 1}{2} p^{\frac{a_2}{2}}, & \text{if } a_2 \equiv 1 \pmod 2 \end{cases} \\ &= -(1 - p^{-2}) \nu_p(T). \end{aligned} \tag{116}$$

On the other hand, [Yan04, Proposition 5.7] gives

$$\alpha_p(1, NT, L_p^{ra}) = 2(p+1); \tag{117}$$

combining these identities with (114) and Proposition 3.2 yields the proposition for $p \nmid N$. The case $p|N$ will be dealt at the end of this section after some preparation. □

In the remainder of the section, we suppose $p|N$; in particular, p is odd. We have

$$\mathcal{L}_p = \left\{ \begin{pmatrix} a & p^{-1} b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_p \right\}, \qquad Q(x) = N \det(x). \tag{118}$$

The Gram matrix of $\mathcal{L}_p$, with respect to the $\mathbb{Z}_p$-basis $\{\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right), \left(\begin{smallmatrix} & N^{-1} \\ 1 & \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\}$ is

$$\mathcal{S} := \begin{pmatrix} 1 & & \\ & -1 & \\ & & -N \end{pmatrix} \tag{119}$$

In particular, $[\mathcal{L}^\vee : \mathcal{L}]^{-1} = p^{-1}$ and hence

$$W_{T,p}(e, s, \Phi_p^{\mathcal{L}}) = \gamma(\mathcal{V}_p)^2 \cdot p^{-1} \alpha_\ell(X, T, \mathcal{L})\big|_{X = p^{-s}}. \tag{120}$$

Our first step is a formula for the representation density $\alpha_p(X, T, \mathcal{L})$, using the explicit formulas in [Yan98]. Recall that for a general $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ and lattice L, Yang decomposes the representation density as

$$\alpha_p(X, T, L) = 1 + R_1(X, T, L) + R_2(X, T, L) \tag{121}$$

for some explicit polynomials $R_1(X, T, L)$ and $R_2(X, T, L)$, defined in [Yan98, §7].

In our case, we compute $\alpha_p(X, T, \mathcal{L})$ via comparison to the representation density $\alpha_p(X, T, L_0)$, where $L_0 = M_2(\mathbb{Z}_p)^{\mathrm{tr}=0}$ with quadratic form $Q(x) = \det x$.

Lemma 3.5. *Suppose $p|N$ and let $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$. Then*

$$\alpha_p(X, T, \mathcal{L}) = X^{-2} \left(p\alpha(X, NT, L_0) \ + (X - p) \left(R_1(X, NT, L_0) + \frac{X + p}{p} \right) \right), \tag{122}$$

where $R_1(X, NT, L_0)$ is the polynomial defined in [Yan98, Theorem 7.1]

Sketch of proof. In [Yan98, §7], the representation density is expressed in terms of polynomials $R_1(X, T, L)$ and $R_2(X, T, L)$, which are further decomposed in terms of explicit polynomials $I_{i,j}(X, T, L)$ with $i = 1, 2$ and $j = 1, \ldots, 8$. Unwinding the definitions of these polynomials, one can verify explicitly that

$$\begin{aligned} I_{1,j}(X, T, \mathcal{S}) &= X^{-1} I_{1,j}(X, NT, S_0) \\ I_{2,j}(X, T, \mathcal{S}) &= pX^{-2} I_{2,j}(X, NT, S_0), \qquad j = 1, \ldots, 7 \\ I_{2,8}(X, T, \mathcal{S}) &= pX^{-2} I_{2,8}(X, NT, S_0) - 1. \end{aligned}$$

The lemma then follows from [Yan98, Theorem 7.1].

□

Explicit formulas for $\alpha(X, NT, L_0)$ and $R_1(X, NT, L_0)$ are as follows:

Proposition 3.6 ([Kit83], [Yan98, §8])**.** *Suppose $p|N$ and T is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to $\left(\begin{smallmatrix} \epsilon_1 p^a & \\ & \epsilon_2 p^b \end{smallmatrix}\right)$. Let $M = N/p \in \mathbb{Z}_p^\times$ and define*

$$v_0^+ = (-M\epsilon_1, p)_p \qquad v_1^+ = \begin{cases} (-M\epsilon_2, p)_p, & \text{if } b \text{ is odd} \\ (-\epsilon_1\epsilon_2, p)_p, & \text{if } b \text{ is even.} \end{cases}$$

(i) If a is odd, then

$$\alpha_p(X, NT, L_0) = (1 - p^{-2}X^2) \left\{ \sum_{0 \le k < \frac{a+1}{2}} p^k (X^{2k} + (v_0^+ X)^{a+b+2-2k}) + p^{\frac{a+1}{2}} \sum_{a+1 \le k \le b+1} (v_0^+ X)^k \right\}$$

and

$$R_1(X, NT, L_0) = (1-p^{-2}) \sum_{0<k\leq \frac{a+1}{2}} p^k X^{2k} + p^{\frac{a+1}{2}}(1-p^{-1}v_0^+ X) \sum_{a+1<k\leq b+1} (v_0^+ X)^k.$$

(ii) If a is even, then

$$\alpha_p(X, NT, L_0) = (1-p^{-2}X^2)\left\{ \sum_{0\leq 0\leq a/2} p^k (X^{2k} + v_1^+ X^{a+b+2-2k}) \right\}$$

and

$$R_1(X, T, L_0) = -1 + (1-p^{-1}X^2) \sum_{0\leq k\leq a/2} p^k X^{2k} + v_1 p^{a/2} X^{b+2}.$$

□

Proposition 3.7. *Suppose that $p|N$ and $p \in \mathrm{Diff}(T, \mathcal{V})$, i.e. T is not represented by $\mathcal{V}_p$. Then*

$$\frac{p}{1-p} \cdot \alpha_p'(1, T, \mathcal{L}) = \nu_p(T).$$

Here $\nu_p(T) = \deg(Z(y))$ is the intersection multiplicity given explicitly in Proposition 3.2, for any tuple $y = (y_1, y_2)$ with $T(y_1, y_2) = NT$.

Proof. This follows from a straightforward, though tedious, computation using Lemma 3.5 and Proposition 3.6. □

We can now conclude the proof of Proposition 3.3:

Proof of Proposition 3.3 for $p|N$: Using (114) and [Yan98, Proposition 8.7], one has the formula

$$W_{NT,p}(e, 0, \Phi_p^{ra}) = \gamma_p(V_p^{ra})^2 \cdot p^{-2} \cdot 2(p+1) = 2p^{-2}(p+1). \tag{123}$$

On the other hand, combining (114) and Proposition 3.7, we have

$$W'_{T,p}(e, s, \Phi_p^{\mathcal{L}}) = \gamma(\mathcal{V}_p)^2 \cdot p^{-1} \cdot \alpha'(1, T, \mathcal{L}) \cdot (-\log p) = \gamma_p(\mathcal{V}_p)^2 \left(\frac{p-1}{p^2}\right) \nu_p(T) \cdot \log p. \tag{124}$$

From this, the proposition follows. □

4. Proof of the main theorem

4.1. **Positive definite T.** Suppose T is positive definite. Our strategy in this case closely mirrors that of [KRY06]. We begin by recalling the following well-known facts about the Fourier coefficients of Siegel Eisenstein series, see e.g. [Kud97, §1]:

Proposition 4.1. *Suppose $T \in \mathrm{Sym}_r(\mathbb{Q})$ is non-degenerate, that V is a quadratic space over $\mathbb{Q}$ of dimension $r+1$, and $\Phi = \otimes_v \Phi_v \in I_r(s, \chi_V)$ is a factorizable section. Then:*

(i) $E_T(g, s, \Phi) = \prod_{v\leq\infty} W_{T,v}(g_v, s, \Phi_v)$.

(ii) Suppose Φ_v is in the image of the Rallis map and that V_v does not represent T at some place v. Then $W_{T,v}(e, 0, \Phi_v) = 0$.

□

For our purposes, we will also require certain auxiliary sections. Fix a prime p, let $B^{(p)}$ denote the rational quaternion algebra ramified exactly at p and ∞, and let $V^{(p)} \subset B^{(p)}$ denote the subset of traceless elements. We equip $V^{(p)}$ with the quadratic form $Q(x) = \mathrm{nm}(x)$, the reduced norm. We fix an order $\mathcal{O}^{(p)} \subset \mathcal{B}^{(p)}$ as follows: if $p \nmid N$, then we take $\mathcal{O}^{(p)}$ to be an Eichler order of level N. If $p|N$, we take $\mathcal{O}^{(p)}$ to be an Eichler order of level N/p. In any case, let

$$L^{(p)} := \mathcal{O}^{(p)} \cap V^{(p)} \tag{125}$$

denote the set of trace-zero elements.

Finally, we set

$$\Phi^{(p)}(s) \in I_2(s, \chi') \tag{126}$$

to be the global standard section associated to $(L^{(p)})^{\oplus 2}$; here $\chi' = \chi_{V^{(p)}}$ is the character $\chi'(x) = (-1, x)_{\mathbb{A}}$.

Lemma 4.2. *Suppose $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$, fix a prime p as above, and let $q \neq p$.*

(i) If $q \nmid N$, then

$$W_{NT,q}(e, s, \Phi_q^{(p)}) \;=\; \left(\frac{\gamma_q(V_q^{(p)})}{\gamma_q(\mathcal{V}_q)}\right)^2 W_{T,q}(e, s, \Phi_q^{\mathcal{L}}).$$

(ii) Suppose $q|N$. Then at $s = 0$, we have

$$W_{NT,q}(e, 0, \Phi_q^{(p)}) \;=\; \left(\frac{\gamma_q(V_q^{(p)})}{\gamma_q(\mathcal{V}_q)}\right)^2 W_{T,q}(e, 0, \Phi_q^{\mathcal{L}}).$$

Proof. Recall that

$$W_{T,q}(e, s, \Phi_q^{\mathcal{L}}) = \gamma_q(\mathcal{V}_q)^2 \cdot |\det \mathcal{L}_q|_q \cdot \alpha_q(X, T, \mathcal{L}_q)|_{X=p^{-s}} \tag{127}$$

and similarly for $W_{NT,q}(e, s, \Phi_q^{(p)})$.

If $q \nmid N$, then we have identifications $\mathcal{L}_q \simeq (M_2(\mathbb{Z}_q)^{tr=0}, N\det(x))$, and $L_q^{(p)} \simeq (M_2(\mathbb{Z}_q)^{tr=0}, \det(x))$ as quadratic spaces. By [Yan98, Proposition 8.6], we have

$$\alpha_q(X, T, \mathcal{L}_q) = \alpha_q(X, N^{-1}T, L^{(p)}) = \alpha_q(X, NT, L_q^{(p)}) \tag{128}$$

from which the first part of the lemma follows.

When $q|N$, we have that

$$L_q^{(p)} = \left\{ \begin{pmatrix} a & b \\ qc & -a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_q \right\}, \qquad Q(x) = \det(x). \tag{129}$$

By [Yan98, Corollary 8.5], we have

$$\alpha(X, NT, L_q^{(p)}) = q^2 \alpha(X, NT, L_0) + (q - q^2) R_1(X, NT, L_0) + 1 - q^2, \tag{130}$$

and so, comparing with Lemma 3.5, taking $X = 1$ in both formulas gives

$$\alpha(1, NT, L_q^{(p)}) = q\alpha(1, T, \mathcal{L}_q). \tag{131}$$

Observing that $|\det \mathcal{L}_q|_q = q^{-1}$ and $|\det L_q^{(p)}|_q = q^{-2}$, the second part of the lemma follows immediately. □

Corollary 4.3. *Suppose* $\mathrm{Diff}(T) = \{p\}$. *Then*

$$E_T'(g_\tau, 0, \Phi^{\mathcal{L}}) \cdot q^{-T} = \frac{\nu_p(T) \cdot \log p}{2c_p} \cdot E_{NT}(g_\tau, 0, \Phi^{(p)}) \cdot q^{-NT}$$

Proof. By Proposition 4.1, we have

$$W_{T,p}(e,0,\Phi_p^{\mathcal{L}}) = 0, \tag{132}$$

and so

$$E'_T(g_\tau,0,\Phi^{\mathcal{L}}) = W'_{T,p}(e,0,\Phi_p^{\mathcal{L}}) \cdot W_{T,\infty}(g_\tau,0,\Phi_\infty^{\mathcal{L}}) \cdot \prod_{\substack{v<\infty \\ v\neq p}} W_{T,v}(e,0,\Phi_v^{\mathcal{L}}). \tag{133}$$

At the infinite place, we have that $\Phi_\infty^{\mathcal{L}} = \Phi_\infty^{3/2} = \Phi_\infty^{(p)}$. The corresponding Whittaker functionals are given explicitly by the formula

$$W_{T,\infty}(g_\tau,0,\Phi_\infty^{3/2}) = -2\sqrt{2}(2\pi)^2\det(v)^{3/4}q^T, \tag{134}$$

cf. [KRY06, Theorem 5.2.7(i)], for any non-degenerate $T \in \mathrm{Sym}_2(\mathbb{Q})$. Thus, in our case we find

$$\frac{W_{T,\infty}(g_\tau,0,\Phi_\infty^{\mathcal{L}})}{W_{NT,\infty}(g_\tau,0,\Phi_\infty^{(p)})} = \frac{q^T}{q^{NT}}. \tag{135}$$

Combining this identity with Lemma 4.2 and Proposition 3.3, we find

$$\begin{aligned} E'_T(g_\tau,0,\Phi^{\mathcal{L}})q^{-T} &= \frac{\nu_p(T)\log p}{2c_p} \cdot \left(\prod_{v<\infty} \frac{\gamma_v(\mathcal{V})^2}{\gamma_v(V^{(p)})^2}\right) \cdot \prod_{v\leq\infty} W_{NT,v}(g_\tau,0,\Phi^{(p)}) \cdot q^{-NT} \\ &= \frac{\nu_p(T)\log p}{2c_p}\left(\prod_{v<\infty} \frac{\gamma_v(\mathcal{V})^2}{\gamma_v(V^{(p)})^2}\right) E_{NT}(g_\tau,0,\Phi^{(p)})\, q^{-NT}. \end{aligned} \tag{136}$$

It remains to show that the product involving Weil indices equals 1. For any quadratic space V, the product formula $\prod_{v\leq\infty}\gamma_v(V) = 1$ holds; thus

$$\prod_{v<\infty} \frac{\gamma_v(\mathcal{V})^2}{\gamma_v(V^{(p)})^2} = \frac{\gamma_\infty(\mathcal{V})^2}{\gamma_\infty(V^{(p)})^2}. \tag{137}$$

By [KRY06, p. 330], we have $\gamma_\infty(\mathcal{V})^2 = -1 = \gamma_\infty(V^{(p)})^2$, so the ratio above is one, and the corollary is proved. ∎

The next step in our proof is to apply the Siegel-Weil formula for the positive definite space $V^{(p)}$. Let $H^{(p)} = O(V^{(p)})$, viewed as an algebraic group over $\mathbb{Q}$. If

$$\varphi^{(p)} \in S((V^{(p)} \otimes_{\mathbb{Q}} \mathbb{A}_f)^{\oplus 2}) \tag{138}$$

is the characteristic function of $(L^{(p)} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^{\oplus 2}$, we define the theta integral

$$I(g,\varphi^{(p)}) = \int_{[H^{(p)}]} \Theta(g,h,\varphi^{(p)})\,dh, \qquad g \in G_{\mathbb{A}} \tag{139}$$

here $\Theta(g,h,\varphi^{(p)})$ is the usual theta function, and dh is the left Haar measure on $[H^{(p)}] = H^{(p)}(\mathbb{Q})\backslash H^{(p)}(\mathbb{A})$ normalized so the total volume is 1.

Then the Siegel-Weil formula implies that for any non-degenerate T, we have

$$E_T(g,0,\Phi^{(p)}) = 2\, I_T(g,\varphi^{(p)}). \tag{140}$$

The computation of $I_T(g,\varphi^{(p)})$ is given in [KRY06, §5.3]; we review the computation here. Note that $O(V^{(p)}) \simeq SO(V^{(p)}) \times \mu_2$, and since $\varphi^{(p)}$ is the characteristic function of a lattice, it is $\mu_2(\mathbb{A}_f)$-invariant. The measure dh decomposes as $dh_1 \times dc$ where the volume of $\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A})$ with respect to dc is equal to $1/2$; fixing a gauge form ω on $SO(V^{(p)})$ as in [KRY06, p. 126], we obtain a decomposition $dh_1 = \prod_{v\leq\infty} dh_{1,v}$.

There is a surjective map $B^{(p)} \to SO(V^{(p)})$, where an element of $B^{(p)}$ acts by conjugation on $V^{(p)}$. Let $K^{(p)} \subset SO(V^{(p)})(\mathbb{A}_f)$ denote the image of $(\widehat{\mathcal{O}^{(p)}})^\times$ under this map, and write

$$SO(V^{(p)})(\mathbb{A}_f) = \coprod_j SO(V^{(p)})(\mathbb{Q}) \cdot h_j \cdot K^{(p)}. \tag{141}$$

Let $\Gamma_j = SO(V^{(p)})(\mathbb{Q}) \cap h_j K^{(p)} h_j^{-1}$. Then [KRY06, Proposition 5.3.6] implies that for $g = (g_\infty, e, \dots)$,

$$I_T(g,\varphi^{(p)}) = \frac{1}{2} O_{T,\infty}(g_\infty, \varphi^{(p)}_\infty) \operatorname{vol}\left(K^{(p)}, dh_{1,f}\right) \left(\sum_j \sum_{\substack{y\in\Omega(T,V^{(p)}) \\ \bmod \Gamma_j}} \varphi^{(p)}(h_j^{-1} y) \right) \tag{142}$$

where

$$O_{T,\infty}(g_\infty,\varphi'_\infty) = \int_{SO(V^{(p)})(\mathbb{R})} \omega(g_\infty)\varphi'_\infty(h_{1,\infty}^{-1}\cdot x_0)\, dh_{1,\infty} \tag{143}$$

and $x_0 \in \Omega(T, V^{(p)})$ is fixed.

The volume appearing in (142) can be computed following [KRY06, Lemma 5.3.9]; we sketch the argument here.

Lemma 4.4. *Suppose* $\operatorname{Diff}(T) = \{p\}$. *Then*

$$O_{T,\infty}(g_\tau, \varphi^{(p)}_\infty) \ \operatorname{vol}(K^{(p)}, dh_{1,f}) = \frac{24}{p-1} \cdot \left(\prod_{\substack{q|N \\ q\neq p}} (1+q)^{-1} \right) \cdot \det(v)^{3/4}\, q^T. \tag{144}$$

Proof. Suppose $v \leq \infty$ and $\varphi_v \in S((V_v^{(q)})^2)$ is any Schwartz function, and define the local orbital integral

$$O_{T,v}(g_v,\varphi_v) := \int_{SO(V^{(q)})(\mathbb{Q}_v)} \omega(g_v)\varphi_v(h^{-1}x_0)\, dh_{1,v} \tag{145}$$

where $x_0 \in \Omega(T, V^{(q)})$ is a fixed tuple in $(V^{(q)})^2$ with $T(x_0) = T$. Then there is a non-zero constant $d_v = d_v(V^{(q)}, dh_{1,v})$ such that

$$O_{T,v}(g_v,\varphi_v) \ = \ d_v\, W_{T,v}(g_v, 0, \Phi(\varphi_v)). \tag{146}$$

By [KRY06, Proposition 5.3.3], this constant only depends on the local measure $dh_{1,v}$, and not on T, and moreover

$$\prod_{v\leq\infty} d_v = 1. \tag{147}$$

Now arguing as in [KRY06, Lemma 5.3.9], we have that for a finite prime q,

$$\operatorname{vol}(K_q^{(p)}, dh_{1,q}) = d_q \cdot \gamma_q(V^{(p)})^2 \cdot |2|_q^{3/2} \cdot \begin{cases} (1-q^{-2}), & q \nmid Np \\ (1+q)^{-1}(1-q^{-2}), & q|N,\ q\neq p \\ p^{-2}(p+1), & q = p. \end{cases} \tag{148}$$

Thus, we have

$$\begin{aligned} &O_{T,\infty}(g_{\tau,\infty}, \varphi_\infty^{(p)}) \cdot \mathrm{vol}(K^{(p)}, dh_{1,f}) \\ &\quad = W_{T,\infty}(g_{\tau,\infty}, 0, \Phi_\infty^{3/2}) \left(\prod_{v<\infty} \gamma_v(V^{(p)})^2 |2|_v^{3/2}\right) \left(\prod_{q<\infty} (1-q^{-2})\right) \\ &\qquad \times \left(\frac{p^{-2}(p+1)}{1-p^{-2}}\right) \left(\prod_{\substack{q|N \\ q\neq p}} (q+1)^{-1}\right) \\ &\quad = W_{T,\infty}(g_{\tau,\infty}, 0, \Phi_\infty^{3/2}) \cdot \left(\gamma_\infty(V^{(p)})^{-2}\, 2^{-3/2}\right) \zeta(2)^{-1} \frac{1}{p-1} \left(\prod_{\substack{q|N \\ q\neq p}} (q+1)^{-1}\right). \end{aligned}$$

Finally, we recall that $\gamma_\infty(V^{(q)})^2 = -1$, see [KRY06, eqn. (5.3.71)] and $\zeta(2) = \pi^2/6$, and

$$W_{T,\infty}(g_\tau, 0, \Phi_\infty^{3/2}) = -2\sqrt{2}(2\pi)^2 \det(v)^{3/4} q^T, \tag{149}$$

cf. [KRY06, Theorem 5.2.7(i)]; this proves the desired formula. □

Recall that we had written the Fourier expansion of the Eisenstein series $E(\tau, s, \Phi^{\mathcal{L}})$ as

$$E(\tau, s, \Phi^{\mathcal{L}}) = \sum_T C_T(v, s, \Phi^{\mathcal{L}}) q^T. \tag{150}$$

Combining Corollary 4.3 and Lemma 4.4 (replacing T by NT in the latter lemma) we obtain the following result.

Corollary 4.5. *Suppose $T > 0$ and $\mathrm{Diff}(T) = \{p\}$. Then $C'_T(0, \Phi^{\mathcal{L}}) = C'_T(0, v, \Phi^{\mathcal{L}})$ is independent of v, and is given by the formula*

$$C'_T(0, \Phi^{\mathcal{L}}) = \nu_p(T) \log p \cdot \left(12 \prod_{q|N} (1+q)^{-1}\right) \cdot \left(\sum_j \sum_{\substack{x \in \Omega(NT, V^{(p)}) \\ \mathrm{mod}\ \Gamma_j}} \varphi^{(p)}(h_j^{-1} x)\right) \tag{151}$$

Proof. By Corollary 4.3 and applying definitions, we have

$$C'_T(0, \Phi^{\mathcal{L}}) = \frac{\nu_p(T) \cdot \log p}{2c_p} \cdot \det(v)^{-3/4} \cdot E_{NT}(g_\tau, 0, \Phi^{(p)}) \cdot q^{-NT}, \tag{152}$$

where $c_p = \frac{p+1}{p-1}$ if $p|N$ and $c_p = \frac{1}{p-1}$ otherwise. Now combining the Siegel-Weil formula, cf. (140), the factorization (142), and Lemma 4.4, we have

$$\begin{aligned} E_{NT}(g_\tau, 0, \Phi^{(p)}) &= 2 I_{NT}(g_\tau, 0, \Phi^{(p)}) \\ &= O_{NT,\infty}(g_\infty, \varphi_\infty^{(p)}) \operatorname{vol}\left(K^{(p)}, dh_{1,f}\right) \left(\sum_j \sum_{\substack{y \in \Omega(T, V^{(p)}) \\ \bmod \Gamma_j}} \varphi^{(p)}(h_j^{-1} y) \right) \\ &= \frac{24\, c_p}{\prod_{q|N}(1+q)} \det(v)^{3/4}\, q^{NT} \left(\sum_j \sum_{\substack{y \in \Omega(T, V^{(p)}) \\ \bmod \Gamma_j}} \varphi^{(p)}(h_j^{-1} y) \right). \end{aligned} \tag{153}$$

The corollary follows immediately. □

Finally, we prove the main identity in the positive definite case:

Theorem 4.6. *Suppose $T > 0$. Then*

$$\widehat{\deg}\, \mathcal{Z}(T) = \frac{\prod_{q|N}(q+1)}{24} \cdot C_T'(0, \Phi^{\mathcal{L}}) \tag{154}$$

Proof. First, suppose $\#\mathrm{Diff}(T) > 1$. By Lemma 2.6, the left hand side vanishes, and by Proposition 4.1, the right hand side vanishes, establishing the result in this case.

We may therefore suppose $\mathrm{Diff}(T) = \{p\}$ for some prime p. Recall that there is an identification[1]

$$\mathcal{X}(\bar{\mathbb{F}}_p)^{ss} \;\simeq\; \left[B^{(p),\times} \Big\backslash \left(B^{(p),\times}(\mathbb{A}_f) \Big/ \widehat{\mathcal{O}^{(p)}}^{\times} \right) \right]. \tag{155}$$

Now suppose $\varphi \colon E \to E' \in \mathcal{X}(\overline{\mathbb{F}}_p)^{ss}$ is a geometric point corresponding to the coset $[b] = b\, \widehat{\mathcal{O}^{(p)}}^{\times}$ as above. Then the lattice $L(\varphi)$, defined in (14), is identified with the lattice

$$b \cdot L^{(p)} := \left(b\, \widehat{L^{(p)}} b^{-1} \right) \cap V^{(p)} \tag{156}$$

where $L^{(p)} = V^{(p)} \cap \mathcal{O}^{(p)}$ as above.

Tracing through definitions, we have an identification

$$\mathcal{Z}(T)(\overline{\mathbb{F}}_p) \simeq \left[B^{(p),\times} \Big\backslash \mathcal{C}(T) \right] \tag{157}$$

where

$$\mathcal{C}(T) = \left\{ (y, [b]) \in (V^{(p)})^2 \times B^{(p),\times}(\mathbb{A}_f) \Big/ \widehat{\mathcal{O}^{(p)}}^{\times} \,\Big|\, y \in (b \cdot L^{(p)})^2,\ T(y) = NT \right\}. \tag{158}$$

[1] Briefly, the right hand side is interpreted as the set of invertible right $\mathcal{O}^{(p)}$-modules. This latter set is in bijection with $\mathcal{X}(\overline{\mathbb{F}}_p)^{ss}$ as follows. Fix a base point $(\pi_0 \colon E_0 \to E_0') \in \mathcal{X}(\overline{\mathbb{F}}_p)^{ss}$. Then, given a point $\pi \colon E \to E' \in \mathcal{X}(\overline{\mathbb{F}}_p)^{ss}$, the corresponding $\mathcal{O}^{(p)}$ module is $\mathrm{Hom}(\pi_0, \pi)$. See e.g. [Rib90, §3] for details.

As described in Section 3, the (arithmetic) degree of the local ring of $\mathcal{Z}(T)$ at each geometric point is the same, and is given by $\nu_p(T)\log p$. Hence

$$\widehat{\deg}\,\mathcal{Z}(T) = \sum_{z\in\mathcal{Z}(T)(\bar{\mathbb{F}}_p)} \frac{\log|\mathcal{O}_{\mathcal{Z}(T),z}|}{|\operatorname{Aut}(z)|} \tag{159}$$

$$= \nu_p(T)\log p\cdot\#\left[B^{(p),\times}\backslash\mathcal{C}(T)\right], \tag{160}$$

where on the right hand side, we have the "stacky" cardinality, i.e. the number of orbits, with each orbit weighted by the reciprocal of the order of the corresponding stabilizer group.

To determine this cardinality, let $H^{(p)} = SO(V^{(p)})$, and recall that there is an exact sequence

$$1 \longrightarrow Z \longrightarrow B^{(p),\times} \xrightarrow{\;c\;} H^{(p)} \longrightarrow 1 \tag{161}$$

where $b\in B^{(p),\times}$ acts on $V^{(p)}$ by conjugation, and $Z\simeq\mathbb{G}_m$ is the centre. Since $Z(\mathbb{A}_f) = Z(\mathbb{Q})\cdot\left(Z(\mathbb{A}_f)\cap(\widehat{\mathcal{O}^{(p)}})^\times\right)$, we have a bijection

$$B^{(p),\times}\Big\backslash B^{(p),\times}(\mathbb{A}_f)\Big/\widehat{\mathcal{O}^{(p)}}^\times \longleftrightarrow H^{(p)}(\mathbb{Q})\Big\backslash H^{(p)}(\mathbb{A}_f)\Big/K^{(p)} \tag{162}$$

where $K^{(p)}$ was, by definition, the image of $(\widehat{\mathcal{O}^{(p)}})^\times$ in $H^{(p)}(\mathbb{A}_f)$. As in (141), we choose representatives $\{h_k\}$ for this double coset space, and set $\Gamma_j = H^{(p)}(\mathbb{Q})\cap h_jK^{(p)}h_j^{-1}$. Moreover, since the components of y span a two dimensional subspace of $V^{(p)}$, it follows that the stablilizer of any point $(y,[b])$ is equal to $Z(\mathbb{Q})\cap\widehat{\mathcal{O}^{(p)}}^\times = \{\pm1\}$. Thus

$$\#\left[B^{(p),\times}\backslash\mathcal{C}(T)\right] \;=\; \frac{1}{2}\sum_j \sum_{\substack{y\in h_j\cdot L^{(p)}\\ T(y)=NT\\ \mathrm{mod}\Gamma_j}} 1 \;=\; \frac{1}{2}\sum_j \sum_{\substack{y\in\Omega(NT,V^{(p)})\\ \mathrm{mod}\ \Gamma_j}} \varphi^{(p)}(h_j^{-1}y) \tag{163}$$

where, as we recall, $\varphi^{(p)}\in S(V^{(p)}(\mathbb{A}_f)^2)$ is the characteristic function of $(\widehat{L^{(p)}})^2$. The theorem follows from a comparison with Corollary 4.5. □

4.2. T **of signature** $(1,1)$ **or** $(0,2)$**.** Recall that when T is non-degenerate but not positive definite, the class $\widehat{\mathcal{Z}}(T,v)$ is purely archimedean:

$$\widehat{\mathcal{Z}}(T,v) \;:=\; (0,\Xi(T,v)) \;\in\; \widehat{\mathrm{CH}}^2_{\mathbb{C}}(\mathcal{X}_0(N)) \tag{164}$$

where $\Xi(T,v)$ is defined in (66). In this case, the arithmetic degree is given by

$$\widehat{\deg}\,\widehat{\mathcal{Z}}(T,v) \;=\; \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})}\Xi(T,v). \tag{165}$$

The computation of this integral was carried out in [GS19] in the case of compact Shimura varieties. A crucial step in the argument is the application of the Siegel-Weil formula to relate the integral of a certain theta function (attached to the Schwartz form denoted by $\nu(x)$ in *loc. cit.*) to a special value of the corresponding Eisenstein series.

In our case, the fact that $\mathcal{V}$ contains isotropic vectors implies that the relevant theta integrals do not necessarily converge for arbitrary Schwartz functions; however, as we prove below, the particular Schwartz form relevant to our setting *does* lead to a convergent theta integral, and work of Kudla and Rallis [KR94] implies that the Siegel-Weil formula holds for this Schwartz form. With this fact established, the arguments of [GS19] carry through verbatim and yield a computation of the integral (165).

To state things more precisely, recall the explicit expression

$$\nu(\lambda) = \psi(\lambda)\, d\mu \tag{166}$$

where

$$\psi(\lambda, z) = \left(-\frac{1}{\pi} + 2\sum_{i=1}^{2} (R(\lambda_i, z) + 2Q(\lambda_i)) \right) e^{-2\pi \sum R(\lambda_i, z) + Q(\lambda_i)}, \qquad \lambda = (\lambda_1, \lambda_2) \in \mathcal{V}_{\mathbb{R}}^2, \tag{167}$$

and

$$d\mu = \frac{dx \wedge dy}{y^2} \tag{168}$$

with $z = x + iy \in \mathbb{H}^{\pm}$.

For future use, we fix the basepoint $i \in \mathbb{H}$ and abbreviate

$$\phi(\lambda) = \psi(\lambda, i) \tag{169}$$

so that $\phi \in S(\mathcal{V}_{\mathbb{R}}^2)$ is a Schwartz function.

This Schwartz function gives rise to a theta function that, at least for the moment, is best described adelically. Let $H = O(\mathcal{V})$, and let ω denote the action of $G_{2,\mathbb{A}} \times H(\mathbb{A})$ on $S(V^2(\mathbb{A}))$ via the Weil representation. Recall that the action of $H(\mathbb{A})$ is the linear action; more precisely, for $h \in H(\mathbb{A})$, $\lambda = (\lambda_1, \lambda_2) \in \mathcal{V}(\mathbb{A})^2$ and $\varphi \in S(\mathcal{V}(\mathbb{A})^2)$, we have

$$\omega(1, h)\varphi(\lambda) = \varphi(h^{-1} \cdot \lambda) = \varphi\left(h^{-1} \cdot \lambda_1,\, h^{-1} \cdot \lambda_2\right). \tag{170}$$

Returning to our Schwartz function ϕ, we consider the adelic Schwartz function $\phi_{\mathbb{A}} = \phi \otimes 1_{\widehat{\omega}}$, where $1_{\widehat{\omega}} \in S(\mathcal{V}(\mathbb{A}_f)^2)$ is the characteristic function of $\widehat{\mathcal{L}}^2 = (\mathcal{L} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^2$. Then the corresponding theta series is

$$\Theta(g', h, \phi_{\mathbb{A}}) := \sum_{\lambda \in \mathcal{V}^2} \omega(g', h)\phi_{\mathbb{A}}(\lambda). \tag{171}$$

Lemma 4.7. *As a function of h, the theta function $\Theta(g', h, \phi)$ is left $H(\mathbb{Q})$ invariant, is absolutely convergent, and descends to a rapidly decreasing function on $H(\mathbb{Q})\backslash H(\mathbb{A})$.*

Proof. The left invariance is immediate from definitions. To prove the convergence and rapid decrease, we make use of the "mixed model" of the Weil representation. In general, suppose V is a quadratic space of Witt rank r, decomposed as

$$V = V_{an} \overset{\perp}{\oplus} V_{r,r} \tag{172}$$

where V_{an} is anisotropic and $V_{r,r}$ is isomorphic to r copies of the hyperbolic plane. Fix a standard basis

$$v_1', \ldots, v_r', v_1'', \ldots, v_r'' \tag{173}$$

for $V_{r,r}$, where $\langle v_i', v_i'' \rangle = 1$ for $i = 1, \ldots, r$, and all other inner products are zero. Setting $V' = \operatorname{span}\{v_1', \ldots, v_r'\}$, the given basis yields an identification $(V')^k \simeq M_{k,r}(\mathbb{Q})$ for any positive integer k by the map

$$A = (a_{ij}) \in M_{k,r}(\mathbb{Q}) \;\longleftrightarrow\; \left(\sum_{i=1}^{r} a_{1i} v_i', \cdots \sum_{i=1}^{r} a_{ki} v_i' \right). \tag{174}$$

Similarly, setting $V'' = \operatorname{span}\{v_1'', \ldots, v_r''\}$, we may identify $(V'')^k \simeq M_{k,r}(\mathbb{Q})$ in the same way.

For $G' = \mathrm{Mp}_4$, the mixed model of the Weil representation $\widehat{\omega}$ is an action $G'_{\mathbb{A}} \times O(V)(\mathbb{A})$ on $S(V(\mathbb{A})^2) \simeq S(V_{an}(\mathbb{A})^2) \otimes S(M_{2,r}(\mathbb{A})) \otimes S(M_{2,r}(\mathbb{A}))$. The two models are related by an intertwining map

$$S(V(\mathbb{A})^2) \xrightarrow{\sim} S(V_{an}(\mathbb{A})^2) \otimes S(M_{2,r}(\mathbb{A})) \otimes S(M_{2,r}(\mathbb{A})), \qquad \varphi \mapsto \widehat{\varphi}, \tag{175}$$

where, by definition,

$$\widehat{\varphi}(\lambda, A, B) := \int_{M_{2,r}(\mathbb{A})} \varphi(\lambda + X + B)\psi(\mathrm{tr} A^t X) dX, \tag{176}$$

where $\psi\colon \mathbb{A} \to \mathbb{C}$ is the standard additive character that is trivial on $\mathbb{Q}$ and $\widehat{\mathbb{Z}}$, and we identify $V'(\mathbb{A})^2 \simeq M_{2,r}(\mathbb{A}) \simeq V''(\mathbb{A})^2$ as above.

With this setup in place, we have the following criterion due to Kudla-Rallis: let $\varphi \in V(\mathbb{A})^2$ and suppose that

$$\widehat{\varphi}(x, A, B) = 0 \qquad \text{whenever } \mathrm{rank}([A\, B]) < r. \tag{177}$$

Then the proof of [KR94, Proposition 5.3.1] implies that $\Theta(g', h, \varphi)$ is absolutely convergent and rapidly decreasing as a function of $h \in H(\mathbb{Q})\backslash H(\mathbb{A})$.

Returning to the case at hand, note that the Witt rank of $\mathcal{V}$ is $r = 1$, and we have an orthogonal basis

$$e = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right), \ f_1 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \ f_2 = \left(\begin{smallmatrix} 0 & 1/N \\ 0 & 0 \end{smallmatrix}\right) \tag{178}$$

with $\mathcal{V}_{an} = \mathbb{Q}e$ and $\mathcal{V}' = \mathbb{Q}f_1$, and $\mathcal{V}'' = \mathbb{Q}f_2$. Applying the criterion above, it will suffice to show that

$$\widehat{\phi}(\lambda_{an}, 0, 0) = \int_{(x_1,x_2)\in\mathbb{R}^2} \phi(a_1 e + x_1 f_1, a_2 e + x_2 f_1)\, dx_1\, dx_2 \tag{179}$$

vanishes for all $\lambda_{an} = (a_1 e, a_2 e) \in \mathcal{V}_{an}(\mathbb{R})^2$, for the Schwartz function ϕ defined in (169).

We compute

$$R\left(ae + xf_1, i\right) = R\left(\begin{pmatrix} a & \\ x & -a \end{pmatrix}, \ i\right) = \frac{N(x^2 + 4a^2)}{2} \tag{180}$$

and $Q(ae + xf_1) = -Na^2$. Unwinding definitions, we find that

$$\widehat{\phi}(\lambda_{an}, 0, 0) = e^{-2\pi N(a_1^2 + a_2^2)} \int_{\mathbb{R}^2} \left(-\frac{1}{\pi} + N(x_1^2 + x_2^2)\right) e^{-\pi N(x_1^2 + x_2^2)}\, dx_1\, dx_2, \tag{181}$$

and a straightforward calculus exercise shows that the integral vanishes, as required. □

Corollary 4.8 ([KR94])**.** *Let $\Phi_\phi \in I_2(s, \chi_{\mathcal{V}})$ denote the standard section attached to $\phi_{\mathbb{A}}$, and let $E(g, s, \Phi_\phi)$ denote the corresponding Siegel Eisenstein series. Then*

$$2\int_{[H]} \Theta(g, h, \phi_{\mathbb{A}}) dh = E(g, 0, \Phi_\phi) \tag{182}$$

where dh is the Haar measure on $[H] = H(\mathbb{Q})\backslash H(\mathbb{A})$ giving total volume 1. □

Remark 4.9. Strictly speaking, [KR94] asserts that the two sides are proportional; the constant of proportionality is shown to be 2 in [GQT14, Theorem 7.3(ii)], see also the references therein.

Theorem 4.10. *Suppose T is of signature $(1, 1)$ or $(0, 2)$. Then*

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(T, v) = \frac{\prod_{q|N}(q + 1)}{24} C'_T(v, 0, \Phi^{\mathcal{L}}).$$

Proof. By definition, the degree is

$$\widehat{\deg}\,\widehat{\mathcal{Z}}(T,v) \;=\; \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} \Xi(T,v). \tag{183}$$

With the Siegel-Weil formula for the Schwartz function $\phi_{\mathbb{A}}$, Corollary 4.8, in hand, the proof of [GS19, Theorem 5.10] goes through verbatim; in our setting, this formula reads:

$$\int_{\mathcal{X}(\mathbb{C})} \Xi(T,v) = \mathrm{vol}(\mathcal{X}(\mathbb{C}), d\Omega)\; C'_T(v,0,\Phi^{\mathcal{L}}) \tag{184}$$

where $d\Omega = \frac{1}{2\pi} du dv/v^2$. The volume is given explicitly by the well-known formula

$$\mathrm{vol}(\mathcal{X}(\mathbb{C}), d\Omega) = \frac{1}{2}\int_{\Gamma_0(N)\backslash\mathbb{H}} \frac{du\,dv}{2\pi v^2} = \frac{1}{2}\cdot\frac{\prod_{q|N}(q+1)}{6}, \tag{185}$$

where the factor $\frac{1}{2}$ emerges from the action of the group $\{\pm 1\}$, cf. Remark 2.7. The theorem follows immediately. □

4.3. T **of rank one.** Here, we follow [KRY06, §5.8]; see also [GS19, §5.2] for a more general discussion. We will make use of Eisenstein series in genus one and two, see Section 2.6 for the notation.

We begin with the observation that there is an embedding

$$\eta\colon G_{1,\mathbb{A}} \to G_{2,\mathbb{A}}, \qquad \left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}, z\right] \mapsto \left[\begin{pmatrix} 1 & & 0 & \\ & a & & b \\ 0 & & 1 & \\ & c & & d\end{pmatrix}, z\right] \tag{186}$$

which induces a map

$$\eta^*\colon I_2(s,\chi_{\mathcal{V}}) \to I_1(s+\tfrac{1}{2},\chi_{\mathcal{V}}) \tag{187}$$

via pullback. By the same formula, we have an embedding $\eta_v\colon G_{1,v} \to G_{2,v}$ at each place v, inducing a pullback $\eta_v^*\colon I_{2,v}(s,\chi_{\mathcal{V}}) \to I_{1,v}(s+\frac{1}{2},\chi_{\mathcal{V}})$.

Lemma 4.11 ([GS19, Lemma 5.4])**.** *Suppose*

$$T = \begin{pmatrix} 0 & \\ & t\end{pmatrix}. \tag{188}$$

with $t \neq 0$*. Then for any* $\Phi \in I_2(\chi, s)$ *and* $g \in G_{2,\mathbb{A}}$*, we have*

$$E_T(g,s,\Phi) = W_t(e, s+\tfrac{1}{2}, (\eta^*\circ r(g))\,\Phi) + W_T(g,s,\Phi) \tag{189}$$

□

Let's compute these Whittaker functionals more explicitly, in the case that

$$\Phi = \Phi_2^{\mathcal{L}} \in I_2(\chi,s) \tag{190}$$

is the incoherent section from Definition 2.13 above. We begin with a simple lemma:

Lemma 4.12. *Suppose* V_v *is a local quadratic space of dimension* m*, and* $\varphi_1, \varphi_1' \in S(V_v)$*. Let* $\varphi_2 = \varphi_1 \otimes \varphi_1' \in S(V_v^{\oplus 2})$*, and let* $\Phi_2(s) \in I_{2,v}(s,\chi_V)$ *and* $\Phi_1'(s) \in I_{1,v}(s,\chi_V)$ *denote the standard sections corresponding to* φ_2 *and* φ_1' *respectively. Then*

$$\eta^*\Phi_2(s) = \varphi_1(0)\cdot\Phi_1'(s+\tfrac{1}{2}) \tag{191}$$

Proof. Let ω_i denote the Weil representation on $S(V^{\oplus i})$ for $i = 1, 2$. A direct calculation, using the explicit formulas in e.g. [KRY06, Lemma 8.6.5], yields the identity

$$[\omega_2(\eta(g))\varphi_2](\mathbf{v}) = \varphi_1(v) \cdot [\omega_1(g)\varphi_1'](v'), \qquad \mathbf{v} = (v_1, v_1') \in V^{\oplus 2} \tag{192}$$

for every $g \in G_{1,v}$.

On the other hand, if $g_2 = n(B)m(A)k \in G_{2,v}$, then by definition,

$$\Phi_2(g_2, s) = |\det(A)|^{s - \frac{m-3}{2}} \cdot [\omega_2(g_2)\varphi_2](0) \tag{193}$$

and for $g_1 = n(b)m(a)k$, we have

$$\Phi_1'(g_1, s) = |\det(a)|^{s - \frac{m-2}{2}} \cdot [\omega_1(g_1)\varphi_1'](0) \tag{194}$$

The lemma follows easily. □

Turning to the archimedean place, suppose

$$v = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{R})_{>0}, \tag{195}$$

and let $g_{v,\infty} \in G_{2,\infty}$ denote the corresponding group element as in (93). Let $\Phi^{3/2}_{2,\infty} \in I_{2,\infty}(s, \chi_{\mathcal{V}})$ and $\Phi^{3/2}_{1,\infty} \in I_{1,\infty}(s, \chi_{\mathcal{V}})$ denote the weight 3/2 sections in genus 2 and 1, respectively. Then a straightforward computation, cf. [GS19, p. 956], yields

$$(\eta^* \circ r(g_{v,\infty}))\, \Phi^{3/2}_{2,\infty}(s) = v_1^{s/2+3/4} \cdot \left(r(g_{v_2,\infty}) \Phi^{3/2}_{1,\infty}(s + 1/2) \right). \tag{196}$$

Now consider the global standard sections $\Phi_1^{\mathcal{L}} \in I_1(s, \chi_{\mathcal{V}})$ and $\Phi_2^{\mathcal{L}} \in I_2(s, \chi_{\mathcal{V}})$ as in Definition 2.13.

Then the above discussion implies that for $v = diag(v_1, v_2)$ as above, and setting

$$g_v = (g_{v,\infty}, e, \dots) \in G_{2,\mathbb{A}}, \qquad g_{v_2} = (g_{v_2,\infty}, e, \dots) \in G_{1,\mathbb{A}} \tag{197}$$

we have

$$\eta^* \Phi_2^{\mathcal{L}}(g_v, s) = v_1^{s/2+3/4}\, \Phi_1^{\mathcal{L}}(g'_{v_2}, s + \tfrac{1}{2}) \tag{198}$$

and hence

$$W_t\left(e, s + \tfrac{1}{2}, [\eta^* \circ r(g_v)]\, \Phi_2^{\mathcal{L}}\right) = v_1^{s/2+3/4}\, W_t\left(g_{v_2}, s + \tfrac{1}{2}, \Phi_1^{\mathcal{L}}\right). \tag{199}$$

We now turn to the second term in (189); to this end, recall the identities

$$W_{T,p}(e, s, \Phi_2^{\mathcal{L}}) = \gamma_p(\mathcal{V})^2 \cdot [\mathcal{L}_p^\vee : \mathcal{L}_p]^{-1} \cdot |2|_p^{\frac{1}{2}} \cdot \alpha_p(X, T, \mathcal{L})|_{X=p^{-s}} \tag{200}$$

and

$$W_{t,p}(e, s + \tfrac{1}{2}, \Phi_1^{\mathcal{L}}) = \chi_{\mathcal{V},p}(-1) \cdot \gamma_p(\mathcal{V}) \cdot [\mathcal{L}_p^\vee : \mathcal{L}_p]^{-\frac{1}{2}} \cdot \alpha_p(X, t, \mathcal{L})|_{X=p^{-s}}, \tag{201}$$

cf. [KRY06, Lemma 5.7.1]. Here

$$\mathcal{L}_p^\vee = \{x \in \mathcal{V}_p \mid \langle x, y \rangle \in \mathbb{Z}_p \text{ for all } y \in \mathcal{L}_p\} \tag{202}$$

is the dual lattice with respect to the bilinear form $\langle x, y \rangle$ with $\langle x, x \rangle = 2Q(x) = 2N\det(x)$, so, recalling that N is odd and squarefree, we have

$$[\mathcal{L}_p^\vee : \mathcal{L}_p] = \begin{cases} p, & \text{if } p | 2N \\ 1, & \text{otherwise.} \end{cases} \tag{203}$$

For our fixed integer $t \neq 0$, write

$$4Nt = c^2 d \tag{204}$$

where $-d$ is a fundamental discriminant. Let χ_{-d} denote the quadratic Dirichlet character attached to the field $k_d := \mathbb{Q}(\sqrt{-d})$.

Lemma 4.13. *Suppose* $T = \begin{pmatrix} 0 & \\ & t \end{pmatrix}$.

(i) If $p \nmid N$ *, then*

$$\frac{W_{T,p}(e,s,\Phi^{\mathcal{L}}_{2,p})}{W_{t,p}(e,-s+\frac{1}{2},\Phi^{\mathcal{L}}_{1,p})} = \gamma_p(\mathcal{V})\,\chi_{\mathcal{V},p}(-1)\,|2|_p \frac{\zeta_p(-2s+2)}{\zeta_p(2s+2)} \cdot \frac{L_p(s,\chi_{-d})}{L_p(1-s,\chi_{-d})} \cdot |c|_p^{2s-1}. \tag{205}$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ *and* $L_p(s,\chi_{-d}) = (1-\chi_{-d}(p)p^{-s})^{-1}$ *as usual.*

(ii) If $p|N$*, then*

$$\frac{W_{T,p}(e,s,\Phi^{\mathcal{L}}_{2,p})}{W_{t,p}(e,-s+\frac{1}{2},\Phi^{\mathcal{L}}_{1,p})} = -\gamma_p(\mathcal{V})\chi_{\mathcal{V},p}(-1)\frac{\zeta_p(s-1)}{\zeta_p(s+1)}\,\frac{L_p(s,\chi_{-d})}{L_p(1-s,\chi_{-d})}|c|_p^{2s-1}|N|_p^{s-\frac{1}{2}} \tag{206}$$

Proof. Suppose $p \nmid N$, and let $(0,a)$ denote the Gross-Keating invariants of the matrix $\operatorname{diag}(-N,t) \in \operatorname{Sym}_2(\mathbb{Q}_p)$, cf. [Yan04, Appendix B]; note that if $p \neq 2$, then $a = \operatorname{ord}_p(t)$. Using [Yan04, Proposition B4], one can check that

$$\operatorname{ord}_p(c) = \begin{cases} \frac{a-1}{2}, & a \text{ odd}, \\ \frac{a}{2}, & a \text{ even} \end{cases} \tag{207}$$

where c is as in (204). In light of (114), it suffices to prove the identity

$$\left.\frac{\alpha_p(X,T,\mathcal{L})}{\alpha_p(X^{-1},t,\mathcal{L})}\right|_{X=p^{-s}} \overset{?}{=} \frac{\zeta_p(-2s+2)}{\zeta_p(2s+2)} \cdot \frac{L_p(s,\chi_{-d})}{L_p(1-s,\chi_{-d})} \cdot |c|_p^{2s-1}, \tag{208}$$

which amounts to an explicit computation using formulas due to Yang. Indeed, by [Yan04, Proposition C2], we have

$$\frac{\alpha_p(X,t,\mathcal{L})}{(1-p^{-2}X^2)} = \begin{cases} \sum_{k=0}^{\frac{a-1}{2}} p^{-k}X^{2k}, & a \text{ odd}, \\ \sum_{k=0}^{\frac{a}{2}} p^{-k}X^{2k} + (\chi_{-d}(p)-p^{-1}X)^{-1}p^{-\frac{a}{2}-1}X^{a+1}, & a \text{ even}. \end{cases} \tag{209}$$

On the other hand, taking the limit $a_3 \to \infty$ in [Yan04, Theorem 5.7], we have

$$\frac{\alpha_p(X,T,\mathcal{L})}{1-p^{-2}X^{-2}} = \begin{cases} \sum_{k=0}^{\frac{a-1}{2}} p^{k}X^{2k}, & a \text{ odd}, \\ \sum_{k=0}^{\frac{a}{2}-1} p^{k}X^{2k} + p^{\frac{a}{2}}X^a(1-\chi_{-d}(p)X)^{-1}, & a \text{ even}. \end{cases} \tag{210}$$

The desired relation follows easily from an explicit computation.

For (ii), suppose that $p|N$, so that in particular $p \neq 2$.

Write $a = \operatorname{ord}_p(t)$. Then a direct computation using [Yan98, Proposition 3.1] gives

$$\alpha_p(X,t,\mathcal{L}) = 1 + (1-p^{-1})X\left(\frac{p^{-\frac{a+1}{2}}X^{a+1}-1}{p^{-1}X^2-1}\right) + \chi_{-d}(p)p^{-\frac{a+1}{2}}X^{a+1} \tag{211}$$

when a is odd, and

$$\alpha_p(X,t,\mathcal{L}) = 1 + (1-p^{-1})X\left(\frac{p^{-\frac{a}{2}}X^{a}-1}{p^{-1}X^2-1}\right) - p^{-\frac{a}{2}-1}X^{a+1} \tag{212}$$

when a is even.

Consider the case that a is odd, which in turn implies that $\mathrm{ord}_p(c) = \frac{a+1}{2}$. Then taking $b \to \infty$ in the formulas in Proposition 3.6 and applying Lemma 3.5 gives

$$\frac{X^2}{X-p}\alpha(X,T,\mathcal{L}) = \left(\frac{p^{\frac{a+1}{2}}X^{a+1}-1}{pX^2-1}\right)\left((p-p^{-1})X^2 - p^{-1}X - 1\right) + \frac{p^{\frac{a+1}{2}}X^{a+1}}{1-\chi_{-d}(p)X}\left(-p^{-1}X^2 + (\chi_{-d}(p)-p^{-1})X - 1\right) + \frac{X+p}{p}; \quad (213)$$

note that in this case, $\chi_{-d}(p) = (-Nt,p)_p = v_0^+$ in the notation of Proposition 3.6.

From this, a straightforward computation shows that

$$\frac{\alpha(X,T,\mathcal{L})}{\alpha_p(X^{-1},t,\mathcal{L})} = -\left(\frac{1-p^{-1}X}{1-pX}\right)\cdot\left(\frac{1-\chi_{-d}(p)p^{-1}X^{-1}}{1-\chi_{-d}(p)X}\right)p^{\frac{a+3}{2}}X^{a+2}, \quad (214)$$

and the lemma follows. A similar computation establishes the case of even a. □

Finally, we have the archimedean counterpart to the previous lemma:

Lemma 4.14. *Suppose $T = \begin{pmatrix} 0 & \\ & t\end{pmatrix}$ and $v = \begin{pmatrix} v_1 & \\ & v_2\end{pmatrix}$. Let*

$$L_\infty(s,\chi_{-d}) := |d|^{\frac{s}{2}}\cdot\begin{cases}\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}), & \text{if } d>0,\\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), & \text{if } d<0.\end{cases} \quad (215)$$

Then

$$\frac{W_{T,\infty}\left(g_v,s,\Phi^{\frac{3}{2}}_{2,\infty}\right)}{W_{t,\infty}\left(g_{v_2},-s+\frac{1}{2},\Phi^{\frac{3}{2}}_{1,\infty}\right)} = v_1^{-s/2+3/4k}(-2i)\left(\frac{1-s}{1+s}\right)\cdot\frac{L_\infty(s,\chi_{-d})}{L_\infty(1-s,\chi_{-d})}\cdot\frac{\zeta_\infty(-2s+2)}{\zeta_\infty(2s+2)} \times c^{2s-1}N^{-s+\frac{1}{2}}. \quad (216)$$

Recall here we are writing $4Nt = c^2 d$, where d is a fundamental discriminant.

Proof. By [KRY06, Proposition 5.7.7], we have[2]

$$\frac{W_{T,\infty}\left(g_v,s,\Phi^{\frac{3}{2}}_{2,\infty}\right)}{W_{t,\infty}\left(g_{v_2},s-\frac{1}{2},\Phi^{\frac{3}{2}}_{1,\infty}\right)} = v_1^{-s/2+3/4}(-i)\frac{2}{s+1}; \quad (217)$$

On the other hand, we may apply [KRY04, Proposition 14.1] to see

$$\frac{W_{t,\infty}\left(g_{v_2},s-\frac{1}{2},\Phi^{\frac{3}{2}}_{1,\infty}\right)}{W_{t,\infty}\left(g_{v_2},-s+\frac{1}{2},\Phi^{\frac{3}{2}}_{1,\infty}\right)} = (\pi|t|)^{s-\frac{1}{2}}\cdot\begin{cases}\Gamma(\frac{-s+3}{2})\Gamma(\frac{s}{2}+1)^{-1}, & t>0\\ \Gamma(-\frac{s}{2})\Gamma(\frac{s-1}{2})^{-1}, & t<0.\end{cases} \quad (218)$$

The lemma follows from a little algebra. □

[2]Note that there is an error in the statement of [KRY06, Proposition 5.7.7]: the factor $\sqrt{2}$ on the right hand side of the equation in that proposition should be 2.

Corollary 4.15. *Suppose $T = \begin{pmatrix} 0 & \\ & t \end{pmatrix}$ and $v = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}$. Then*

$$W_T(g_v, s, \Phi_2^{\mathcal{L}}) = v_1^{-\frac{s}{2}+\frac{3}{4}} N^{-2s+1} \left(\frac{s-1}{s+1}\right) \frac{\Lambda(-2s+2)}{\Lambda(2s+2)} \left(\prod_{p|N} \beta_p(s)\right) \times W_t(g_{v_2}, -s+\tfrac{1}{2}, \Phi_1^{\mathcal{L}}) \tag{219}$$

where $\Lambda(s) = \prod_{v\leq\infty} \zeta_v(s)$ and

$$\beta_p(s) = -\frac{\zeta_p(s-1)\zeta_p(2s+2)}{\zeta_p(s+1)\zeta_p(-2s+2)}. \tag{220}$$

Proof. Combining Lemma 4.13 and Lemma 4.14, we have

$$\frac{W_T(g_v, s, \Phi_2^{\mathcal{L}})}{W_t(g_{v_2}, -s+\frac{1}{2}, \Phi_1^{\mathcal{L}})} = v_1^{\frac{s}{2}+\frac{3}{4}} (-2i) N^{-s+\frac{1}{2}} \frac{\Lambda(-2s+2)}{\Lambda(2s+2)} \times \left(\prod_{p<\infty} \gamma_p(\mathcal{V}) \chi_{\mathcal{V},p}(-1) |2|_p |N|_p^{s-\frac{1}{2}}\right) \cdot \prod_{p|N} \beta_p(s); \tag{221}$$

here we used the functional equation $\Lambda(s, \chi_{-d}) = \Lambda(1-s, \chi_{-d})$ for the completed L-function $\Lambda(s, \chi_{-d}) = \prod_{v\leq\infty} L_v(s, \chi_{-d})$.

On the other hand, the product formula gives

$$\prod_{p<\infty} \gamma_p(\mathcal{V}) \chi_{\mathcal{V},p}(-1) = \gamma_\infty(\mathcal{V})^{-1} \cdot (-1, -N)_{\mathbb{R}} = -\gamma_\infty(\mathcal{V})^{-1}. \tag{222}$$

By [KRY06, p. 330], we have $\gamma_\infty(\mathcal{V}) = -i$, and the corollary follows easily. □

Corollary 4.16. *Suppose $T = \begin{pmatrix} 0 & \\ & t \end{pmatrix}$ and $v = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}$. Then*

$$v_1^{-3/4} E'_T(g_v, 0, \Phi_2^{\mathcal{L}}) = 2\, W'_t(g_{v_2}, \tfrac{1}{2}, \Phi_1^{\mathcal{L}}) + \left(\log v_1 + 2 + \frac{4\Lambda'(2)}{\Lambda(2)} + \sum_{p|N} \frac{p-1}{p+1} \log p\right) W_t(g_{v_2}, \tfrac{1}{2}, \Phi_1^{\mathcal{L}}). \tag{223}$$

Proof. This follows immediately from the previous corollary, and equations (189) and (199). □

Our next step is to compare this expression with the arithmetic heights of special divisors, as computed in [DY19]. Recall that we have written the Eisenstein in "classical" coordinates as

$$E(\tau, s, \Phi_2^{\mathcal{L}}) = \det(v)^{-\frac{3}{4}} E(g_\tau, s, \Phi_2^{\mathcal{L}}) = \sum_T C_T(v, s, \Phi_2^{\mathcal{L}})\, q^T. \tag{224}$$

Proposition 4.17. *Suppose $T = \begin{pmatrix} 0 & \\ & t \end{pmatrix}$ and $v = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}$. Then*

$$4\langle \widehat{\mathcal{Z}}(t, v_2), \widehat{\omega}_N \rangle + \log v_1 \deg Z(t) = -\frac{\prod_{p|N}(p+1)}{12} C'_T(v, 0, \Phi_2^{\mathcal{L}}), \tag{225}$$

where $\langle \widehat{\mathcal{Z}}(t, v_2), \widehat{\omega}_N \rangle = \widehat{\deg}(\mathcal{Z}(t, v_2) \cdot \widehat{\omega}_N)$ is the intersection pairing, as in Section 2.3

Proof. This follows from combining Corollary 4.16 with [DY19, Theorem 1.3]. Let

$$\mathcal{E}_{DY}(\tau, s) = A_1(s)E(\tau, s-\tfrac{1}{2}, \Phi_1^{\mathcal{L}}), \qquad A_1(s) := -\frac{s}{4\pi}\Lambda(2s)\left(\prod_{p|N}(1-p^{-2s})\right)N^{\frac{1}{2}+\frac{3}{2}s} \tag{226}$$

denote the normalized genus 1 Eisenstein series in [DY19] (note that the variable s is shifted by 1/2 here, as compared to loc. cit.). Then for $\tau_2 = u_2 + iv_2 \in \mathbb{H}$, Theorem 1.3 of loc. cit. states

$$\varphi(N)\cdot\langle\widehat{\mathcal{Z}}(t,v_2),\widehat{\omega}_N\rangle q_2^t = \mathcal{E}'_{DY,t}(\tau_2,1) - \sum_{p|N}\frac{p}{p-1}\log p\cdot\mathcal{E}_{DY,t}(\tau_2,1) \tag{227}$$

$$= A_1(1)\left(E'_t(\tau_2,\tfrac{1}{2},\Phi_1^{\mathcal{L}}) + \left\{\frac{A'_1(1)}{A_1(1)} - \sum_{p|N}\frac{p}{p-1}\log p\right\}E_t(\tau_2,\tfrac{1}{2},\Phi_1^{\mathcal{L}})\right). \tag{228}$$

Now

$$A_1(1) = -\frac{1}{4\pi}\Lambda(2)N^2\prod_{p|N}(1-p^{-2}) = -\frac{1}{24}\prod_{p|N}(p^2-1) \tag{229}$$

and

$$\frac{A'_1(1)}{A_1(1)} - \sum_{p|N}\frac{p}{p-1}\log p = 1 + \frac{2\Lambda'(2)}{\Lambda(2)} + \frac{3}{2}\log N + \sum_{p|N}\left(\frac{2p^{-2}}{1-p^{-2}} - \frac{p}{p-1}\right)\log p \tag{230}$$

$$= 1 + \frac{2\Lambda'(2)}{\Lambda(2)} + \sum_{p|N}\frac{p-1}{2(p+1)}\log p. \tag{231}$$

Thus for $\tau = \begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}$, we have $q_{\tau_2}^t = q^T$ and, comparing with Corollary 4.16, we find

$$2(v_2)^{\frac{3}{4}}\cdot\frac{\varphi(N)}{A_1(1)}\cdot\langle\widehat{\mathcal{Z}}(t,v_2),\widehat{\omega}_N\rangle\, q^T \tag{232}$$

$$= 2W'_t(g_{\tau_2},\tfrac{1}{2},\Phi_1^{\mathcal{L}}) + 2\left\{1 + \frac{2\Lambda'(2)}{\Lambda(2)} + \sum_{p|N}\frac{p-1}{2(p+1)}\log p\right\}W_t(g_{\tau_2},\tfrac{1}{2},\Phi_1^{\mathcal{L}}) \tag{233}$$

$$= v_1^{-\frac{3}{4}}E'_T(g_\tau,0,\Phi_2^{\mathcal{L}}) - \log v\cdot W_t(g_{\tau_2},\tfrac{1}{2},\Phi_1^{\mathcal{L}}). \tag{234}$$

Finally, [DY19, Theorem 1.3] asserts that

$$\frac{1}{2}\deg(Z(t))q_{\tau_2}^t = \frac{1}{\varphi(N)}\mathcal{E}_{DY,t}(\tau_2,1) = \frac{A_1(1)}{\varphi(N)}\cdot v_2^{-\frac{3}{4}}\cdot W_t(g_{v_2},\tfrac{1}{2},\Phi_1^{\mathcal{L}}). \tag{235}$$

□

We arrive at the main theorem for T of rank 1.

Theorem 4.18. *Suppose* $\operatorname{rank}(T) = 1$. *Then*

$$\widehat{\deg}\mathcal{Z}(T,v) = \frac{\prod_{p|N}(p+1)}{24}C'_T(v,0,\Phi^{\mathcal{L}}) \tag{236}$$

Proof. It follows from definitions that for any $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, we have

$$C_{\gamma T {}^t\gamma}(v, s, \Phi^{\mathcal{L}}) = C_T({}^t\gamma v \gamma, s, \Phi^{\mathcal{L}}). \tag{237}$$

As $\widehat{\mathcal{Z}}(T, v)$ satisfies the same invariance, cf. (80), we may use Lemma 2.10 to assume that $T = \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)$. Furthermore, given any $v \in \mathrm{Sym}_2(\mathbb{R})$, one can find a matrix $\theta = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ such that $\theta v {}^t\theta$ is diagonal, and we have

$$C_{\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)}(\theta v {}^t\theta, s, \Phi^{\mathcal{L}}) = C_{\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)}(v, s, \Phi^{\mathcal{L}}) \qquad \text{and} \qquad \widehat{\mathcal{Z}}(\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right), \theta v {}^t\theta) = \widehat{\mathcal{Z}}(\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right), v). \tag{238}$$

Thus we may assume $T = \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} v_1 & \\ & v_2 \end{smallmatrix}\right)$. In this case, we have, by definition,

$$\widehat{\mathcal{Z}}(T, v) = -2\widehat{\mathcal{Z}}(t, v_2) \cdot \widehat{\omega}_N + \left(0, \log v_1 \delta_{Z(t)}\right), \tag{239}$$

cf. Definition 2.11. The theorem follows from the previous proposition. ☐

4.4. $T = 0$**.** Our first task is computing the constant term $E_0(g, s, \Phi_2^{\mathcal{L}})$. To this end, for $g \in G_{2,\mathbb{A}}$, we abbreviate

$$B(g, s) := W_0\left(e, s + \tfrac{1}{2}, (\eta^* \circ r(g))\Phi_2^{\mathcal{L}}\right), \qquad B_v(g, s) := W_{0,v}\left(e, s + \tfrac{1}{2}, (\eta^* \circ r(g))\Phi_{2,v}^{\mathcal{L}}\right) \tag{240}$$

where W_0 is the Whittaker functional in genus 1, and η^* is the map defined in (187). Arguing as in [KRY06, §5.9], for $g \in G_{2,\mathbb{A}}$, we have

$$E_0(g, s, \Phi_2^{\mathcal{L}}) = W_0(g, s, \Phi_2^{\mathcal{L}}) + \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} B(m(\gamma)g, s) + \Phi_2^{\mathcal{L}}(g, s). \tag{241}$$

For $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, take $g = g_z \in G_{2,\mathbb{A}}$ as in (93).

For the first term, [KRY06, (5.9.3)] gives[3]

$$W_{0,\infty}(g_v, s, \Phi_{2,\infty}^{\mathcal{L}}) = -2^{3/2} \frac{(s-1)\zeta_\infty(2s-1)}{(s+1)\zeta_\infty(2s+2)} \det(v)^{-s/2+3/4} \tag{242}$$

where $\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$, and

$$W_{0,p}(e, s, \Phi_{2,p}^{\mathcal{L}}) = |2|_p^{3/2} \frac{\zeta_p(2s-1)}{\zeta_p(2s+2)} \gamma(\mathcal{V}_p)^2 \tag{243}$$

for $p \nmid N$.

When $p|N$, we may take $a, b \to \infty$ in Proposition 3.6, apply Lemma 3.5 and simplify to find

$$W_{0,p}(e, s, \Phi_{2,p}^{\mathcal{L}}) = \frac{\zeta_p(2s-1)}{\zeta_p(2s+2)} \cdot \frac{p^{-1}(1 + p^{-s+1})}{1 + p^{-s-1}} \gamma(\mathcal{V}_p)^2. \tag{244}$$

Combining these identities with the fact that $\gamma_\infty(\mathcal{V})^2 = -1$, and applying the product formula $\prod_{v \le \infty} \gamma_v(\mathcal{V}) = 1$, we have

$$W_0(g_v, s, \Phi_2^{\mathcal{L}}) = \det(v)^{-s/2+3/4} \frac{(s-1)\Lambda(2s-1)}{(s+1)\Lambda(2s+2)} \cdot A(s), \tag{245}$$

where

$$A(s) := \prod_{p|N} p^{-1} \frac{1 + p^{-s+1}}{1 + p^{-s-1}}. \tag{246}$$

[3]Note that the power of 2 appearing on the right hand side of [KRY06, (5.9.3)] is misstated.

Note that $A(0) = 1$ and

$$A'(0) = \sum_{p|N} \frac{1-p}{1+p} \log p, \tag{247}$$

and so

$$W_0'(g_v, 0, \Phi_2^{\mathcal{L}}) = (\det v)^{3/4} \cdot \left(\frac{1}{2} \log \det v + 2 - 4\frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N} \frac{1-p}{1+p} \log p \right). \tag{248}$$

Next, we consider the middle term in (241). For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and prime p, we have

$$r(m(\gamma))\Phi_{2,p}^{\mathcal{L}} = \Phi_{2,p}^{\mathcal{L}}; \tag{249}$$

hence Lemma 4.12 gives

$$\eta^* \left(r(m(\gamma))\Phi_{2,p}^{\mathcal{L}}(s) \right) = \Phi_{1,p}^{\mathcal{L}} \left(s + \tfrac{1}{2} \right), \tag{250}$$

which is in particular independent of γ. Thus we have

$$B_p(m(\gamma), s) = W_{0,p}(e, s + \frac{1}{2}, \Phi_1^{\mathcal{L}}), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{251}$$

Applying (201) and taking $a \to \infty$ in (209) for $p \nmid N$, or (211) and (212) for $p|N$, a short computation gives

$$B_p(m(\gamma), s) = \chi_{\mathcal{V},p}(-1)\gamma_p(\mathcal{V})|2|_p^{1/2}\zeta_p(2s+1) \begin{cases} \zeta_p(2s+2)^{-1}, & p \nmid N \\ p^{-s-3/2}\zeta_p(s)^{-1}, & p|N \end{cases} \tag{252}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

At the infinite place, the proof of [KRY06, Proposition 5.9.2] gives

$$B_\infty(m(\gamma)g_v, s) = \sqrt{2}\,\gamma_\infty(\mathcal{V})\ \det(v)^{s/2+3/4} Q_v(c,d)^{-\frac{1}{2}-s} \frac{s\zeta_\infty(2s+1)}{(s+1)\zeta_\infty(2s+2)} \tag{253}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $Q_v(c,d) = (c,d) \cdot v \cdot {}^t(c,d)$. Consider the series

$$G(s,v) := \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} Q_v(c,d)^{-s} \tag{254}$$

for $Re(s) \gg 0$. Note that

$$\zeta(2s)G(s,v) = Z(s,v) \tag{255}$$

where

$$Z(s,v) := \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} Q_v(m,n)^{-s} \tag{256}$$

is the Epstein zeta function attached to the positive definite quadratic form determined by v. By [Sie80, §1.5], the series $Z(s,v)$ extends meromorphically to $s \in \mathbb{C}$, with only a simple pole at $s = 1$; in particular, $Z(s,v)$ is holomorphic at $s = \frac{1}{2}$. Since $\zeta(2s)$ has a pole at $s = \frac{1}{2}$, it follows that $G(s,v)$ vanishes at $s = \frac{1}{2}$.

The preceding discussion implies that

$$\sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} B(m(\gamma)g_v, s) = \det(v)^{s/2+3/4} \cdot \frac{s\Lambda(2s+1)}{(s+1)\Lambda(2s+2)} \cdot G(s + \tfrac{1}{2}, v) \cdot B(s) \tag{257}$$

where

$$B(s) = \prod_{p|N} p^{-s-3/2} \frac{\zeta_p(2s+2)}{\zeta_p(s)}. \tag{258}$$

Notice that

$$\operatorname*{ord}_{s=0} B(s) = \sigma_0(N) \tag{259}$$

where $\sigma_0(N)$ is the number of prime divisors of N, and hence the sum (257) vanishes at $s = 0$ to order at least $\sigma_0(N) + 1$.

Finally, by definition we have that

$$\Phi_2^{\mathcal{L}}(g_v, s) = \det(v)^{s/2+3/4}. \tag{260}$$

To summarize the discussion, we obtain:

Proposition 4.19. *Let* $C_0(v, s, \Phi_2^{\mathcal{L}}) = \det(v)^{-3/4} E_0(g_v, s, \Phi_2^{\mathcal{L}})$*. Then*

$$C_0'(v, 0, \Phi_2^{\mathcal{L}}) = \log\det v + 2 - 4\frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N} \frac{1-p}{1+p}\log p. \tag{261}$$

□

Corollary 4.20.

$$\widehat{\deg}\, \widehat{\mathcal{Z}}(0, v) \;=\; \left(\frac{\prod_{p|N}(p+1)}{24}\right) \cdot C_0'(v, 0, \Phi_2^{\mathcal{L}}). \tag{262}$$

Proof. Recall that by definition,

$$\widehat{\mathcal{Z}}(0, v) = \widehat{\omega} \cdot \widehat{\omega} + (0, \log\det v \cdot [\Omega])\,, \tag{263}$$

where

$$\widehat{\omega} = -2\widehat{\omega}_N - \sum_{p|N} \widehat{\mathcal{X}}_p^0 + (0, \log N), \tag{264}$$

cf. (54), and we set $\Omega = \frac{dx\wedge dy}{2\pi y^2}$.

The arithmetic degree $\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = \widehat{\deg}\, \widehat{\omega}_N \cdot \widehat{\omega}_N$ was computed independently by Kühn [Küh01] and Bost; adjusting for the normalization of the metric (52), which differs by a multiplicative constant from the normalization of [Küh01], and a factor of 2 because our degree is 'stacky', the result is

$$\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = \frac{\prod_{p|N}(p+1)}{24}\left(\frac{1}{2} - \frac{\Lambda'(-1)}{\Lambda(-1)}\right). \tag{265}$$

Next, let W_N denote the Atkin-Lehner involution on $\mathcal{X}_0(N)$, and note that $W_N^*(\widehat{\omega}_N) = \widehat{\omega}_N$ and $W_N^*(\widehat{\mathcal{X}}_p^0) = \widehat{\mathcal{X}}_p^\infty$, so that

$$\langle \widehat{\omega}_N \cdot \widehat{\mathcal{X}}_p^0 \rangle = \langle W_N^*\widehat{\omega}_N, W_N^*\widehat{\mathcal{X}}_p^0 \rangle = \langle \widehat{\omega}_N \cdot \widehat{\mathcal{X}}_p^\infty \rangle = \frac{1}{2}\langle \widehat{\omega}_N, \widehat{\mathcal{X}}_p^0 + \widehat{\mathcal{X}}_p^\infty \rangle. \tag{266}$$

On the other hand, we have

$$\widehat{\mathcal{X}}_p^0 + \widehat{\mathcal{X}}_p^\infty = (\mathcal{X}_0(N)_{\mathbb{F}_p}, 0) = \widehat{\operatorname{div}}(p) + (0, 2\log p) = (0, 2\log p) \in \widehat{\operatorname{CH}}^1(\mathcal{X}); \tag{267}$$

here we view p as a rational function on $\mathcal{X}_0(N)$, and so its arithmetic divisor $\widehat{\operatorname{div}}(p) = (\mathcal{X}_{/\mathbb{F}_p}, -\log p^2)$ vanishes in $\widehat{\operatorname{CH}}^1(\mathcal{X})$.

Thus

$$\langle \widehat{\omega}_N, \sum_{p|N} \widehat{\mathcal{X}}_p^0 - (0, \log n) \rangle = \sum_{p|N} \langle \widehat{\omega}_N, \mathcal{X}_p^0 - (0, \log p) \rangle = 0. \tag{268}$$

Finally, we note that

$$\langle \widehat{\mathcal{X}}_p^0, (0, \log N)\rangle = \langle (0, \log N), (0, \log N)\rangle = 0 \tag{269}$$

and $\langle \widehat{\mathcal{X}}_p^0, \mathcal{X}_q^0\rangle = 0$ if $p \neq q$. Moreover, by [DY19, Lemma 7.2], we have

$$\langle \widehat{\mathcal{X}}_p^0, \widehat{\mathcal{X}}_p^0\rangle = -\frac{\prod_{q|N}(q+1)}{24}\frac{p-1}{p+1}\log p; \tag{270}$$

again, this formula differs from *loc. cit.* by a factor of 2, because we are using the 'stacky' degree.

Putting everything together, we find that

$$\langle \widehat{\omega}, \widehat{\omega}\rangle = 4\langle \widehat{\omega}_N, \widehat{\omega}_N\rangle + \sum_{p|N}\langle \widehat{\mathcal{X}}_p^0, \widehat{\mathcal{X}}_p^0\rangle = \frac{\prod_{q|N}(1+q)}{24}\left(2 - 4\frac{\Lambda'(-1)}{\Lambda(-1)} - \sum_{p|N}\frac{1-p}{1+p}\log p\right). \tag{271}$$

Finally, we observe that

$$\widehat{\deg}(0, \log\det v \cdot [\Omega]) = \frac{\log\det v}{2}\int_{[\Gamma_0(N)\backslash\mathbb{H}]}\frac{dx\wedge dy}{2\pi y^2} = \log\det v\cdot\frac{\prod_{q|N}(q+1)}{24}, \tag{272}$$

and the theorem follows. □

5. Comparison of two Eisenstein series

In addition to the lattice $\mathcal{L}$, there is another lattice which naturally parameterizes the modular curve $X_0(N)$ over $\mathbb{C}$. It is

$$L = \{A = \begin{pmatrix} a & b \\ Nc & -a\end{pmatrix} : a, b, c \in \mathbb{Z}\}, \quad Q(A) = \det A = -a^2 - Nbc.$$

We set $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Our aim in this section is to compare the incoherent Eisenstein series attached to L with the Eisenstein series $E(\tau, s, \Phi_2^{\mathcal{L}})$ appearing in our main theorem.

For $r = 1, 2$, let $\Phi_r^L(s) = \otimes_{v\leq\infty}\Phi_{r,v}^L(s)$ be the incoherent standard section in $I_r(s, \chi')$ with $\Phi_{r,\infty}^L = \Phi_{r,\infty}^{3/2}$ and, for $v < \infty$, the component $\Phi_{r,v}^L$ is the standard section attached to L_v via the Rallis map, cf. Definition 2.13; here the character χ' is given by $\chi'(x) = (-1, x)_{\mathbb{A}}$.

We will also require some auxilliary sections. For each prime ℓ dividing N let $L^{(\ell)}$ denote the quadratic lattice defined in (125); recall that by definition, $L^{(\ell)}$ is the set of traceless elements in an Eichler order of level N/ℓ contained in the definite quaternion algebra ramified exactly at ℓ and ∞, and the quadratic form is the reduced norm.

For $r = 1, 2$, let $\Phi_r^{(\ell)}(s) \in I_r(s, \chi')$ denote the section with $\Phi_{r,\infty}^{(\ell)}(s) = \Phi_{r,\infty}^{3/2}(s)$, and for $v < \infty$, the local section $\Phi_{r,v}^{(\ell)}(s)$ is the standard section associated to $L_v^{(\ell)}$. Note that this section is coherent, in the sense of [Kud97].

Finally, for $\ell | N$, we set

$$L^{sp,(\ell)} = \{A = \begin{pmatrix} a & b \\ (N/\ell)c & -a\end{pmatrix} : a, b, c \in \mathbb{Z}\}, \quad Q(A) = \det A = -a^2 - (N/\ell)bc. \tag{273}$$

In the same manner as above, we obtain an incoherent section $\Phi_r^{sp,(\ell)}$ with $\Phi_{r,\infty}^{sp,(\ell)} = \Phi_{r,\infty}^{3/2}$ and where at finite places, $\Phi_{r,v}^{sp,(\ell)}$ is the standard section associated to $L_v^{sp,(\ell)}$.

For convenience, we write

$$\mathcal{V} = \mathcal{L}\otimes_{\mathbb{Z}}\mathbb{Q}, \qquad V = L\otimes_{\mathbb{Z}}\mathbb{Q}, \qquad V^{(\ell)} = L^{(\ell)}\otimes_{\mathbb{Z}}\mathbb{Q} \qquad \text{and} \qquad V^{sp,(\ell)} = L^{sp,(\ell)}\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{274}$$

Our first step is a comparison of Whittaker functionals in genus two.

Proposition 5.1. *Suppose $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$.*

(i) If $q \nmid N$, then

$$\frac{W_{T,q}(e,s,\Phi^{\mathcal{L}}_{2,q})}{\gamma_q(\mathcal{V})^2} = \frac{W_{NT,q}(e,s,\Phi^{L}_{2,q})}{\gamma_q(V)^2} = \frac{W_{NT,q}(e,s,\Phi^{(\ell)}_{2,q})}{\gamma_q(V^{(\ell)})^2}$$

with the last identity holding for any $\ell | N$.

(ii) If $q|N$, then

$$\frac{W_{T,q}(e,0,\Phi^{\mathcal{L}}_{2,q})}{\gamma_q(\mathcal{V})^2} = \frac{W_{NT,q}(e,0,\Phi^{L}_{2,q})}{\gamma_q(V)^2} = \frac{W_{NT,q}(e,0,\Phi^{(\ell)}_{2,q})}{\gamma_q(V^{(\ell)})^2}$$

with the last identity holding for $\ell \neq q$.

(iii) If $q|N$, then

$$\begin{aligned} &\frac{W'_{T,q}(e,0,\Phi^{\mathcal{L}}_{2,q})}{\gamma_q(\mathcal{V})^2} - \frac{W'_{NT,q}(e,0,\Phi^{L}_{2,q})}{\gamma_q(V)^2} \\ &= \left(\frac{q-1}{2(q+1)} \cdot \frac{W_{NT,q}(e,0,\Phi^{(q)}_{q})}{\gamma_q(V^{(q)})^2} + \frac{3q-1}{2(q-1)} \frac{W_{NT,q}(e,0,\Phi^{L}_{q})}{\gamma_q(V)^2} - \frac{2}{q^2-1} \frac{W_{NT,q}(e,0,\Phi^{sp,(q)}_{q})}{\gamma_q(V^{sp,(q)})^2} \right) \log q \end{aligned}$$

(iv) Let $\tau \in \mathbb{H}_2$ and $g_{\tau,\infty} \in G_{2,\infty}$ as in (93). Then for any $\Phi \in I_{2,\infty}(s,\chi'_\infty)$, we have

$$W_{T,\infty}(g_{N\tau,\infty},s,\Phi) = N^{-s+\frac{3}{2}} W_{NT,\infty}(g_{\tau,\infty},s,\Phi).$$

(v) Globally, we have

$$\begin{aligned} N^{-\frac{3}{2}} W'_T(g_{N\tau},0,\Phi^{\mathcal{L}}_2) &= W'_{NT}(g_\tau,0,\Phi^{L}_2) + \left(\sum_{\ell | N} \frac{\ell+1}{2(\ell-1)} \log \ell \right) W_{NT}(g_\tau,0,\Phi^{L}_2) \\ &\quad + \sum_{\ell|N} \left(\frac{\ell-1}{2(\ell+1)} W_{NT}(g_\tau,0,\Phi^{(\ell)}_2) - \frac{2}{\ell^2-1} W_{NT}(g_\tau,0,\Phi^{sp,(\ell)}_2) \right) \log \ell \end{aligned}$$

Proof. For any primes ℓ, q with $q \neq \ell$, the lattices L_q and $L^{(\ell)}_q$ are isometric. Hence parts (i) and (ii) of the proposition follow immediately from Lemma 4.2.

To prove *(iii)*, we work at the level of representation densities. First, a direct computation using Lemma 3.5 and (130) gives

$$(X-q)\alpha(X,NT,L_q) + (q^2-q)X^2\alpha(X,T,\mathcal{L}_q) = (X-1)\left[q^2\alpha(X,NT,L^0_q) + (q-1)(X-q)\right] \quad (275)$$

where, changing notation slightly, we let $L^0_q = M_2(\mathbb{Z}_q)^{\mathrm{tr}=0}$ with quadratic form $Q(x) = \det x$. On the other hand, [Yan98, Corollary 8.4] implies that

$$q^2\alpha_q(X,NT,L^0_q) = \left(\frac{1-q}{2}\right)\alpha_q(X,NT,L^{(q)}_q) + \left(\frac{1+q}{2}\right)\alpha_q(X,NT,L_q) + q^2 - 1. \quad (276)$$

Substituting this into (275) and rearranging gives

$$\left(\frac{X+1}{2}\right)\alpha_q(X,NT,L_q) - qX^2\alpha_q(X,T,\mathcal{L}_q) = \left(\frac{X-1}{2}\right)\left[\alpha_q(X,NT,L^{(q)}_q) - 2(X+1)\right]. \quad (277)$$

Differentiating with respect to X and substituting $X=1$, we have

$$\alpha_q'(1,NT,L_q)-q\alpha_q'(1,T,\mathcal{L}_q)=\frac{\alpha_q(1,NT,L_q^{(q)})-\alpha_q(1,NT,L_q)}{2}+2q\alpha_q(1,T,\mathcal{L}_q)-2. \tag{278}$$

To manipulate this expression further, we take $X=1$ in (275) and (276), which yields the relations

$$\alpha_q(1,NT,L_q)=q\alpha_q(1,T,\mathcal{L}_q) \tag{279}$$

and we can rewrite (276) as

$$2=2\frac{q^2}{q^2-1}\alpha_q(1,NT,L_q^0)+\left(\frac{1}{q+1}\right)\alpha_q(1,NT,L_q^{(q)})-\left(\frac{1}{q-1}\right)\alpha_q(1,NT,L_q). \tag{280}$$

Substituting these identities back into (278) gives

$$\begin{aligned}\alpha_q'(1,NT,L_q)-q\alpha_q'(1,T,\mathcal{L}_q)=\frac{q-1}{2(q+1)}\alpha_q(1,NT,L_q^{(q)})+\frac{3q-1}{2(q-1)}\alpha_q(1,NT,L_q)\\ -\frac{2q^2}{q^2-1}\alpha_q(1,NT,L_q^0).\end{aligned} \tag{281}$$

Finally, applying (114), we obtain *(iii)*.

To prove *(iv)*, we first observe that for any $b\in\mathrm{Sym}_2(\mathbb{R})$, we have

$$w\,n(b)\,g_{N\tau}=w\,n(b)\,m(\sqrt{N})g_\tau=w\,m(\sqrt{N})\,n(N^{-1}b)g_\tau=m(\sqrt{N^{-1}})\,w\,n(N^{-1}b)g_\tau, \tag{282}$$

where the notation is as in (84). Thus, for any $\Phi\in I_{2,\infty}(s,\chi_\infty')$, we have

$$\begin{aligned}W_{T,\infty}(g_{N\tau},s,\Phi)&=\int_{b\in\mathrm{Sym}_2(\mathbb{R})}\Phi(wn(b)g_{N\tau},s)e^{-2\pi i\mathrm{tr}Tb}\,db\\&=\int_{b\in\mathrm{Sym}_2(\mathbb{R})}\Phi(m(\sqrt{N^{-1}})\,w\,n(N^{-1}b)g_\tau,,s)\,e^{-2\pi i\mathrm{tr}Tb}\,db\\&=\chi_{\mathcal{V},\infty}(N^{-1})N^{-s-\frac{3}{2}}\int_{b\in\mathrm{Sym}_2(\mathbb{R})}\Phi(wn(N^{-1}b)g_\tau,s)\,e^{-2\pi i\mathrm{tr}Tb}\,db\\&=N^{-s-\frac{3}{2}}\cdot N^3\int_{\mathrm{Sym}_2(\mathbb{R})}\Phi(wn(b)g_\tau,s)\,e^{-2\pi N\mathrm{tr}Tb}\,db\\&=N^{-s+\frac{3}{2}}W_{NT,\infty}(g_\tau,s,\Phi),\end{aligned} \tag{283}$$

where in the second-to-last line, we used the fact that $\chi_{\mathcal{V},\infty}(\sqrt{N^{-1}})=(\sqrt{N^{-1}},-N)_\infty=1$, and applied the change of variables $b\mapsto Nb$.

To prove *(v)*, recall that the local Weil indices, for any quadratic space W over $\mathbb{Q}$, satisfy the product formula $\prod_{q\le\infty}\gamma_q(W)=1$. Thus, applying part *(i)* – *(iv)* of the proposition, we have

$$\begin{aligned}&N^{-\frac{3}{2}}\,W_T'(g_{N\tau},0,\Phi_2^{\mathcal{L}})\\&=\left\{W_{NT}'(g_\tau,0,\Phi_2^L)+\left(\sum_{\ell|N}\frac{\ell+1}{2(\ell-1)}\log\ell\right)W_{NT}(g_\tau,0,\Phi_2^L)\right\}\left(\frac{\gamma_\infty(V)}{\gamma_\infty(\mathcal{V})}\right)^2\\&\quad+\sum_{\ell|N}\left(\frac{\ell-1}{2(\ell+1)}W_{NT}(g_\tau,0,\Phi_2^{(\ell)})\left(\frac{\gamma_\infty(V^{(\ell)})}{\gamma_\infty(\mathcal{V})}\right)^2-\frac{2}{\ell(\ell^2-1)}W_{NT}(g_\tau,0,\Phi_2^{sp,(\ell)})\left(\frac{\gamma_\infty(V}{\gamma_\infty(\mathcal{V})}\right)^2\right)\log\ell\end{aligned} \tag{284}$$

Note that $V_\infty \simeq \mathcal{V}_\infty \simeq V_\infty^{sp,(\ell)}$, and by [KRY06, p. 330], we have $\gamma_\infty(V) = -i = -\gamma_\infty(V^{(\ell)})$. Thus

$$\left(\frac{\gamma_\infty(V)}{\gamma_\infty(\mathcal{V})}\right)^2 = \left(\frac{\gamma_\infty(V^{(\ell)})}{\gamma_\infty(\mathcal{V})}\right)^2 = 1 \tag{285}$$

and the proposition follows. □

To investigate the degenerate terms, we require the genus one analogue of Proposition 5.1.

Proposition 5.2. *For the lattice $\mathcal{L}$, let $\mathcal{V} = \mathcal{L} \otimes \mathbb{Q}$, and for a prime $q \leq \infty$, define*

$$c_q(\mathcal{V}) = \chi_{\mathcal{V},q}(-1)\gamma_q(\mathcal{V}).$$

We define $c_q(V)$, $c_q(V^{(\ell)})$ and $c_q(V^{sp,(\ell)})$ in a similar way.

Let $t \in \mathbb{Z}$. Then

(i) Suppose $q \nmid N$ and $q < \infty$. Then

$$\frac{W_{t,q}(e,s,\Phi^{\mathcal{L}}_{1,q})}{c_q(\mathcal{V})} = \frac{W_{Nt,q}(e,s,\Phi^{L}_{1,q})}{c_q(V)} = \frac{W_{Nt,q}(e,s,\Phi^{(\ell)}_{1,q})}{c_q(V^{(\ell)})}$$

with the last identity holding for any $\ell | N$.

(ii) Suppose $q|N$. Then

$$\frac{W_{t,q}(e,\frac{1}{2},\Phi^{\mathcal{L}}_{1,q})}{c_q(\mathcal{V})} = q^{\frac{1}{2}}\frac{W_{Nt,q}(e,\frac{1}{2},\Phi^{L}_{1,q})}{c_q(V)} = q^{\frac{1}{2}}\frac{W_{Nt,q}(e,\frac{1}{2},\Phi^{(\ell)}_{1,q})}{c_q(V^{(\ell)})}$$

with the last identity holding for all $\ell \neq q$.

(iii) Suppose $q|N$. Then

$$q^{-\frac{1}{2}}\frac{W'_{t,q}(e,\frac{1}{2},\Phi^{\mathcal{L}}_{1,q})}{c_q(\mathcal{V})} = \frac{W'_{Nt,q}(e,\frac{1}{2},\Phi^{L}_{1,q})}{c_q(V)}$$
$$+\left(\frac{q+1}{2(q-1)}\frac{W_{Nt,q}(e,\frac{1}{2},\Phi^{L}_{1,q})}{c_q(V)} - \frac{q-1}{2(q+1)}\frac{W_{Nt,q}(e,\frac{1}{2},\Phi^{(q)}_{1,q})}{c_q(V^{(q)})} - \frac{2}{q^2-1}\frac{W_{Nt,q}(e,\frac{1}{2},\Phi^{sp,(q)}_{1,q})}{c_q(V^{sp,(q)})}\right)\log q$$

(iv) Let $\tau = u + iv \in \mathbb{H}$ and $g_{\tau,\infty} = [n(u)m(\sqrt{v}),1] \in G_{1,\infty}$ denote the corresponding group element. Then for any $\Phi \in I_{1,\infty}(s,\chi'_\infty)$, we have

$$W_{t,\infty}(g_{N\tau,\infty},s,\Phi) = N^{-\frac{s}{2}+\frac{1}{2}}W_{Nt,\infty}(g_{\tau,\infty},s,\Phi).$$

(v) Globally,

$$N^{-3/4}W'_t(g_{N\tau},\tfrac{1}{2},\Phi^{\mathcal{L}}_1) = W'_{Nt}(g_\tau,\tfrac{1}{2},\Phi^L_1) + \left(\sum_{\ell|N}\frac{\log\ell}{\ell-1}\right)W_{Nt}(g_\tau,\frac{1}{2},\Phi^L_1)$$
$$+\sum_{\ell|N}\left(\frac{\ell-1}{2(\ell+1)}W_{Nt}(g_\tau,\tfrac{1}{2},\Phi^{(\ell)}_1) - \frac{2}{\ell^2-1}W_{Nt}(g_\tau,\tfrac{1}{2},\Phi^{sp,(\ell)}_1)\right)\log\ell$$

Proof. First, suppose that $q \nmid N$. The representation density $\alpha_q(X,t,\mathcal{L}_q)$ was computed previously, cf. (209). On the other hand, we have that for any $\ell|N$, the quadratic spaces L_q and $L_q^{(\ell)}$ are both isometric to $(M_2(\mathbb{Z}_q)^{tr=0}, \det)$. Applying [Yan98, Proposition 8.3], a short computation reveals

$$\alpha_q(X,t,\mathcal{L}_q) = \alpha_q(X,Nt,L_q) = \alpha_q(X,Nt,L_q^{(\ell)}). \tag{286}$$

Finally, applying the identity (201), which relates the representation densities to Whittaker functionals, yields *(i)*.

When $q|N$, the representation density $\alpha_q(X,t,\mathcal{L}_q)$ can be computed explicitly using [Yan98, Theorem 3.1]; the result is recorded in equations (211) and (212) above. Comparing these formulas with the explicit formulas in Corollary 8.2 and Proposition 8.3 of [Yan98], we find

$$\alpha_q(X,t,\mathcal{L}_q) = 1 - X^{-1} + X^{-1}\alpha_q(X,Nt,L_q) \tag{287}$$

Evaluating at $X=1$ gives

$$\alpha_q(1,t,\mathcal{L}_q) = \alpha_q(1,Nt,L_q) = \alpha_q(1,Nt,L_q^{(\ell)}) \tag{288}$$

for $\ell \neq q$, and hence, applying (201) again gives part *(ii)* of the proposition.

To prove (iii) we differentiate (287) and evaluate at $X=1$, which yields

$$\alpha_q'(1,t,\mathcal{L}_q) = \alpha_q'(1,Nt,L_q) - \alpha_q(1,Nt,L_q) + 1. \tag{289}$$

By [Yan98, Corollary 8.2], we have $\alpha_q(X,Nt,L_q)+\alpha_q(X,Nt,L_q^{(q)}) = 2$; substituting this identity above, we conclude that

$$\alpha_q'(1,t,\mathcal{L}_q) = \alpha_q'(1,Nt,L_q) - \frac{1}{2}\alpha_q(1,Nt,L_q) + \frac{1}{2}\alpha_q(1,Nt,L_q^{(q)}). \tag{290}$$

On the other hand, the same corollary implies that

$$\frac{2q}{q^2-1}\alpha_q(1,Nt,L_q^0) - \frac{\alpha_q(1,Nt,L_q)}{q-1} - \frac{\alpha_q(1,Nt,L_q^{(q)})}{q+1} = 0. \tag{291}$$

Thus

$$\begin{aligned}\alpha_q'(1,t,\mathcal{L}_q) - \alpha_q'(1,Nt,L) = &-\frac{q+1}{2(q-1)}\alpha_q(1,Nt,L_q) + \frac{q-1}{2(q+1)}\alpha_q(1,Nt,L_q^{(q)}) \\ &+ \frac{2q}{q^2-1}\alpha_q(1,Nt,L_q^0).\end{aligned} \tag{292}$$

Another appeal to (201) gives part *(iii)* of the proposition.

Part *(iv)* follows in the same way as the proof of Proposition 5.1(iv).

Finally, we observe that for any quadratic space W, the constants $c_q(W)$ satisfy the product formula $\prod_{q\leq\infty} c_q(W) = 1$. Combining this observation with the preceding parts of the proposition, a short computation yields

$$\begin{aligned}N^{-\frac{3}{2}}W_t'(g_\tau,\frac{1}{2},\Phi^{\mathcal{L}}) = &\left(W_{Nt}'(g_\tau,\tfrac{1}{2},\Phi_1^L) + \left(\sum_{\ell|N}\frac{\log\ell}{\ell-1}\right)W_{Nt}(g_\tau,\frac{1}{2},\Phi_1^L)\right)\frac{c_\infty(V)}{c_\infty(\mathcal{V})} \\ &+ \sum_{\ell|N}\left(-\frac{\ell-1}{2(\ell+1)}W_{Nt}(g_\tau,\tfrac{1}{2},\Phi_1^{(\ell)})\frac{c_\infty(V^{(\ell)})}{c_\infty(\mathcal{V})}\right)\log\ell \\ &+ \left(-\frac{2}{\ell^2-1}W_{Nt}(g_\tau,\tfrac{1}{2},\Phi_1^{sp,(\ell)})\frac{c_\infty(V^{sp,\ell})}{c_\infty(\mathcal{V})}\right)\log\ell\end{aligned}$$

Since $\mathcal{V}_\infty \simeq V_\infty \simeq V_\infty^{sp,(\ell)}$, we have $c_\infty(\mathcal{V}) = c_\infty(V)c_\infty(V^{sp,(\ell)})$. Moreover, we have

$$\chi_{\mathcal{V},\infty}(-1) = (-1,-1)_\infty = -1 = \chi_{\mathcal{V}^{(\ell)},\infty}(-1) \qquad \text{and} \qquad \gamma_\infty(\mathcal{V}) = -\gamma_\infty(V^{(\ell)}) \tag{293}$$

cf. [KRY06, p. 330]. Thus $c_\infty(V^\ell) = -c_\infty(\mathcal{V})$, and the proposition follows.

∎

For a section $\Phi \in I_2(s,\chi')$, let $E(g,s,\Phi) = \sum_T E_T(g,s,\Phi)$ denote the Fourier expansion of the corresponding Eisenstein series, as in Section 2.6.

Proposition 5.3. *Let $\tau \in \mathbb{H}_2$. Then for any $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ we have*

$$N^{-\frac{3}{2}} E'_T(g_{N\tau}, 0, \Phi_2^{\mathcal{L}}) = E'_{NT}(g_\tau, 0, \Phi_2^L) + \sum_{\ell | N} \frac{\ell - 1}{2(\ell+1)} E_{NT}(g_\tau 0, \Phi_2^{(\ell)}) \log \ell. \tag{294}$$

Proof. We note that in general, if $\tau = u + iv \in \mathbb{H}_2$, we have

$$E_T(g_\tau, s, \Phi) = e^{2\pi i tr(Tu)} E_T(g_v, s, \Phi). \tag{295}$$

Thus, we may assume that $\tau = iv$ for $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, and write $g_\tau = g_v$ in the remainder of the proof.

Now suppose T is non-degenerate. Then for any section $\Phi \in I_2(s,\chi')$, we have

$$E_T(g,s,\Phi) = W_T(g,s,\Phi), \tag{296}$$

cf. Proposition 4.1. On the other hand, recall that Eisenstein series $E(g,s,\Phi_2^L)$ and $E(g,s,\Phi_2^{sp,(\ell)})$ are incoherent, in the sense of [Kud97], and so they vanish at $s=0$. In particular,

$$W_{NT}(g_v, 0, \Phi^L) = 0 \qquad \text{and} \qquad W_{NT}(g_v, 0, \Phi^{sp,(\ell)}) = 0. \tag{297}$$

Thus, the proposition follows from Proposition 5.1(v) in this case.

Next, suppose that T has rank 1. By Lemma 2.10, we may choose $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ that $T = {}^t\gamma \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right) \gamma$ for some $t \neq 0$. Since

$$E_T(g,s,\Phi) = E_{{}^t\gamma \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)\gamma}(g,s,\Phi) = E_{\left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)}(m(\gamma)g,s,\Phi) \tag{298}$$

for any section $\Phi \in I_2(s,\chi')$, we may assume without loss of generality that $T = \left(\begin{smallmatrix} 0 & \\ & t \end{smallmatrix}\right)$. Similarly, replacing v with $\theta v {}^t\theta$ for an appropriate choice of $\theta = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$, we may further assume that

$$v = \left(\begin{smallmatrix} v_1 & \\ & v_2 \end{smallmatrix}\right) \tag{299}$$

is diagonal.

For $r = 1$ or 2, let Φ_r^* denote any one of the sections $\Phi_r^{\mathcal{L}}$, Φ_r^L, $\Phi_r^{(\ell)}$ or $\Phi_r^{sp,(\ell)}$. Note that in all cases, $\Phi_{r,\infty}^* = \Phi_{r,\infty}^{\frac{3}{2}}$. In this case, combining Lemma 4.11, Lemma 4.12 and (196), we find

$$E_T(g_v, s, \Phi_2^*) = v_1^{\frac{s}{2}+\frac{3}{4}} W_t\left(g_{v_2}, s+\tfrac{1}{2}, \Phi_1^*\right) + W_T(g_v, s, \Phi_2^*). \tag{300}$$

In particular,

$$N^{-\frac{3}{2}} E'_T(g_{Nv}, 0, \Phi_2^{\mathcal{L}}) = v_1^{\frac{3}{4}} \left(\frac{\log(Nv_1)}{2}\right) N^{-\frac{3}{4}} W_t(g_{Nv_2}, \tfrac{1}{2}, \Phi_1^{\mathcal{L}}) + v_1^{\frac{3}{4}} N^{-\frac{3}{4}} W'_t(g_{Nv_2}, \tfrac{1}{2}, \Phi_1^{\mathcal{L}}) + N^{-\frac{3}{2}} W'_T(g_v, 0, \Phi_2^{\mathcal{L}}). \tag{301}$$

Applying (300), Proposition 5.1 and Proposition 5.2, a short computation gives

$$N^{-\frac{3}{2}}E'_T(g_{Nv},0,\Phi_2^{\mathcal{L}}) - E'_{NT}(g_v,0,\Phi_2^L) - \sum_{\ell|N}\frac{\ell-1}{2(\ell+1)}E_{NT}(g_v,0,\Phi_2^{(\ell)})\log\ell$$
$$= \left(\sum_{\ell|N}\frac{\ell+1}{2(\ell-1)}\log\ell\right)\left\{v_1^{\frac{3}{4}}W_{Nt}(g_{v_2},\tfrac{1}{2},\Phi_1^L) + W_{NT}(g_v,0,\Phi_2^L)\right\}$$
$$- \sum_{\ell|N}\frac{2}{\ell^2-1}\left\{v_1^{\frac{3}{4}}W_{Nt}(g_{v_2},\tfrac{1}{2},\Phi_1^{sp,(\ell)}) + W_{NT}(g_v,0,\Phi_2^{sp,(\ell)})\right\}\log\ell$$
$$= \left(\sum_{\ell|N}\frac{\ell+1}{2(\ell-1)}\log\ell\right)\left\{E_{NT}(g_v,0,\Phi_2^L)\right\} - \sum_{\ell|N}\frac{2}{\ell^2-1}\left\{E_{NT}(g_v,0,\Phi_2^{sp,(\ell)})\right\}\log\ell$$
$$= 0$$

where the last line follows since the sections Φ_2^L and $\Phi_2^{sp,(\ell)}$ are incoherent, so the Eisenstein series $E(g,s,\Phi_2^L)$ and $E(g,s,\Phi_2^{sp,(\ell)})$ vanish at $s=0$. This proves the proposition in the case that T has rank 1.

Finally, suppose $T=0$. Recall that for any section $\Phi_2 \in I_2(s,\chi)$, an argument along the lines of [KRY06, §5.9] (see also [KR88, Lemma 2.4]) gives

$$E_0(g,s,\Phi) = \Phi(g,s) + \mathbb{B}(g,s,\Phi) + W_0(g,s,\Phi) \tag{302}$$

where

$$\mathbb{B}(g,s,\Phi) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} B(m(\gamma)g,s,\Phi) \tag{303}$$

with

$$B(g,s,\Phi) = W_0\left(e, s+\tfrac{1}{2}, (\eta^*\circ r(g))\,\Phi\right). \tag{304}$$

A similar argument to the proof of the Proposition 4.19 shows that

$$\mathbb{B}(g_v,0,\Phi^{(\ell)}) = \mathbb{B}(g_v,0,\Phi^{sp,(\ell)}) = 0 \tag{305}$$

and

$$\mathrm{ord}_{s=0}\mathbb{B}(g_v,s,\Phi^L) \geq 2. \tag{306}$$

Moreover, if Φ_2^* is any of the sections $\Phi_2^{\mathcal{L}}$, Φ_2^L, $\Phi_2^{(\ell)}$ or $\Phi_2^{sp,(\ell)}$, the definitions imply the formula

$$\Phi_2^*(g_v,s) = \det(v)^{\frac{s}{2}+\frac{3}{4}}. \tag{307}$$

Thus, we find

$$E'_0(g_{Nv},0,\Phi_2^{\mathcal{L}}) = N^{\frac{3}{2}}\det(v)^{\frac{3}{4}}\left(\log N + \frac{1}{2}\log\det v\right) + W'_0(g_{Nv},0,\Phi_2^{\mathcal{L}}), \tag{308}$$

and

$$E'_0(g_v,0,\Phi_2^L) = \det(v)^{\frac{3}{4}}\left(\frac{1}{2}\log\det v\right) + W'_0(g_v,0,\Phi_2^L); \tag{309}$$

in addition, since Φ_2^L and $\Phi^{sp,(\ell)}$ are incoherent, we have

$$E_0(g_v,0,\Phi_2^L) = E_0(g_v,0,\Phi_2^{sp,(\ell)}) = 0 \tag{310}$$

so

$$W_0(g_v, 0, \Phi^L) = W_0(g_v, 0, \Phi^{sp,(\ell)}) = -\det(v)^{\frac{3}{4}}. \tag{311}$$

Thus, Proposition 5.1 implies that

$$N^{-\frac{3}{2}} E_0'(g_{Nv}, 0, \Phi_2^{\mathcal{L}}) - E_0'(g_v, 0, \Phi_2^L) - \sum_{\ell | N} \frac{\ell - 1}{2(\ell+1)} E_0(g_v, 0, \Phi_2^{(\ell)}) \log \ell \tag{312}$$

$$= \det(v)^{\frac{3}{4}} \log N + \left(\sum_{\ell | N} \frac{\ell+1}{2(\ell-1)} \log \ell \right) W_0(g_v, 0, \Phi_2^L)$$

$$- \sum_{\ell} \frac{2}{\ell^2 - 1} W_0(g_v, 0, \Phi_2^{sp,(\ell)}) \log \ell - \sum_{\ell} \frac{\ell-1}{2(\ell+1)} \log \ell \cdot \det(v)^{\frac{3}{4}} \tag{313}$$

$$= \det(v)^{\frac{3}{4}} \left(\sum_{\ell} 1 - \frac{\ell+1}{2(\ell-1)} + \frac{2}{\ell^2-1} - \frac{\ell-1}{2(\ell+1)} \right) \log \ell \tag{314}$$

$$= 0.$$

This implies the proposition for the case $T = 0$. □

Remark 5.4. The preceding proposition can be phrased in more classical language, as follows, cf. Section 2.6. If Φ_2^* is one of the sections $\Phi_2^{\mathcal{L}}, \Phi_2^L$ or $\Phi_2^{(\ell)}$, and $\tau \in \mathbb{H}_2$, we set

$$E(\tau, s, \Phi_2^*) = \det(v)^{-\frac{3}{4}} E(g_\tau, s, \Phi_2^*). \tag{315}$$

Let

$$\Gamma_0(4N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_r(\mathbb{Z}) \mid C \equiv 0 \pmod{4N} \right\} \tag{316}$$

denote the usual congruence subgroup. Then it can be verified that $E(\tau, s, \Phi_2^*)$ transforms like a Siegel modular form of scalar weight 3/2 and level $\Gamma_0(4N)$ with character $\chi(\gamma) = \mathrm{sgn}\det(d)$, see [KRY06, §8.5.6] for a more precise formulation.

Let $M_{\frac{3}{2}}(\Gamma_0(4N))$ denote the space of (non-holomorphic) Siegel modular forms of scalar weight 3/2 and level $\Gamma_0(4N)$ and character χ, and consider the Hecke operator

$$U_N \colon M_{\frac{3}{2}}(\Gamma_0(4N)) \to M_{\frac{3}{2}}(\Gamma_0(4N)) \tag{317}$$

given by

$$U_N(F)(\tau) = \sum_{u \in \mathrm{Sym}_2(\mathbb{Z}/N\mathbb{Z})} F|_{3/2} \left(\begin{smallmatrix} 1 & u \\ & N \end{smallmatrix} \right) = N^{\frac{3}{2}} \sum_u F(N^{-1}\tau + N^{-1}u) \tag{318}$$

Then the previous proposition can be rewritten in classical terms as the identity

$$E'(\tau, 0, \Phi_2^{\mathcal{L}}) = U_N \left(E'(\tau, 0, \Phi_2^L) + \sum_{\ell | N} \frac{\ell - 1}{2(\ell+1)} E(\tau, 0, \Phi^{(\ell)}) \log \ell \right). \tag{319}$$

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