Arithmetic special cycles and Jacobi forms

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Abstract

We consider families of special cycles, as introduced by Kudla, on compact Shimura varieties attached to quadratic spaces over totally real fields. By augmenting these cycles with Green currents, we obtain classes in the arithmetic Chow groups of the canonical models of these Shimura varieties (viewed as arithmetic varieties over their reflex field). Our main result asserts that certain generating series built from these cycles can be identified with the Fourier expansions of Hilbert-Jacobi modular forms. This result provides evidence for an arithmetic analogue of Kudla's conjecture relating these cycles to Siegel modular forms.

Contents

1 Introduction

The main result of this paper is a modularity result for certain generating series of "special" cycles that live in the arithmetic Chow groups of Shimura varieties of orthogonal type.

We begin by introducing the main players: let F be a totally real extension of $\mathbb{Q}$ with $d = [F : \mathbb{Q}]$, and let $\sigma_1, \ldots, \sigma_d$ denote the Archimedean places of F. Suppose V is a quadratic space over F that is of signature $((p, 2), (p + 2, 0), (p + 2, 0), \cdots, (p + 2, 0))$ with $p > 0$. In other words, we assume that

$V \otimes_{F,\sigma_1} \mathbb{R}$ is a real quadratic space of signature $(p, 2)$ and that V is positive definite at all other real places.

We assume throughout that V is anisotropic over F. Note that the signature condition guarantees that V is anisotropic whenever $d > 1$.

Let $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V)$. The corresponding Hermitian symmetric space $\mathbb{D}$ has two connected components; fix one component $\mathbb{D}^+$ and let $H^+(\mathbb{R})$ denote its stabilizer in $H(\mathbb{R})$. For a neat compact open subgroup $K_f \subset H(\mathbb{A}_f)$, let $\Gamma := H^+(\mathbb{R}) \cap H(\mathbb{Q}) \cap K_f$, and consider the quotient

$$X := \Gamma \backslash \mathbb{D}^+. \tag{1.1}$$

This space is a (connected) Shimura variety; in particular, it admits a canonical model over a number field $E \subset \mathbb{C}$ depending on K_f, see [Kud97] for details; moreover, since V is anisotropic, X is a projective variety.

Fix a Γ-invariant lattice $L \subset V$ such that the restriction of the bilinear form $\langle \cdot, \cdot, \rangle$ to L is valued in $\mathcal{O}_F$, and consider the dual lattice

$$L' = \{\mathbf{x} \in L \mid \langle \mathbf{x}, L \rangle \subset \partial_F^{-1}\} \tag{1.2}$$

where ∂_F^{-1} is the inverse different.

For an integer n with $1 \leq n \leq p$, let $S(V(\mathbb{A}_f)^n)$ denote the Schwartz space of compactly supported, locally constant functions on $V(\mathbb{A}_f)^n$, and consider the subspace

$$S(L^n) := \{\varphi \in S(V(\mathbb{A}_f)^n)^\Gamma \mid \mathrm{supp}(\varphi) \subset (\widehat{L'})^n \text{ and } \varphi(\mathbf{x} + l) = \varphi(\mathbf{x}) \text{ for all } l \in L^n\} \tag{1.3}$$

For every $T \in \mathrm{Sym}_n(F)$ and Γ-invariant Schwartz function $\varphi \in S(L^n)$, there is an E-rational "special" cycle

$$Z(T, \varphi) \tag{1.4}$$

of codimension n on X, defined originally by Kudla [Kud97]; see Section 2.3 below.

It was conjectured by Kudla that these cycles are closely connected to automorphic forms; more precisely, he conjectured that upon passing to the Chow group of X, the generating series formed by the classes of these special cycles can be identified with the Fourier expansions of Hilbert-Siegel modular forms. When $F = \mathbb{Q}$, the codimension one case follows from results of Borcherds [Bor99], and the conjecture for higher codimension was established by Zhang and Bruinier-Raum [BWR15, Zha09]; when $F \neq \mathbb{Q}$, conditional proofs have been given by Yuan-Zhang-Zhang [YZZ09] and Kudla [Kud19].

More recently, attention has shifted to the arithmetic analogues of this result, where one replaces the Chow groups with an "arithmetic" counterpart, attached to a model $\mathcal{X}$ of X defined over a subring of the reflex field of E; these arithmetic Chow groups were introduced by Gillet-Soulé [GS90] and subsequently generalized by Burgos-Kramer-Kühn [BGKK07]. Roughly speaking, in this framework cycles are represented by pairs $(\mathcal{Z}, g_{\mathcal{Z}})$, where $\mathcal{Z}$ is a cycle on $\mathcal{X}$, and $g_{\mathcal{Z}}$ is a *Green object*, a purely differential-geometric datum that encodes cohomological information about the archimedean fibres of $\mathcal{Z}$.

In this paper, we consider the case where the model $\mathcal{X}$ is taken to be X itself. In order to promote the special cycles to the arithmetic setting, we need to choose the Green objects: for this, we employ the results of [GS19], where a family $\{\mathfrak{g}(T, \varphi; \mathbf{v})\}$ of Green forms was constructed. Note that these forms depend on an additional parameter $\mathbf{v} \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$, which should be regarded as the imaginary part of a variable in the Hilbert-Siegel upper half space.

We thereby obtain classes

$$\widehat{Z}(T, \mathbf{v}) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X) \otimes_{\mathbb{C}} S(L^n)^{\vee}, \tag{1.5}$$

where $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X)$ is the Gillet-Soulé arithmetic Chow group attached to X, by the formula

$$\widehat{Z}(T, \mathbf{v})(\varphi) = \big(Z(T, \varphi), \mathfrak{g}(T, \varphi; \mathbf{v})\big) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X). \tag{1.6}$$

For reasons that will emerge in the course of the proof our main theorem, we will also need to consider a larger arithmetic Chow group $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}})$, constructed by Burgos-Kramer-Kühn [BGKK07]. This group appears as an example of their general cohomological approach to the theory of Gillet-Soulé. There is a natural injective map $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X) \hookrightarrow \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}})$; abusing notation, we identify the special cycle $\widehat{Z}(T, \mathbf{v})$ with its image under this map.

Theorem 1.1. *(i) Suppose $1 < n \leq p$. Fix $T_2 \in \mathrm{Sym}_{n-1}(F)$, and define the formal generating series*

$$\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau}) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Z}(T, \mathbf{v}) \, q^T \tag{1.7}$$

where $\boldsymbol{\tau} \in \mathbb{H}_n^d$ lies in the Hilbert-Siegel upper half space of genus n, and $\mathbf{v} = \mathrm{Im}(\tau)$. Then $\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau})$ is the q-expansion of a (non-holomorphic) Hilbert-Jacobi modular form of weight $p/2 + 1$ and index T_2, taking values in $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}}) \otimes S(L^n)^{\vee}$ via the Weil representation.

(ii) When $n = 1$, the generating series

$$\widehat{\phi}_1(\boldsymbol{\tau}) = \sum_{t \in F} \widehat{Z}(t, \mathbf{v}) q^t \tag{1.8}$$

is the q-expansion of a (non-holomorphic) Hilbert modular form of weight $p/2 + 1$, valued in $\widehat{\mathrm{CH}}_{\mathbb{C}}^1(X) \otimes S(L)^{\vee}$.

Some clarification is required in interpreting this theorem. The issue is that there is no apparent topology on the arithmetic Chow groups for which the series (1.7) and (1.8) can be said to converge in a reasonable sense; in a similar vein, the Green forms $\mathfrak{g}(T, \mathbf{v})$ vary smoothly in the parameter $\mathbf{v}$, but there does not appear to be a natural way in which the family of classes $\widehat{Z}(T, \mathbf{v})$ can be said to vary smoothly. What is being asserted in the theorem is the existence of:

(i) finitely many classes $\widehat{Z}_1, \ldots \widehat{Z}_r \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$ (or in $\widehat{\mathrm{CH}}_{\mathbb{C}}^1(X)$ when $n = 1$),

(ii) finitely many Jacobi modular forms (in the usual sense) $f_1, \ldots, f_r$,

(iii) and a Jacobi form $g(\boldsymbol{\tau})$ valued in the space of currents on X that is locally uniformly bounded in $\boldsymbol{\tau}$,

such that the T'th coefficient of the Jacobi form $\sum_i f_i(\boldsymbol{\tau})\widehat{Z}_i + a(g(\boldsymbol{\tau}))$ coincides with $\widehat{Z}(T, \mathbf{v})$. Here $a(g(\boldsymbol{\tau})) \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$ is an "archimedean class" associated to the current $g(\boldsymbol{\tau})$. A more precise account may be found in Section 2.6.

To prove the theorem, we first prove the $n = 1$ case, using a modularity result due to Bruinier [Bru12] that involves a different set of Green functions; our theorem follows from a comparison between his Green functions and ours.

For $n > 1$, we exhibit a decomposition

$$\widehat{Z}(T, \mathbf{v}) = \widehat{A}(T, \mathbf{v}) + \widehat{B}(T, \mathbf{v}) \tag{1.9}$$

in $\widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}}) \otimes S(L^n)^\vee$; this decomposition is based on a mild generalization of the star product formula [GS19, Theorem 4.10]. The main theorem then follows from the modularity of the series

$$\widehat{\phi}_A(\boldsymbol{\tau}) := \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{A}(T, \mathbf{v}) \, q^T \qquad \text{and} \qquad \widehat{\phi}_B(\boldsymbol{\tau}) = \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{B}(T, \mathbf{v}) \, q^T, \tag{1.10}$$

proven in Corollary 6.3 and Theorem 5.1 respectively. The classes $\widehat{B}(T, \mathbf{v})$ are purely archimedean, and the modularity of $\widehat{\phi}_B(\boldsymbol{\tau})$ follows from an explicit computation involving the Kudla-Millson Schwartz form, [KM90]. The classes $\widehat{A}(T, \mathbf{v})$ are expressed as linear combinations of pushforwards of special cycles along sub-Shimura varieties of X, weighted by the Fourier coefficients of classical theta series; the modularity of $\widehat{\phi}_A(\boldsymbol{\tau})$ follows from this description and the $n = 1$ case.

This result provides evidence for the arithmetic version of Kudla's conjecture, namely that the generating series

$$\widehat{\phi}_n(\boldsymbol{\tau}) = \sum_{T \in \mathrm{Sym}_n(F)} \widehat{Z}(T, \mathbf{v}) \, q^T \tag{1.11}$$

is a Hilbert-Siegel modular form; indeed, the series $\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau})$ is a (formal) Fourier-Jacobi coefficient of $\widehat{\phi}_n(\boldsymbol{\tau})$. Unfortunately, there does not seem to be an obvious path by which one can infer the more general result from the results in this paper; the decomposition $\widehat{Z}(T, \mathbf{v})$ depends on the lower-right matrix T_2, and it is not clear how to compare the decompositions for various T_2.

Acknowledgements

2 Preliminaries

2.1 Notation

- Throughout, we fix a totally real field F with $[F : \mathbb{Q}] = d$. Let $\sigma_1, \ldots, \sigma_d$ denote the real embeddings. We identify $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{R}^d$, and denote by $\sigma_i(\mathbf{t})$ the ith component of $\mathbf{t} \in F \otimes_{\mathbb{Q}} \mathbb{R}$.

- For any matrix A, we denote the transpose by A'.

- If $A \in \mathrm{Mat}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$, we write

$$e(A) := \prod_{i=1}^{d} \exp\left(2\pi i \operatorname{tr}\left(\sigma_i(A)\right)\right) \tag{2.1}$$

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- If (V, Q) is a quadratic space over F, let $\langle \mathbf{x}, \mathbf{y} \rangle$ denote the corresponding bilinear form. Here we take the convention $Q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$. If $\mathbf{x} \in V$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n) \in V^n$, we set $\langle \mathbf{x}, \mathbf{y} \rangle = (\langle \mathbf{x}, \mathbf{y}_1 \rangle, \ldots, \langle \mathbf{x}, \mathbf{y}_n \rangle) \in \mathrm{Mat}_{1,n}(F)$.

  For $i = 1, \ldots d$, we set $V_i = V \otimes_{F, \sigma_i} \mathbb{R}$.

- Let
$$\mathbb{H}_n^d = \{\boldsymbol{\tau} = \mathbf{u} + i\mathbf{v} \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R}) \mid \mathbf{v} \gg 0\} \tag{2.2}$$

  denote the Hilbert-Siegel upper half-space of genus n attached to F. Via the fixed embeddings $\sigma_1, \ldots, \sigma_d$, we may identify $\mathrm{Sym}_n(F \otimes \mathbb{R}) \simeq \mathrm{Sym}_n(\mathbb{R})^d$; we let $\sigma_i(\boldsymbol{\tau}) = \sigma_i(\mathbf{u}) + i\sigma_i(\mathbf{v})$ denote the corresponding component, so that, in particular, $\sigma_i(\mathbf{v}) \in \mathrm{Sym}_n(\mathbb{R})_{>0}$ for $i = 1, \ldots, d$.

  If $\boldsymbol{\tau} \in \mathbb{H}_n^d$ and $T \in \mathrm{Sym}_n(F)$, we write
$$q^T = e(\boldsymbol{\tau} T). \tag{2.3}$$

2.2 Arithmetic Chow groups

In this section, we recall the theory of arithmetic Chow groups $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X)$, as conceived by Gillet and Soulé; note here and throughout this paper, we work with complex coefficients. Recall that X is defined over a number field E endowed with a fixed complex embedding $\sigma : E \to \mathbb{C}$; we view X as an arithmetic variety over the "arithmetic ring" $(E, \sigma, \text{complex conjugation})$ in the terminology of [GS90, §3.1.1].

An *arithmetic cycle* is a pair (Z, g), where Z is a formal $\mathbb{C}$-linear combination of codimension n subvarieties of X, and g is a *Green current* for Z; more precisely, g is a current of degree $(p-1, p-1)$ on $X(\mathbb{C})$ such Green's equation
$$\mathrm{dd}^c g \ + \ \delta_{Z(\mathbb{C})} = \omega \tag{2.4}$$

holds, where the right hand side is the current defined by integration against some smooth form ω (here and throughout this paper, we will abuse notation and write ω both for the form and the current it defines). Given a codimension $n - 1$ subvariety Y and a rational function $f \in k(Y)^\times$ on Y, we define the principal arithmetic divisor
$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|^2 \delta_Y); \tag{2.5}$$

the arithmetic Chow group $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X)$ is quotient of the space of arithmetic cycles by the subspace spanned by (a) the principal arithmetic divisors and (b) classes of the form $(0, \eta)$ with $\eta \in \mathrm{im}(\partial) + \mathrm{im}(\bar{\partial})$. For more details, see [GS90, Sou92]

In their paper [BGKK07], Burgos, Kramer and Kühn give an abstract reformulation and generalization of this theory: their main results describe the construction of arithmetic Chow groups $\widehat{\mathrm{CH}}^*(X, \mathcal{C})$ for any "Gillet complex" $\mathcal{C}$ satisfying certain properties. One of the examples they describe is the complex of currents $\mathcal{D}_{\mathrm{cur}}$; we will content ourselves with the superficial description given below, which will suffice for our purposes, and the reader is invited to consult [BGKK07, §6.2] for a thorough treatment.

Unwinding the formal definitions in [BGKK07], one finds that classes in $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}})$ are represented by tuples $(Z, [T, g])$, with Z as before, but now T and g are currents of degree (n, n)

and $(n-1, n-1)$ such that[1]

$$\mathrm{dd}^c g + \delta_{Z(\mathbb{C})} = T + \mathrm{dd}^c(\eta) \tag{2.6}$$

for some current η with support contained in $Z(\mathbb{C})$; we can view this as a relaxation of the condition that the right hand side of (2.4) is smooth. A nice consequence is that any codimension n cycle Z on X gives rise to a *canonical class* (see [BGKK07, Definition 6.37])

$$\widehat{Z}^{\mathrm{can}} := (Z, [\delta_Z, 0]) \tag{2.7}$$

By [BGKK07, Theorem 6.35], the natural map

$$\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X) \to \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}}), \qquad (Z, g) \mapsto (Z, [\omega, g]) \tag{2.8}$$

is injective. Moreover while $\widehat{\mathrm{CH}}_{\mathbb{C}}^*(X, \mathcal{D}_{\mathrm{cur}})$ is not a ring in general, it is a module over $\widehat{\mathrm{CH}}_{\mathbb{C}}^*(X)$. As a special case of this product, let $(Z, g) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X)$ be an arithmetic divisor, where g is a Green function with logarithmic singularities along the divisor Z. Suppose $\widehat{Y}^{\mathrm{can}} \in \widehat{\mathrm{CH}}^m(X, \mathcal{D}_{\mathrm{cur}})$ is the canonical class attached to a cycle Y that intersects Z properly; then by inspecting the proofs of [BGKK07, Theorem 6.23, Proposition 6.32] we find

$$(Z, g) \cdot \widehat{Y}^{\mathrm{can}} = \left(Z \cdot Y, [\omega \wedge \delta_{Y(\mathbb{C})}, g \wedge \delta_{Y(\mathbb{C})}] \right) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^{m+1}(X, \mathcal{D}_{\mathrm{cur}}). \tag{2.9}$$

Remark 2.1. One consequence of our setup is the vanishing of certain "archimedean rational" classes in $\widehat{\mathrm{CH}}^n(X)$ and $\widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$. More precisely, if Y is a codimension $n-1$ subvariety, then we have

$$(0, \delta_{Y(\mathbb{C})}) = 0 \ \in \ \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X). \tag{2.10}$$

To see this, let $c \in \mathbb{Q}$ be any rational number such that $c \neq 0$ or 1, and view c as a rational function on Y; its divisor is trivial, and so

$$0 = \widehat{\mathrm{div}}(c) = (0, -\log|c|^2 \delta_{Y(\mathbb{C})}) = -\log|c|^2 \cdot (0, \delta_{Y(\mathbb{C})}). \tag{2.11}$$

and hence $(0, \delta_{Y(\mathbb{C})}) = 0$. As a special case, we have $(0, 1) = 0 \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X)$.

2.3 Special cycles

Here we review Kudla's construction of the family $\{Z(T)\}$ of special cycles on X, [Kud97]. First, recall that the symmetric space $\mathbb{D}$ has a concrete realization

$$\mathbb{D} = \{z \in \mathbb{P}^1(V \otimes_{\sigma_1, F} \mathbb{C}) \mid \langle z, z \rangle = 0, \langle z, \overline{z} \rangle < 0\} \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ is the $\mathbb{C}$-bilinear extension of the bilinear form on V; the two connected components of $\mathbb{D}$ are interchanged by conjugation.

Given a collection of vectors $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in V^n$, let

$$\mathbb{D}_{\mathbf{x}}^+ := \{z \in \mathbb{D}^+ \mid z \perp \sigma_1(\mathbf{x}_i) \ \text{for} \ i = 1, \ldots, n\}. \tag{2.13}$$

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[1] The reader is cautioned that in [BGKK07], the authors normalize delta currents and currents defined via integration by powers of $2\pi i$, resulting in formulas that look slightly different from those presented here; because we are working with $\mathbb{C}$-coefficients, the formulations are equivalent.

where, abusing notation, we denote by $\sigma_1\colon V \to V_1 = V \otimes_{F,\sigma_1} \mathbb{R}$ the map induced by inclusion in the first factor.

Let $\Gamma_{\mathbf{x}}$ denote the pointwise stabilizer of $\mathbf{x}$ in Γ; then the inclusion $\mathbb{D}_{\mathbf{x}}^+ \subset \mathbb{D}^+$ induces a map

$$\Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}^+ \to \Gamma \backslash \mathbb{D}^+ = X, \tag{2.14}$$

which defines a complex algebraic cycle that we denote $Z(\mathbf{x})$. If the span of $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is not totally positive definite, then $\mathbb{D}_{\mathbf{x}}^+ = \emptyset$ and $Z(\mathbf{x}) = 0$; otherwise, the codimension of $Z(\mathbf{x})$ is the dimension of this span.

Now suppose $T \in \mathrm{Sym}_n(F)$ and $\varphi \in S(L^n)$, and set

$$Z(T, \varphi)^{\natural} := \sum_{\substack{\mathbf{x} \in \Omega(T) \\ \mathrm{mod}\ \Gamma}} \varphi(\mathbf{x}) \cdot Z(\mathbf{x}), \tag{2.15}$$

where

$$\Omega(T) := \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in V^n \mid \langle \mathbf{x}_i, \mathbf{x}_j \rangle = T_{ij}\}. \tag{2.16}$$

This cycle is rational over E. If $Z(T, \varphi)^{\natural} \neq 0$, then T is necessarily totally positive semidefinite, and in this case $Z(T, \varphi)^{\natural}$ has codimension equal to the rank of T.

Finally, we define a $S(L^n)^{\vee}$-valued cycle $Z(T)^{\natural}$ by the rule

$$Z(T)^{\natural}\colon \varphi \mapsto Z(T, \varphi)^{\natural}. \tag{2.17}$$

for $\varphi \in S(V^n)$.

2.4   The cotautological bundle

Let $\mathcal{E} \to X$ denote the tautological bundle: over the complex points $X(\mathbb{C}) = \Gamma \backslash \mathbb{D}^+$, the fibre $\mathcal{E}_z$ at a point $z \in \mathbb{D}^+$ is the line corresponding to z in the model (2.12). There is a natural Hermitian metric $\|\cdot\|_{\mathcal{E}}^2$ on $\mathcal{E}(\mathbb{C})$, defined at a point $z \in \mathbb{D}^+$ by the formula $\|v_z\|_{\mathcal{E},z}^2 = -\langle v_z, v_z \rangle$ for $v_z \in z$. Consider the arithmetic class

$$\widehat{\omega} = -\widehat{c}_1(\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X); \tag{2.18}$$

concretely, $\widehat{\omega} = -(\mathrm{div}\, s, -\log\|s\|_{\mathcal{E}}^2)$, where s is any meromorphic section of $\mathcal{E}$. Finally, for future use, we set

$$\Omega := -c_1(\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \in A^{1,1}(X(\mathbb{C})) \tag{2.19}$$

where $-\Omega = c_1(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is the first Chern form attached to $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$; here the Chern form is normalized as in [Sou92, §4.2]. Note that $-\Omega$ is a Kähler form, cf. [GS19, §2.2].

Remark 2.2. Elsewhere in the literature, one often finds a different normalization (i.e. an overall multiplicative constant) for the metric $\|\cdot\|_{\mathcal{E}}$ that is better suited to certain arithmetic applications; for example, see [KRY06, §3.3]. In our setting, however, Remark 2.1 implies that rescaling the metric does not change the Chern class in $\widehat{\mathrm{CH}}_{\mathbb{C}}^1(X)$.

2.5   Green forms and arithmetic cycles

In this section, we sketch the construction of a family of Green forms for the special cycles, following [GS19].

We begin by recalling that for any tuple $x = (x_1, \ldots x_n) \in V_1^n = (V \otimes_{\sigma_1, F} \mathbb{R})^n$, Kudla and Millson (see [KM90]) have defined a Schwartz form $\varphi_{\mathrm{KM}}(x)$, which is valued in the space of closed (n, n) forms on $\mathbb{D}^+$, and is of exponential decay in $\mathbf{x}$. Let $T(x) \in \mathrm{Sym}_n(\mathbb{R})$ denote the matrix of inner products, i.e. $T(x)_{ij} = \langle x_i, x_j \rangle$, and consider the normalized form

$$\varphi_{\mathrm{KM}}^o(x) \; := \; \varphi_{\mathrm{KM}}(x) \, e^{2\pi \mathrm{tr} T(x)} \tag{2.20}$$

The form φ_{KM} is of weight $p/2 + 1$; more precisely, let $\widetilde{U}(n) \subset \mathrm{Mp}_n(\mathbb{R})$ denote the double cover of $U(n)$, which admits a character $\det^{\frac{1}{2}}$ whose square factors through $U(n)$ as the determinant. Then $\omega(\widetilde{k})\varphi_{\mathrm{KM}} = (\det^{\frac{1}{2}}(\widetilde{k}))^{p+2}\varphi_{\mathrm{KM}}$, where ω is the Weil representation attached to V_1, cf. [KM90, Theorem 5.2].

In [GS19, §2.2], another form $\nu^o(x)$, valued in the space of smooth $(n-1, n-1)$ forms on $\mathbb{D}^+$ is defined (this form is denoted by $\nu^o(x)_{[2n-2]}$ there). It satisfies the relation

$$\mathrm{dd}^c \nu^o(\sqrt{u}x)_{[2n-2]} \; = \; -u \frac{\partial}{\partial u} \varphi_{\mathrm{KM}}^o(\sqrt{u}x), \qquad u \in \mathbb{R}_{>0}. \tag{2.21}$$

For a complex parameter $\rho \gg 0$, let

$$\mathfrak{g}^o(x; \rho) \; := \; \int_1^\infty \nu^o(\sqrt{u}x) \frac{du}{u^{1+\rho}}; \tag{2.22}$$

then $\mathfrak{g}^o(x, \rho)$ defines a smooth form for $Re(\rho) \gg 0$. The corresponding current admits a meromorphic continuation to a neighbourhood of $\rho = 0$ and we set

$$\mathfrak{g}^o(x) := \underset{\rho=0}{CT} \, \mathfrak{g}^o(x; \rho). \tag{2.23}$$

Note that, for example,

$$\mathfrak{g}^o(0) \; = \; \nu^o(0) \, \underset{\rho=0}{CT} \int_1^\infty \frac{du}{u^{1+\rho}} = 0. \tag{2.24}$$

This current satisfies the equation

$$\mathrm{dd}^c \mathfrak{g}^o(x) + \delta_{\mathbb{D}_{\mathbf{x}}^+} \wedge \Omega^{n-r(x)} = \varphi_{\mathrm{KM}}^o(x) \tag{2.25}$$

where $r(x) = \dim \mathrm{span}(x) = \dim \mathrm{span}(x_1, \ldots, x_n)$; for details regarding all these facts, see [GS19, §2.6].

Now suppose $T \in \mathrm{Sym}_n(F)$. Following [GS19, §4], we define an $S(L^n)^\vee$-valued current $\mathfrak{g}^o(T, \mathbf{v})$, depending on a parameter $\mathbf{v} \in \mathrm{Sym}_n(F \otimes_\mathbb{Q} \mathbb{R})_{\gg 0}$, as follows: let $v = \sigma_1(\mathbf{v})$, where $\sigma_1 \colon F \to \mathbb{R}$ is the distinguished real embedding, and choose any matrix $a \in \mathrm{GL}_n(\mathbb{R})$ such that $v = aa'$. Then $\mathfrak{g}^o(T, \mathbf{v})$ is defined by the formula

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) \; := \; \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \mathfrak{g}^o\left(\sigma_1(\mathbf{x})a\right), \tag{2.26}$$

where $\sigma_1(\mathbf{x}) \in V_1^n$; by [GS19, Proposition 2.12], this is independent of the choice of $a \in \mathrm{GL}_n(\mathbb{R})$. Then $\mathfrak{g}^o(T, \mathbf{v})$ is a Γ-invariant current on $\mathbb{D}^+$ and hence descends to $X(\mathbb{C}) = \Gamma \backslash \mathbb{D}^+$.

Consider the $S(L^n)^\vee$-valued differential form $\omega(T, \mathbf{v})$, defined by the formula

$$\omega(T, \mathbf{v})(\varphi) := \sum_{\mathbf{x} \in \Omega(T)} \varphi(x)\, \varphi^o(\sigma_1(\mathbf{x})a), \qquad \sigma_1(\mathbf{v}) = aa'; \tag{2.27}$$

it is a q-coefficient of the *Kudla-Millson theta series*

$$\Theta_{\mathrm{KM}}(\boldsymbol{\tau}) = \sum_{T \in \mathrm{Sym}_n(F)} \omega(T, \mathbf{v})\, q^T, \tag{2.28}$$

where $\tau \in \mathbb{H}_n^d$, and $\mathbf{v} = \mathrm{Im}(\boldsymbol{\tau})$. We then have the equation of currents

$$\mathrm{dd^c}\mathfrak{g}^o(T, \mathbf{v}) + \delta_{Z(T)(\mathbb{C})} \wedge \Omega^{n-\mathrm{rank}T} = [\omega(T, \mathbf{v})] \tag{2.29}$$

on X, cf. [GS19, Proposition 4.4]

In particular, if T is non-degenerate, then $\mathfrak{g}^o(T, \mathbf{v})$ is a Green current for the cycle $Z(T)^\natural$; in this case, we obtain an arithmetic special cycle

$$\widehat{Z}(T, \mathbf{v}) := (Z(T)^\natural, \mathfrak{g}^o(T, \mathbf{v})) \in \widehat{\mathrm{CH}}_\mathbb{C}^n(X) \otimes_\mathbb{C} S(V^n)^\vee. \tag{2.30}$$

Now suppose $T \in \mathrm{Sym}_n(F)$ is arbitrary, let $r = \mathrm{rank}(T)$, and fix $\varphi \in S(V^n)$. We may choose a pair (Z_0, g_0) representing the power $\widehat{\omega}^{n-r}$ of the cotautological bundle, such that Z_0 intersects $Z(T, \varphi)^\natural$ properly and g_0 has logarithmic type [Sou92, §II.2]. We then define

$$\widehat{Z}(T, \mathbf{v}, \varphi) := \left(Z(T, \varphi)^\natural \cdot Z_0,\ \mathfrak{g}^o(T, \mathbf{v}, \varphi) + g_0 \wedge \delta_{Z(T,\varphi)(\mathbb{C})}\right) \in \widehat{\mathrm{CH}}_\mathbb{C}^n(X). \tag{2.31}$$

The reader may consult [GS19, §5.4] for more detail on this construction, including the fact that it is well-defined. Finally, we define $\widehat{Z}(T, \mathbf{v}) \in \widehat{\mathrm{CH}}_\mathbb{C}^n(X) \otimes S(V^n)^\vee$ by the rule

$$\widehat{Z}(T, \mathbf{v})(\varphi) = \widehat{Z}(T, \mathbf{v}, \varphi). \tag{2.32}$$

Remark 2.3. In [GS19], the Green current $\mathfrak{g}^o(T, \mathbf{v})$ is augmented by an additional term, depending on $\log(\det \mathbf{v})$, when T is degenerate see [GS19, Definition 4.5]. This term was essential in establishing the archimedean arithmetic Siegel-Weil formula in the degenerate case; however, in the setting of the present paper, Remark 2.1 implies that this additional term vanishes upon passing to $\widehat{\mathrm{CH}}_\mathbb{C}^n(X)$; thus it can be omitted from the discussion without consequence. In particular, according to our definitions, we have

$$\widehat{Z}(0_n, \mathbf{v})(\varphi) = \varphi(0) \cdot \widehat{\omega}^n \tag{2.33}$$

2.6 Hilbert-Jacobi modular forms

In this section, we briefly review the basic definitions of vector-valued (Hilbert) Jacobi modular forms, mainly to fix notions. For convenience, we work in "classical" coordinates and only with parallel scalar weight. Throughout, we fix an integer $n \geq 1$.

We begin by briefly recalling the theory of metaplectic groups and the Weil representation; a convenient summary for the facts mentioned here, in a form useful to us, is [JS07, §2]. For a

place $v \leq \infty$, let $\widetilde{\mathrm{Sp}}_n(F_v)$ denote the metaplectic group, a two-fold cover of $\mathrm{Sp}_n(F_v)$; as a set, $\widetilde{\mathrm{Sp}}_n(F_v) = \mathrm{Sp}_n(F_v) \times \{\pm 1\}$. When $F_v = \mathbb{R}$, the group $\widetilde{\mathrm{Sp}}_n(\mathbb{R})$ is isomorphic to the group of pairs (g, ϕ), where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\phi \colon \mathbb{H}_n \to \mathbb{C}$ is a function such that $\phi(\tau)^2 = \det(C\tau + D)$; in this model, multiplication is given by

$$(g, \phi(\tau)) \cdot (g', \phi'(\tau)) = (gg', \phi(g'\tau)\phi'(\tau)). \tag{2.34}$$

At a non-dyadic finite place, there exists a canonical embedding $\mathrm{Sp}_n(\mathcal{O}_v) \to \widetilde{\mathrm{Sp}}_n(F_v)$. Consider the restricted product $\prod'_{v \leq \infty} \widetilde{\mathrm{Sp}}_n(F_v)$ with respect to these embeddings; the global double cover $\widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ of $\mathrm{Sp}_n(\mathbb{A})$ is the quotient $\prod'_{v \leq \infty} \widetilde{\mathrm{Sp}}_n(F_v)/I$ of this restricted direct product by the subgroup

$$I := \{(1, \epsilon_v)_{v \leq \infty} \mid \prod_v \epsilon_v = 1,\ \epsilon_v = 1 \text{ for almost all } v\}; \tag{2.35}$$

there is a splitting

$$\iota_F \colon \mathrm{Sp}_n(F) \hookrightarrow \widetilde{\mathrm{Sp}}_{n,\mathbb{A}}, \qquad \gamma \mapsto \prod_v (\gamma, 1)_v \cdot I \tag{2.36}$$

Let $\widetilde{\Gamma}'$ denote the full inverse image of $\mathrm{Sp}_n(\mathcal{O}_F)$ under the covering map $\prod_{v|\infty} \widetilde{\mathrm{Sp}}_n(F_v) \to \mathrm{Sp}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$. We obtain an action ρ of $\widetilde{\Gamma}'$ on the space $S(V(\mathbb{A}_f)^n)$ as follows. Let ω denote the[2] Weil representation of $\widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ on $S(V(\mathbb{A})^n)$. Given $\widetilde{\gamma} \in \widetilde{\Gamma}'$, choose $\widetilde{\gamma}_f \in \prod'_{v < \infty} \widetilde{\mathrm{Sp}}_n(F_v)$ such that $\widetilde{\gamma}\widetilde{\gamma}_f \in \mathrm{im}(\iota_F)$ and set

$$\rho(\widetilde{\gamma}) := \omega(\widetilde{\gamma}_f). \tag{2.37}$$

Recall that we had fixed a lattice $L \subset V$. The subspace $S(L^n) \subset S(V(\mathbb{A}_f)^n)$, as defined in (1.3), is stable under the action of $\widetilde{\Gamma}'$; when we wish to emphasize this lattice, we denote the corresponding action by ρ_L.

Given $\kappa \in \frac{1}{2}\mathbb{Z}$, we define a (parallel, scalar) weight κ slash operator, for the group $\widetilde{\Gamma}'$ acting on the space of functions $f \colon \mathbb{H}_n^d \to S(L^n)^\vee$, by the formula

$$f|_\kappa[\widetilde{\gamma}](\boldsymbol{\tau}) = \prod_{v|\infty} \phi_v(\sigma_v(\boldsymbol{\tau}))^{-2\kappa} \rho_L^\vee(\widetilde{\gamma}^{-1}) \cdot f(g\boldsymbol{\tau}), \qquad \widetilde{\gamma} = (g_v, \phi_v(\tau))_{v|\infty} \tag{2.38}$$

where $g = (g_v)_v$.

If $n > 1$, consider the Jacobi group $G^J = G^J_{n,n-1} \subset \mathrm{Sp}_n$; for any ring R, its R-points are given by

$$G^J(R) := \left\{ g = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda^t & 1_{n-1} & \mu^t & 0 \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1_{n-1} \end{pmatrix} \ \middle|\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R),\ \mu, \tau \in M_{1,n-1}(R) \right\}. \tag{2.39}$$

Define $\widetilde{\Gamma}^J \subset \widetilde{\Gamma}'$ to be the inverse image of $G^J(\mathcal{O}_F)$ in $\widetilde{G}'_{\mathbb{R}}$.

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[2] Here we take the Weil representation for the standard additive character $\psi_F \colon \mathbb{A}_F/F \to \mathbb{C}$, which we suppress from the notation.

Definition 2.4. Suppose

$$f \colon \mathbb{H}_n^d \to S(L^n)^\vee \tag{2.40}$$

is a smooth function. Given $T_2 \in \mathrm{Sym}_{n-1}(F)$, we say that $f(\boldsymbol{\tau})$ transforms like a Jacobi modular form of genus n, weight κ and index T_2 if the following conditions hold.

(a) For all $\mathbf{u}_2 \in \mathrm{Sym}_{n-1}(F_\mathbb{R})$,

$$f\left(\boldsymbol{\tau} + \left(\begin{smallmatrix} 0 & \\ & \mathbf{u}_2 \end{smallmatrix}\right)\right) \ = \ e(T_2 \mathbf{u}_2) f(\boldsymbol{\tau}). \tag{2.41}$$

(b) For all $\widetilde{\gamma} \in \widetilde{\Gamma}^J$,

$$f|_\kappa[\widetilde{\gamma}](\boldsymbol{\tau}) \ = \ f(\boldsymbol{\tau}). \tag{2.42}$$

Let $A_{\kappa, T_2}(\rho_L^\vee)$ denote the space of $S(L^n)^\vee$-valued smooth functions that transform like a Jacobi modular form of weight κ and index T_2.

Remark 2.5.   1. If desired, one can impose further analytic properties of f (holomorphic, real analytic, etc.).

2. If $n = 1$, then we simply say that a function $f \colon \mathbb{H}_1^d \to S(L)^\vee$ transforms like a (Hilbert) modular form of weight κ if it satisfies $f|_\kappa[\widetilde{\gamma}](\boldsymbol{\tau}) = f(\boldsymbol{\tau})$ as usual.

3. An $S(L^n)^\vee$-valued Jacobi modular form f, in the above sense, has a Fourier expansion of the form

$$f(\boldsymbol{\tau}) \ := \ \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} c_f(T, \mathbf{v}) \, q^T \tag{2.43}$$

where the coefficients $c_f(T, \mathbf{v})$ are smooth functions $c_f(T, \mathbf{v}) \colon \mathrm{Sym}_n(F \otimes_\mathbb{Q} \mathbb{R})_{\gg 0} \to S(V^n)^\vee$. The dependence on $\mathbf{v}$ arises from the natural expectation that the Fourier-Jacobi coefficients of non-holomorphic Siegel modular forms should be Jacobi forms.

We now clarify what it should mean for generating series with coefficients in arithmetic Chow groups, as in Theorem 1.1 to be modular.

First, let $D^{n-1}(X)$ denote the space of currents on $X(\mathbb{C})$ of complex bidegree $(n-1, n-1)$, and note that there is a map

$$a \colon D^{n-1}(X) \ \to \ \widehat{\mathrm{CH}}_\mathbb{C}^n(X, \mathcal{D}_{\mathrm{cur}}), \qquad a(g) = (0, [\mathrm{dd}^c g, g]). \tag{2.44}$$

Definition 2.6. Define the space $A_{\kappa, T_2}(\rho^\vee; D^{n-1}(X))$ of "Jacobi forms valued in $S(L^n)^\vee \otimes_\mathbb{C} D^{n-1}(X)$" as the space of functions

$$\xi \colon \mathbb{H}_n^d \to D^{n-1}(X) \otimes_\mathbb{C} S(L^n)^\vee \tag{2.45}$$

such that the following two conditions hold.

(a) For every smooth form α on X, the function $\xi(\boldsymbol{\tau})(\alpha)$ is an element of $A_{\kappa, T_2}(S(V^n)^\vee)$, and in particular, is smooth in the variable $\boldsymbol{\tau}$.

(b) Fix an integer $k \geq 0$ and let $\| \cdot \|_k$ be an algebra seminorm, on the space of smooth differential forms on X, such that given a sequence $\{\alpha_i\}$, we have $\|\alpha_i\|_k \to 0$ if and only if α_i, together with all partial derivatives of order $\leq k$, tends uniformly to zero. We then require that for every compact subset $C \subset \mathbb{H}_n^d$, there exists a constant $c_{k,V}$ such that

$$|\xi(\boldsymbol{\tau})(\alpha)| \leq c_{k,C} \|\alpha\|_k \tag{2.46}$$

for all $\boldsymbol{\tau} \in V$ and all smooth forms α.

The second condition ensures that any such function admits a Fourier expansion as in (2.43) whose coefficients are continuous in the sense of distributions, i.e. they are again $S(L^n)^\vee$-valued currents.

Definition 2.7. Given a collection of classes $\widehat{Y}(T, \mathbf{v}) \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L^n)^\vee$, consider the formal generating series

$$\widehat{\Phi}_{T_2}(\boldsymbol{\tau}) := \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Y}(T, \mathbf{v}) \, q^T. \tag{2.47}$$

Roughly speaking, we say that $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ is modular (of weight κ and index T_2) if there is an element

$$\widehat{\phi}(\boldsymbol{\tau}) \in A_{\kappa, T_2}(\rho_L^\vee) \otimes_{\mathbb{C}} \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}}) \; + \; a\left(A_{\kappa, T_2}(\rho_L^\vee; D^{n-1}(X))\right) \tag{2.48}$$

whose Fourier expansion coincides with $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$. More precisely, we define the modularity of $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ to mean that there are finitely many classes

$$\widehat{Z}_1, \ldots, \widehat{Z}_r \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}}) \tag{2.49}$$

and Jacobi forms

$$f_1, \ldots f_r \in A_{\kappa, T_2}(\rho^\vee), \qquad g \in A_{\kappa, T_2}(\rho^\vee; D^{n-1}(X)) \tag{2.50}$$

such that

$$\widehat{Y}(T, \mathbf{v}) = \sum_i c_{f_i}(T, \mathbf{v}) \, \widehat{Z}_i + a\left(c_g(T, \mathbf{v})\right) \; \in \; \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(V^n)^\vee \tag{2.51}$$

for all $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$.

Remark 2.8. 1. If $\widehat{Z}_1, \ldots, \widehat{Z}_r \in \widehat{\mathrm{CH}}_{\mathbb{C}}^n(X)$ and $g(\boldsymbol{\tau})$ takes values in the space of (currents represented by) smooth differential forms on X, then we say that $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ is valued in $\widehat{\mathrm{CH}}_{\mathbb{C}}^n(X) \otimes S(V^n)^\vee$; indeed, in this case, the right hand side of (2.51) lands in this latter group.

2. As before, one may also impose additional analytic conditions on the forms f_i, g appearing above if desired.

3. Elsewhere in the literature (e.g. [Bor99, Bru12, Zha09]), one finds a notion of modularity that amounts to omitting the second term in (2.48); this notion is well-adapted to the case that the generating series of interest are holomorphic, i.e. the coefficients are independent of the imaginary part of τ.

   In contrast, the generating series that figure in our main theorem depend on these parameters in an essential way. Indeed, the Green forms $\mathfrak{g}^o(T, \mathbf{v})$ vary smoothly in v; however, to the best of the author's knowledge, there is no natural topology on $\widehat{\mathrm{CH}}^n(X)$, or $\widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$, for which the corresponding family $\widehat{Z}(T, \mathbf{v})$ varies smoothly in v. As we will see in the course of the proof of the main theorem, the additional term in (2.48) will allow us enough flexibility to reflect the non-holomorphic behaviour of the generating series. Similar considerations appear in [ES18] in the codimension one case.

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3   The genus one case

In this section, we prove the main theorem in the case $n = 1$; later on, this will be a key step in the proof for general n. The proof of this theorem amounts to a comparison with a generating series of special divisors equipped with a different family of Green functions, defined by Bruinier. A similar comparison appears in [ES18] for unitary groups over imaginary quadratic fields; in the case at hand, however, the compactness of X allows us to apply spectral theory and simplify the argument considerably.

Suppose $t \gg 0$. In [Bru12], Bruinier constructs an $S(L)^\vee$-valued Green function $\Phi(t)$ for the divisor $Z(t) = Z(t)^\natural$. To be a bit more precise about this, recall the Kudla-Millson theta function $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$ from (2.28). As a function of τ, the theta function Θ_{KM} is non-holomorphic and transforms as a Hilbert modular form of parallel weight $\kappa = p/2 + 1$. It is moreover of moderate growth, [Bru12, Prop. 3.4] and hence can be paired, via the Petersson pairing, with cusp forms. Let $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau}) \in S_\kappa(\rho_L)$ denote the cuspidal projection, defined by the property

$$\langle \Theta_{\mathrm{KM}}, g \rangle^{\mathrm{Pet}} = \langle \Lambda_{\mathrm{KM}}, g \rangle^{\mathrm{Pet}} \tag{3.1}$$

for all cusp forms $g \in S_\kappa(\rho)$.

Writing the Fourier expansion

$$\Lambda_{\mathrm{KM}}(\boldsymbol{\tau}) = \sum_t c_\Lambda(t)\ q^t, \qquad \Lambda_{\mathrm{KM},t} \in A^{1,1}(X) \otimes S(L)^\vee \tag{3.2}$$

it follows from [Bru12, Corollary 5.16, Theorem 6.4] that $\Phi(t)$ satisfies the equation

$$\mathrm{dd}^c[\Phi(t)]\ +\ \delta_{Z(t)}\ =\ [c_\Lambda(t) + B(t) \cdot \Omega] \tag{3.3}$$

of currents on X, where

$$B(t)\ :=\ -\frac{\deg(Z(t))}{\mathrm{vol}(X, (-\Omega)^p)}\ \in\ S(L)^\vee. \tag{3.4}$$

recall here that $(-\Omega)^p$ induces a volume form on X.

Finally, define classes $\widehat{Z}_{\mathrm{Br}}(t) \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes S(L)^\vee$ as follows:

$$\widehat{Z}_{\mathrm{Br}}(t)\ =\ \begin{cases} (Z(t), \Phi(t)), & \text{if } t \gg 0 \\ \widehat{\omega} \otimes \mathrm{ev}_0, & \text{if } t = 0. \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\mathrm{ev}_0 \in S(L)^\vee$ is the functional $\varphi \mapsto \varphi(0)$.

We then have the generating series

$$\widehat{\phi}_{\mathrm{Br}}(\tau) = \sum_t \widehat{Z}_{\mathrm{Br}}(t)\, q^t. \tag{3.6}$$

Theorem 3.1 (Bruinier). *The generating series $\widehat{\phi}_B(\tau)$ is a (holomorphic) Hilbert modular form of parallel weight $\kappa = p/2 + 1$. More precisely, there are finitely many classes $\widehat{Z}_1, \dots \widehat{Z}_r \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$ and holomorphic Hilbert modular forms $f_1, \dots, f_r$ such that $\widehat{Z}_{\mathrm{Br}}(t) = \sum c_{f_i}(t)\, \widehat{Z}_i$.*

Proof. The proof follows the same argument as [Bru12, Theorem 7.1], whose main steps we recall here. Bruinier defines a space $M_k^!(\rho_L)$ of weakly holomorphic forms [Bru12, §4] of a certain "dual" weight k; each $f \in M_k^!(\rho_L)$ is defined by a finite collection of vectors $c_f(m) \in S(L)^\vee$ indexed by $m \in F$. Applying Bruinier's criterion for the modularity of a generating series, cf. [Bru12, (7.1)], we need to show that

$$\sum_m c_f(m)\widehat{Z}_{\mathrm{Br}}(m) = 0 \in \widehat{\mathrm{CH}}^1(X) \tag{3.7}$$

for all $f \in M_k^!(\rho_L)$. Let $c_0 = c_f(0)(0)$, and assume $c_0 \in \mathbb{Z}$. By [Bru12, Theorem 6.8], after replacing f by a sufficiently large integer multiple, there exists an analytic meromorphic section Ψ^{an} of $(\omega^{an})^{-c_0}$ such that

$$\mathrm{div}\,\Psi^{an} = \sum_{m \neq 0} c_f(m) \cdot Z(m)^{an}. \tag{3.8}$$

and with

$$-\log\|\Psi^{an}\|^2 = \sum_{m \neq 0} c_f(m) \cdot \Phi(m). \tag{3.9}$$

Recall that X is projective; by GAGA and the fact that the $Z(m)$'s are defined over E, there is an E-rational section ψ of ω^{c_0} and a constant $C \in \mathbb{C}$ such that

$$\mathrm{div}(\psi) = \sum_{m \neq 0} c_f(m)Z(m), \qquad -\log\|\psi^{an}\|^2 = -\log\|\Psi^{an}\|^2 + C. \tag{3.10}$$

Thus

$$-c_0 \cdot \widehat{\omega} \;=\; \widehat{\mathrm{div}}(\psi) = \sum_{m \neq 0} c_f(m) \cdot \widehat{Z}_{\mathrm{Br}}(m) \;+\; (0, C) \;\in\; \widehat{\mathrm{CH}}^1(X). \tag{3.11}$$

However, as in Remark 2.1, the class $(0, C) = 0$, and thus we find

$$\sum_m c_f(m)\widehat{Z}_{\mathrm{Br}}(m) \;=\; c_0 \cdot \widehat{\omega} + \sum_{m \neq 0} c_f(m)\widehat{Z}_{\mathrm{Br}}(m) \;=\; 0 \tag{3.12}$$

as required. $\qquad\square$

Now we consider the difference

$$\widehat{\phi}_1(\boldsymbol{\tau}) - \widehat{\phi}_{\mathrm{Br}}(\boldsymbol{\tau}) = \sum_t (0, \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t))\,q^t \tag{3.13}$$

whose terms are classes represented by purely archimedean cycles. Comparing the Green equations (2.25) and (3.3), we have that for $t \neq 0$ and any smooth form η,

$$\mathrm{dd}^c[\mathfrak{g}^o(t, v) - \Phi(t)](\eta) = \int_X (\mathfrak{g}^o(t, \mathbf{v}) - \Phi(t))\,\mathrm{dd}^c\eta = \int_X (\omega(t, \mathbf{v}) - c_\Lambda(t) - B(t)\Omega) \wedge \eta \tag{3.14}$$

where $\omega(t, \mathbf{v})$ is the t'th q-coefficient of $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$; in particular, elliptic regularity implies that the difference $\mathfrak{g}^o(t, \mathbf{v}) - \Phi(t)$ is smooth on X.

Theorem 3.2. *There exists a smooth $S(L)^\vee$-valued function $s(\boldsymbol{\tau}, z)$ on $\mathbb{H}_1^d \times X$ such that the following holds.*

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(i) For each fixed $z \in X$, the function $s(\boldsymbol{\tau}, z)$ transforms like a Hilbert modular form in $\boldsymbol{\tau}$.

(ii) Let

$$s(\boldsymbol{\tau}, z) = \sum_t c_s(t, \mathbf{v}, z) \, q^t \tag{3.15}$$

denote its q-expansion in $\boldsymbol{\tau}$; then for each t, we have

$$(0, \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t)) \;=\; (0, c_s(t, \mathbf{v}, z)) \;\in\; \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes_{\mathbb{C}} S(L)^{\vee} \tag{3.16}$$

Combining this theorem with Theorem 3.1, we obtain:

Corollary 3.3. *The generating series $\widehat{\phi}_1(\boldsymbol{\tau})$ is modular, valued in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes S(L)^{\vee}$, in the sense of Remark 2.8(i).* $\qquad\square$

Proof of Theorem 3.2. Recall that the $(1,1)$ form $-\Omega$ is a Kähler form on X. Let $-\Delta_X$ denote the corresponding Laplacian; the eigenvalues of $-\Delta_X$ are non-negative, discrete in $\mathbb{R}_{\geq 0}$, and each eigenspace is finite dimensional.

Write $\Delta_X = 2(\partial\partial^* + \partial^*\partial)$ and let $L \colon \eta \mapsto -\eta \wedge \Omega$ denote the Lefschetz operator. From the Kähler identities $[L, \partial] = [L, \overline{\partial}] = [L, \Delta_S] = 0$ and $[L, \partial^*] = i\overline{\partial}$, an easy induction argument shows that

$$\partial^* \circ L^k \;=\; L^k \circ \partial^* \;-\; ik\,\overline{\partial} \circ L^{k-1} \tag{3.17}$$

for $k \geq 1$.

Thus, for a smooth function ϕ on X, we have

$$\Delta_X(\phi) \cdot (-\Omega)^p = \Delta_X \circ L^p(\phi) = 2\partial\partial^* \circ L^p(\phi) \tag{3.18}$$

$$= 2\,\partial \circ \left(L^p \circ \partial^* - ip\overline{\partial} \circ L^{p-1} \right)(\phi) \tag{3.19}$$

$$= -2ip\,\partial\overline{\partial}\left(\phi \wedge (-\Omega)^{p-1}\right) \tag{3.20}$$

$$= -4\pi p\,\mathrm{dd}^{\mathrm{c}}\left(\phi \wedge (-\Omega)^{p-1}\right); \tag{3.21}$$

note here that $p = \dim_{\mathbb{C}}(X)$.

Consider the Hodge pairing

$$\langle f, g \rangle_{L^2} = \int_X f\,\overline{g}\,(-\Omega)^p = (-1)^p \int_X f\,\overline{g}\,\Omega^p. \tag{3.22}$$

If $\lambda > 0$ and ϕ_λ is a Laplace eigenfunction, we have that for any $t \neq 0$,

$$\langle \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t), \phi_\lambda \rangle_{L^2} = \lambda^{-1}\langle \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t), -\Delta_X\phi_\lambda \rangle_{L^2} \tag{3.23}$$

$$= (-1)^p \lambda^{-1} \int_X \left(\mathfrak{g}^o(t, \mathbf{v}) - \Phi(t)\right) \cdot \left(\overline{-\Delta_X\phi_\lambda}\right) \cdot \Omega^p \tag{3.24}$$

$$= (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X \left(\mathfrak{g}^o(t, \mathbf{v}) - \Phi(t)\right) \cdot \mathrm{dd}^{\mathrm{c}}\left(\overline{\phi}_\lambda\,\Omega^{p-1}\right) \tag{3.25}$$

$$= (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X \left(\omega(t, \mathbf{v}) - c_\Lambda(t) - B(t)\Omega\right) \wedge \overline{\phi}_\lambda\Omega^{p-1} \tag{3.26}$$

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Note that $\int_X \overline{\phi}_\lambda \Omega^p = \langle 1, \phi_\lambda \rangle_{L^2} = 0$, as $\lambda > 0$ and so ϕ_λ is orthogonal to constants; thus the term involving $B(m)\Omega$ vanishes, and so

$$\langle \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t), \phi_\lambda \rangle_{L^2} = (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X (\omega(t, \mathbf{v}) - c_\Lambda(t)) \wedge \overline{\phi}_\lambda \Omega^{p-1} \tag{3.27}$$

for all $t \neq 0$. This equality also holds for $t = 0$, as both sides of this equation vanish. Indeed, for the left hand side we have $\mathfrak{g}^o(0, \mathbf{v}) = 0$, cf. (2.24), and $\Phi(0) = 0$ by definition; on the right hand side, $c_\Lambda(0) = 0$ as $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau})$ is cuspidal, and the constant term of the Kudla-Millson theta function is given by

$$\omega(0, \mathbf{v}) = \Omega \otimes \mathrm{ev}_0. \tag{3.28}$$

Now define

$$h(\boldsymbol{\tau}, z) = (L^*)^{p-1} \circ * \, (\Theta_{\mathrm{KM}}(\boldsymbol{\tau}) - \Lambda_{\mathrm{KM}}(\boldsymbol{\tau})) \tag{3.29}$$

where $*$ is the Hodge star operator, and L^* is the adjoint of the Lefschetz map L. Clearly $h(\boldsymbol{\tau}, z)$ is a smooth function, and transforms like a modular form in $\boldsymbol{\tau}$, since both $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$ and $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau})$ do; writing its Fourier expansion

$$h(\boldsymbol{\tau}, z) = \sum_t c_h(t, \mathbf{v}, z) \, q^t, \tag{3.30}$$

we have

$$\langle c_h(t, \mathbf{v}, z), \phi \rangle_{L^2} = (-1)^{p-1} \int_X (\omega(t, \mathbf{v}) - c_\Lambda(t)) \wedge \overline{\phi} \, \Omega^{p-1} \tag{3.31}$$

for any smooth function ϕ.

Note that for any integer N and L^2 normalized eigenfunction ϕ_λ with $\lambda \neq 0$,

$$\langle h, \phi_\lambda \rangle_{L^2} = \lambda^{-N} \langle -\Delta_X^N(h), \varphi_\lambda \rangle \leq \lambda^{-N} \| -\Delta_X^N(h) \|_{L^2}^2. \tag{3.32}$$

Choose an orthonormal basis $\{\phi_\lambda\}$ of $L^2(X)$ consisting of eigenfunctions, and consider the sum

$$s(\tau, z) = 4\pi p \sum_{\lambda > 0} \lambda^{-1} \langle h, \phi_\lambda \rangle_{L^2} \, \phi_\lambda; \tag{3.33}$$

by Weyl's law, there are positive constants C_1 and C_2 such that

$$\#\{\lambda \mid \lambda < x\} \sim x^{C_1} \tag{3.34}$$

and $\|\phi_\lambda\|_{L^\infty} = O(\lambda^{C_2})$. Thus taking N sufficiently large in (3.32), we conclude that the sum (3.33) converges uniformly, and hence defines a smooth function in $(\boldsymbol{\tau}, z)$. Writing its Fourier expanison as

$$s(\boldsymbol{\tau}, z) = \sum_t c_s(t, \mathbf{v}, z) q^t, \tag{3.35}$$

we have

$$\langle c_s(t, \mathbf{v}, z), \phi_\lambda \rangle_{L^2} = \langle \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t), \phi_\lambda \rangle_{L^2} \tag{3.36}$$

for any eigenfunction ϕ_λ with $\lambda \neq 0$. Thus $c_s(t, \mathbf{v}, z)$ and $g^o(t, \mathbf{v}) - \Phi(t)$ differ by a function that is constant in z; as $(0, 1) = 0 \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X)$, we have

$$(0, \mathfrak{g}^o(t, \mathbf{v}) - \Phi(t)) = (0, c_s(t, \mathbf{v}, z)) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X) \otimes_{\mathbb{C}} S(L)^\vee, \tag{3.37}$$

which concludes the proof of the theorem.

$\square$

4 Decomposing Green currents

We now suppose $n > 1$ and fix $T_2 \in \mathrm{Sym}_{n-1}(F)$.

The aim of this section is to establish a decomposition $\widehat{Z}(T, \mathbf{v}) = \widehat{A}(T, \mathbf{v}) + \widehat{B}(T, \mathbf{v})$, where $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$. Our first step is to decompose Green forms in a useful way; the result can be seen as an extension of the star product formula [GS19, Theorem 4.10] to the degenerate case.

Let $x = (x_1, \ldots, x_n) \in (V_1)^n = (V \otimes_{F, \sigma_1} \mathbb{R})^n$ and set $y = (x_2, \ldots, x_n) \in V_1^{n-1}$. By [GS19, Proposition 2.6.(a)], we may decompose

$$\mathfrak{g}^o(x, \rho) = \int_1^\infty \nu^o(\sqrt{t}\, x_1) \wedge \varphi_{\mathrm{KM}}^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}}\ +\ \int_1^\infty \varphi_{\mathrm{KM}}^o(\sqrt{t}\, x_1) \wedge \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}} \tag{4.1}$$

for $Re(\rho) \gg 0$.

By the transgression formula (2.21), we may rewrite the second term in (4.1) as

$$\int_1^\infty \varphi_{\mathrm{KM}}^o(\sqrt{t}\, x_1) \wedge \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}} \tag{4.2}$$

$$= \int_1^\infty \left(\int_1^t \frac{\partial}{\partial u} \varphi_{\mathrm{KM}}^o(\sqrt{u}\, x_1)\, du \right) \wedge \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}} + \varphi_{\mathrm{KM}}^o(x_1) \wedge \int_1^\infty \nu^o(\sqrt{t}\, y) \frac{dt}{t^{1+\rho}} \tag{4.3}$$

$$= \int_1^\infty \left(\int_1^t -\mathrm{dd}^c \nu^o(\sqrt{u}\, x_1)\, \frac{du}{u} \right) \wedge \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}} + \varphi_{\mathrm{KM}}^o(x_1) \wedge \mathfrak{g}^o(y, \rho). \tag{4.4}$$

For $t > 1$, define smooth forms

$$\alpha_t(x_1, y) := \int_1^t \overline{\partial} \nu^o(\sqrt{u}\, x_1)\, \frac{du}{u} \wedge \nu^o(\sqrt{t}\, y) \tag{4.5}$$

and

$$\beta_t(x_1, y) := \int_1^t \nu^o(\sqrt{u}\, x_1)\, \frac{du}{u} \wedge \partial \nu^o(\sqrt{t}\, y) \tag{4.6}$$

so that

$$(4.4) = \frac{i}{2\pi} \int_1^\infty \partial \alpha_t(x_1, y) + \overline{\partial} \beta_t(x_1, y)\, \frac{dt}{t^{1+\rho}}\ -\ \int_1^\infty \left[\int_1^t \nu^o(\sqrt{u}\, x_1)\, \frac{du}{u} \right] \wedge \mathrm{dd}^c \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}}$$
$$+ \varphi_{\mathrm{KM}}^o(x_1) \wedge \mathfrak{g}^o(y, \rho). \tag{4.7}$$

Finally, we consider the second integral above; as $Re(\rho)$ is large, we may interchange the order of integration and obtain

$$\int_1^\infty \left(\int_1^t \nu^o(\sqrt{u}\, x)\, \frac{du}{u} \right) \wedge \mathrm{dd}^c \nu^o(\sqrt{t}\, y)\, \frac{dt}{t^{1+\rho}} \tag{4.8}$$

$$= \int_1^\infty \nu^o(\sqrt{u}\, x) \wedge \left(\int_u^\infty \mathrm{dd}^c \nu^o(\sqrt{t}\, y) \frac{dt}{t^{1+\rho}} \right) \frac{du}{u} \tag{4.9}$$

$$= \int_1^\infty \nu^o(\sqrt{u}\, x) \wedge \left(\int_u^\infty -\frac{\partial}{\partial t} \varphi_{\mathrm{KM}}^o(\sqrt{t}\, y) \frac{dt}{t^\rho} \right) \frac{du}{u} \tag{4.10}$$

$$= \int_1^\infty \nu^o(\sqrt{u}\, x) \wedge \varphi_{\mathrm{KM}}^o(\sqrt{u}\, y)\, \frac{du}{u^{1+\rho}} - \rho \int_1^\infty \nu^o(\sqrt{u}\, x) \wedge \left(\int_u^\infty \varphi_{\mathrm{KM}}^o(\sqrt{t}\, y) \frac{dt}{t^{1+\rho}} \right) \frac{du}{u} \tag{4.11}$$

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Note that the first term here coincides with the first term in (4.1). Combining these computations, it follows that

$$\mathfrak{g}^o(x,\rho) = \varphi^o_{\mathrm{KM}}(x_1) \wedge \mathfrak{g}^o(y,\rho) + \frac{i}{2\pi} \int_1^\infty \partial\alpha_t(x_1,y) + \overline{\partial}\beta_t(x_1,y) \frac{dt}{t^{1+\rho}}$$

$$+ \rho \int_1^\infty \nu^o(\sqrt{u}\,x) \wedge \left(\int_u^\infty \varphi^o_{\mathrm{KM}}(\sqrt{t}\,y) \frac{dt}{t^{1+\rho}} \right) \frac{du}{u}. \tag{4.12}$$

This identity holds for arbitrary $x = (x_1, y) \in V_1^n$ and $Re(\rho) \gg 0$, and is an identity of smooth differential forms on $\mathbb{D}$.

To continue, we view the above line as an identity of currents, and consider meromorphic continuation.[3] Note that (as currents)

$$\rho \int_1^\infty \nu^o(\sqrt{u}\,x) \wedge \left(\int_u^\infty \varphi^o_{\mathrm{KM}}(\sqrt{t}\,y) \frac{dt}{t^{1+\rho}} \right) \frac{du}{u} \tag{4.13}$$

$$= \rho \int_1^\infty \nu^o(\sqrt{u}\,x) \wedge \int_u^\infty \left(\varphi^o_{\mathrm{KM}}(\sqrt{t}\,y) - \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \right) \frac{dt}{t^{1+\rho}}$$

$$+ \int_1^\infty \nu^o(\sqrt{u}x) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \frac{du}{u^{1+\rho}} \tag{4.14}$$

where $r(y) = \dim \operatorname{span}(y)$. The first term vanishes at $\rho = 0$; indeed, the double integral in the first term is holomorphic at $\rho = 0$, as can easily seen by by Bismut's asymptotic [?, Theorem 3.2]

$$\varphi^o_{\mathrm{KM}}(\sqrt{t}y) - \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} = O(t^{-1/2}) \tag{4.15}$$

as $t \to \infty$.

Next, let

$$\alpha(x,y;\rho) := \int_1^\infty \alpha_t(x,y) \frac{dt}{t^{1+\rho}}, \qquad \beta(x,y;\rho) := \int_1^\infty \beta_t(x,y) \frac{dt}{t^{1+\rho}}. \tag{4.16}$$

A straightforward modification of the proof of [GS19, Proposition 2.12.(iii)] can be used to show that $\alpha(x_1, y; \rho)$ and $\beta(x_1, y; \rho)$ have meromorphic extensions, as currents, to a neighbourhood of $\rho = 0$. We denote the constant terms in the Laurent expansion at $\rho = 0$ by $\alpha(x_1, y)$ and $\beta(x_1, y)$ respectively. Thus, as currents on $\mathbb{D}$, we have

$$\mathfrak{g}^o(x_1,y) = \varphi^o_{\mathrm{KM}}(x_1) \wedge \mathfrak{g}^o(y) + d\alpha(x_1,y) + d^c\beta(x_1,y) \tag{4.17}$$

$$+ CT_{\rho=0} \int_1^\infty \nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \frac{du}{u^{1+\rho}} \tag{4.18}$$

for all $x_1 \in V_1$ and $y \in (V_1)^{n-1}$.

As a final observation, note that if $x_1 \in \operatorname{span}(y)$, then $\nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} = \delta_{\mathbb{D}_y^+}$, see [GS19, Lemma 2.4]. Thus

$$\gamma(x_1,y) := CT_{\rho=0} \int_1^\infty \nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \frac{du}{u^{1+\rho}} \tag{4.19}$$

$$= \begin{cases} \mathfrak{g}^o(x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)}, & \text{if } x_1 \notin \operatorname{span}(y), \\ 0, & \text{if } x_1 \in \operatorname{span}(y) \end{cases} \tag{4.20}$$

---

[3]More precisely, we mean that for every smooth form α, the function $[\mathfrak{g}^o(x,\rho)](\alpha) = \int_X \mathfrak{g}^o(x,\rho) \wedge \alpha$ admits a meromorphic continuation in ρ, such that the Laurent coefficients are continuous in α in the sense of currents.

In particular, we recover the star product formula [GS19, Theorem 2.16] in the non-degenerate case.

Now we discuss a decomposition of the global Green current $\mathfrak{g}^o(T, \mathbf{v})$, for $\mathbf{v} \in \mathrm{Sym}_n(F_\mathbb{R})_{\gg 0}$. Write

$$v := \sigma_1(\mathbf{v}) = \begin{pmatrix} v_1 & v_{12} \\ v_{12}' & v_2 \end{pmatrix} \tag{4.21}$$

with $v_1 \in \mathbb{R}_{>0}$ and $v_{12} \in M_{1,n-1}(\mathbb{R})$; recall that $\sigma_1 \colon F \to \mathbb{R}$ is the distinguished real embedding. Set

$$v_2^* := v_2 - v_{12}' v_{12}/v_1 \in \mathrm{Sym}_{n-1}(\mathbb{R})_{>0}, \tag{4.22}$$

and fix a matrix $a_2^* \in \mathrm{GL}_{n-1}(\mathbb{R})$ such that $v_2^* = a_2^* \cdot (a_2^*)'$.

Proposition 4.1. *Let* $T \in \mathrm{Sym}_n(F)$ *and* $\mathbf{v} \in \mathrm{Sym}_n(F_\mathbb{R})_{\gg 0}$ *as above, and define* $S(L)^\vee$*-valued currents* $\mathfrak{a}(T, \mathbf{v})$ *and* $\mathfrak{b}(T, \mathbf{v})$ *on* X *by the formulas*

$$\mathfrak{a}(T, \mathbf{v})(\varphi) := \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \gamma(\sqrt{v_1} x_1, y), \tag{4.23}$$

where we have written $\sigma_1(\mathbf{x}) = (x_1, y) \in V_1 \oplus (V_1)^{n-1}$, *and*

$$\mathfrak{b}(T, \mathbf{v})(\varphi) = \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \varphi^o_{\mathrm{KM}}\left(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}^o(y a_2^*). \tag{4.24}$$

Then

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) \equiv \mathfrak{a}(T, \mathbf{v})(\varphi) + \mathfrak{b}(T, \mathbf{v})(\varphi) \pmod{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}}. \tag{4.25}$$

Proof. First, the fact that the sums defining $\mathfrak{a}(T, \mathbf{v})$ and $\mathfrak{b}(T, \mathbf{v})$ converge to currents on X follows from the same argument as [GS19, Proposition 4.3].

Now recall that

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) = \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \mathfrak{g}^o(xa) \tag{4.26}$$

where $x = \sigma_1(\mathbf{x})$, and $a \in \mathrm{GL}_n(\mathbb{R})$ satisfies $v = aa'$. Note that

$$v = \begin{pmatrix} v_1 & v_{12} \\ v_{12}' & v_2 \end{pmatrix} = \theta \begin{pmatrix} v_1 & \\ & v_2^* \end{pmatrix} \theta', \qquad \text{where } \theta = \begin{pmatrix} 1 & \\ v_{12}'/v_1 & 1_{n-1} \end{pmatrix}. \tag{4.27}$$

Thus, we may take

$$a = \theta \cdot \begin{pmatrix} \sqrt{v_1} & \\ & a_2^* \end{pmatrix}, \tag{4.28}$$

and so, applying (4.18), we find

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) = \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \mathfrak{g}^o\left((x_1, y) \theta \begin{pmatrix} \sqrt{v_1} & \\ & a_2^* \end{pmatrix}\right) \qquad x = (x_1, y) \tag{4.29}$$

$$= \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \mathfrak{g}^o\left(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, \, y a_2^*\right) \tag{4.30}$$

$$= \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \left(\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}^o(y a_2^*) + \partial\alpha(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, \, y a_2^*)\right.$$

$$\left. + \overline{\partial}\beta(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, \, y a_2^*) + \gamma(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, \, y a_2^*)\right). \tag{4.31}$$

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Again, an argument as in [GS19, Proposition 4.3] shows that the sums

$$\eta_1 := \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \alpha(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) \tag{4.32}$$

and

$$\eta_2 := \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \beta(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) \tag{4.33}$$

converge to Γ-invariant currents on $\mathbb{D}$, and hence define currents on X. Moreover, it follows easily from the definitions that

$$\gamma(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) = \gamma(\sqrt{v_1} x_1, y). \tag{4.34}$$

Thus, we find

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) = \mathfrak{a}(T, \mathbf{v})(\varphi) + \mathfrak{b}(T, \mathbf{v})(\varphi) + \partial \eta_1 + \overline{\partial} \eta_2, \tag{4.35}$$

as required. $\qquad\square$

Next, we define an $S_L^\vee$-valued current $\psi(T, \mathbf{v})$ as follows. For $\mathbf{x} \in \Omega(T)$, write $\sigma_1(\mathbf{x}) = x = (x_1, y) \in V_1 \oplus V_1^{n-1}$ as above; then

$$\psi(T, \mathbf{v})(\varphi) := \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \varphi_{\mathrm{KM}}^o(\sqrt{v_1} x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \tag{4.36}$$

defines a Γ-equivariant current on $\mathbb{D}^+$, and hence descends to a current (also denoted $\psi(T, \mathbf{v})$) on X.

Lemma 4.2. *(i) Let $\omega(T, \mathbf{v})$ be the Tth coefficient of the Kudla-Millson theta function, as in* (2.27)*; then*

$$\mathrm{dd}^c \mathfrak{b}(T, \mathbf{v}) = \omega(T, \mathbf{v}) - \psi(T, \mathbf{v}). \tag{4.37}$$

(ii) We have

$$\mathrm{dd}^c \mathfrak{a}(T, \mathbf{v}) + \delta_{Z(T)(\mathbb{C})} \wedge \Omega^{n-r(T)} = \psi(T, \mathbf{v}), \tag{4.38}$$

where $r(T) = \mathrm{rank}(T)$.

Proof. With $v = \sigma_1(\mathbf{v})$ and taking $a = \theta \cdot \begin{pmatrix} \sqrt{v_1} & \\ & a_2^* \end{pmatrix}$ as (4.28), we have

$$\omega(T, \mathbf{v})(\varphi) = \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \varphi_{\mathrm{KM}}^o(xa) \tag{4.39}$$

$$= \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \varphi_{\mathrm{KM}}^o\left(\sqrt{v_1} x_1 + \frac{y \, v_{12}'}{\sqrt{v_1}}, \, ya_2^*\right) \tag{4.40}$$

$$= \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \, \varphi_{\mathrm{KM}}^o\left(\sqrt{v_1} x_1 + \frac{y v_{12}'}{\sqrt{v_1}}\right) \wedge \varphi_{\mathrm{KM}}^o(ya_2^*) \tag{4.41}$$

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for $\varphi \in S(L^n)$, where the last line follows from [KM90, Theorem 5.2(i)]. Therefore,

$$\mathrm{dd}^c \mathfrak{b}(T,\mathbf{v})(\varphi) = \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\,v'_{12}}{\sqrt{v_1}}\right) \wedge \mathrm{dd}^c\mathfrak{g}^o(ya_2^*) \tag{4.42}$$

$$= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\,v'_{12}}{\sqrt{v_1}}\right) \wedge \left\{ -\delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r_2} + \varphi^o_{\mathrm{KM}}(ya_2^*)\right\} \tag{4.43}$$

$$= - \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\,v'_{12}}{\sqrt{v_1}}\right) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r_2} + \omega(T,\mathbf{v})(\varphi). \tag{4.44}$$

For $v \in V_1$, the restriction $\varphi^o_{\mathrm{KM}}(v) \wedge \delta_{\mathbb{D}_y^+}$ depends only on the orthogonal projection of v onto $\mathrm{span}(y)^\perp$; see, for example, [GS19, Lemma 2.4]. In particular,

$$\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\,v'_{12}}{\sqrt{v_1}}\right) \wedge \delta_{\mathbb{D}_y^+} = \varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1\right) \wedge \delta_{\mathbb{D}_y^+}. \tag{4.45}$$

The first part of the lemma follows upon applying the definition of $\gamma(T,\mathbf{v})$ in (4.36).

The second part then follows from the first, together with Proposition 4.1 and (2.29). $\qquad\square$

We finally arrived at the promised decomposition of $\widehat{Z}(T,\mathbf{v})$. Recall that in defining the cycle $\widehat{Z}(T,\mathbf{v})$ in Section 2.5, we fixed a representative (Z_0, g_0) for $\widehat{\omega}^{n-r(T)}$ such that Z_0 intersects $Z(T)$ properly. By the previous proposition,

$$\mathrm{dd}^c\left(\mathfrak{a}(T,\mathbf{v}) + g_0 \wedge \delta_{Z(T)(\mathbb{C})}\right) + \delta_{Z(T)\cap Z_0(\mathbb{C})} \;=\; \psi(T,\mathbf{v}); \tag{4.46}$$

we then obtain classes in $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L^n)$ by setting

$$\widehat{A}(T,\mathbf{v}) \;:=\; \left(Z(T) \cdot Z_0, \, [\psi(T,\mathbf{v}),\, \mathfrak{a}(T,\mathbf{v}) + g_0 \wedge \delta_{Z(T)(\mathbb{C})}]\right) \tag{4.47}$$

and

$$\widehat{B}(T,\mathbf{v}) \;:=\; \left(0, \, [\omega(T,\mathbf{v}) - \psi(T,\mathbf{v}),\, \mathfrak{b}(T,\mathbf{v})]\right), \tag{4.48}$$

so that

$$\widehat{Z}(T,\mathbf{v}) = \widehat{A}(T,\mathbf{v}) + \widehat{B}(T,\mathbf{v}) \;\in\; \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L)^\vee \tag{4.49}$$

Remark 4.3. Suppose $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$ as above; if T_2 is not totally positive semidefinite, then $\mathbb{D}_y^+ = \emptyset$ for any $\mathbf{y} \in \Omega(T_2)$, and hence $\widehat{A}(T,\mathbf{v}) = 0$.

5 Modularity I

In this section, we establish the modularity of the generating series

$$\widehat{\phi}_B(\boldsymbol{\tau}) = \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{B}(T,\mathbf{v})\, q^T. \tag{5.1}$$

Note that

$$\widehat{B}(T,\mathbf{v}) = (0, [\mathrm{dd}^c\mathfrak{b}(T,\mathbf{v}), \mathfrak{b}(T,\mathbf{v})]) = a(\mathfrak{b}(T,v)); \tag{5.2}$$

thus in light of Definition 2.7, it suffices to establish the following theorem.

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Theorem 5.1. *Fix $T_2 \in \mathrm{Sym}_{n-1}(F)$, and consider the generating series*

$$\xi(\boldsymbol{\tau}) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix} \right)} \mathfrak{b}(T, \mathbf{v}) \, q^T, \tag{5.3}$$

Then $\xi(\boldsymbol{\tau})$ is an element of $A_{\kappa, T_2}(\rho_L^\vee; D^*(X))$, see Definition 2.6.

Proof. We begin by showing the convergence of the series (5.3). By linearity, we may assume $\varphi = \varphi_1 \otimes \varphi_2$ where $\varphi_1 \in S(L)$ and $\varphi_2 \in S(L^{n-1})$, so that

$$\sum_{T = \left(\begin{smallmatrix} T_1 & T_{12} \\ T'_{12} & T_2 \end{smallmatrix} \right)} \mathfrak{b}(T, \mathbf{v})(\varphi) \, q^T = \sum_{T} \sum_{(\mathbf{x}_1, \mathbf{y}) \in \Omega(T)} \varphi_2(\mathbf{y}) \, \varphi_1(\mathbf{x}_1) \, \varphi_{\mathrm{KM}}^o \left(\sqrt{v_1} x_1 + \frac{y \cdot v'_{12}}{\sqrt{v_1}} \right) \wedge \mathfrak{g}^o(y a_2^*) \, q^T \tag{5.4}$$

where $x_1 = \sigma_1(\mathbf{x}_1)$ and $y = \sigma_1(\mathbf{y})$; here we work with Γ-equivariant currents on $\mathbb{D}^+$. For $v \in V_1$, consider the normalized Kudla-Millson form

$$\varphi_{\mathrm{KM}}(v) := e^{-2\pi \langle v, v \rangle} \, \varphi_{\mathrm{KM}}^o(v) \tag{5.5}$$

which is a Schwartz form on V_1, valued in closed forms on $\mathbb{D}^+$; more precisely, for any integer k and relatively compact open subset $U \subset \mathbb{D}^+$, there exists a positive definite quadratic form Q_U on V_1 such that

$$\|\varphi_{\mathrm{KM}}(v)\|_{k, \overline{U}} \ll e^{-Q_U(v)} \tag{5.6}$$

where $\| \cdot \|_{k, \overline{U}}$ is an algebra seminorm measuring uniform convergence of all derivatives of order $\leq k$ on the space of smooth forms supported on $\overline{U}$, and the implied constant depends on k and $\overline{U}$. Similarly, for $y \in V_1^{n-1}$, write

$$\mathfrak{g}(y) = e^{-2\pi \sum \langle y_i, y_i \rangle} \mathfrak{g}^o(y); \tag{5.7}$$

if $\mathbb{D}_y^+ \cap \overline{U} = \emptyset$, then $\mathfrak{g}(y)$ is smooth on U, and the form Q_U may be chosen so that

$$\|\mathfrak{g}(y)\|_{k, \overline{U}} \ll e^{-\sum_{i=1}^{n-1} Q_U(y_i)}, \qquad y = (y_1, \ldots, y_{n-1}) \in V_1^{n-1}, \tag{5.8}$$

see [GS19, §2.1.5].

Finally, for the remaining real embeddings $\sigma_2, \ldots \sigma_d$, let $\varphi_{\infty_i} \in S(V_i^n)$ denote the standard Gaussian on the positive definite space $V_i = V \otimes_{F, \sigma_i} \mathbb{R}$, defined by $\varphi_{\infty_i}(x_1, \ldots, x_n) = e^{-2\pi \sum \langle x_i, x_i \rangle}$. Then a brief calculation gives

$$\xi(\boldsymbol{\tau})(\varphi) = \sum_{T} \sum_{(\mathbf{x}_1, \mathbf{y}) \in \Omega(T)} \varphi_1(\mathbf{x}_1) \varphi_2(\mathbf{y}) \varphi_{\mathrm{KM}} \left(\sqrt{v_1} x_1 + \frac{y \cdot v'_{12}}{\sqrt{v_1}} \right) \wedge \mathfrak{g}(y a_2^*) \cdot \prod_{i=2}^{d} \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1, \mathbf{y}) a_i) \, e(T\mathbf{u}) \tag{5.9}$$

where we have chosen matrices $a_i \in GL_n(\mathbb{R})$ for $i = 2, \ldots, d$, such that $\sigma_i(\mathbf{v}) = a_i \cdot a_i'$. Let

$$S_1 := \{\mathbf{y} \in (L')^{n-1} \, | \langle \mathbf{y}, \mathbf{y} \rangle = T_2 \text{ and } \mathbb{D}_y^+ \cap \overline{U} \neq \emptyset\} \tag{5.10}$$

which is a finite set, and let

$$S_2 := \{\mathbf{y} \in (L')^{n-1} \, | \langle \mathbf{y}, \mathbf{y} \rangle = T_2 \text{ and } \mathbb{D}_y^+ \cap \overline{U} = \emptyset\}. \tag{5.11}$$

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Using the estimates (5.6) and (5.8), and standard arguments for convergence of theta series, it follows that for $\varphi = \varphi_1 \otimes \varphi_2$, the sum

$$\sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \sum_{\substack{(\mathbf{x}_1,\mathbf{y}) \in \Omega(T) \\ \mathbf{y} \in S_2}} \varphi_1(\mathbf{x}_1)\varphi_2(\mathbf{y})\varphi_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y \cdot v'_{12}}{\sqrt{v_1}}\right) \wedge \mathfrak{g}(ya_2^*) \cdot \prod_{i=2}^d \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y})a_i) \; e(T\mathbf{u})$$

(5.12)

converges absolutely to a smooth form on $\mathbb{H}_n \times U$. The (finitely many) remaining terms, corresponding to $\mathbf{y} \in S_1$, can be written as

$$\sum_{\mathbf{y} \in S_1} f_{\mathbf{y}}(\boldsymbol{\tau})(\varphi_1 \otimes \varphi_2) \wedge \mathfrak{g}(ya_2^*)$$

(5.13)

where, for any $\mathbf{y} \in V^{n-1}$ and $\varphi \in S(L^n)$, we set

$$f_{\mathbf{y}}(\boldsymbol{\tau})(\varphi) \;=\; \sum_{\mathbf{x}_1 \in V} \varphi(\mathbf{x}_1,\mathbf{y})\varphi_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y \cdot v'_{12}}{\sqrt{v_1}}\right) \prod_{i=2}^d \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y})a_i) \; e(T(\mathbf{x}_1,\mathbf{y})\mathbf{u}).$$

(5.14)

where $T(\mathbf{x}_1,\mathbf{y}) = \left(\begin{smallmatrix} \langle \mathbf{x}_1,\mathbf{x}_1 \rangle & \langle \mathbf{x}_1,\mathbf{y} \rangle \\ \langle \mathbf{x}_1,\mathbf{y} \rangle' & \langle \mathbf{y},\mathbf{y} \rangle \end{smallmatrix}\right)$. Again, the estimate (5.6) shows that the series defining $f_{\mathbf{y}}(\boldsymbol{\tau})$ converges absolutely to a smooth form on $\mathbb{H}_d^n \times \mathbb{D}^+$. Moreover, for a fixed $y \in V_1^{n-1}$ and any compactly supported test form α on $\mathbb{D}^+$, the value of the current $\mathfrak{g}^o(ya_2^*)[\alpha]$ varies smoothly in the entries of a_2^* (this fact follows easily from the discussion in [GS19, §2.1.4]).

Taken together, the above considerations imply that the series $\xi(\boldsymbol{\tau})(\varphi_1 \otimes \varphi_2)$ converges absolutely to a Γ-invariant current on $\mathbb{D}^+$, and therefore descends to a current on X that moreover satisfies part (b) of Definition 2.6 as $\boldsymbol{\tau}$ varies. By linearity, the same is true for arbitrary $\varphi \in S(L^n)$. In addition, we have shown that for any test form α, the value of the current $\xi(\boldsymbol{\tau})[\alpha]$ is smooth in $\boldsymbol{\tau}$.

It remains to show that $\xi(\boldsymbol{\tau})$ transforms like a Jacobi modular form, i.e. is invariant under the slash operators (2.38). To this end, note that (by Vaserstein's theorem), every element of $\widetilde{\Gamma}^J$ can be written as a product of the following elements.

(i) For each $i = 1, \ldots, d$, let

$$\widetilde{\epsilon}(i) = (\widetilde{\epsilon}(i))_v \in \prod_{v|\infty} \widetilde{\mathrm{Sp}}_n(F_v)$$

(5.15)

be the element whose v'th component is $(\mathrm{Id}, 1)$ if $v \neq \sigma_i$, and $(\mathrm{Id}, -1)$ if $v = \sigma_i$.

(ii) For $\mu, \lambda \in \mathrm{M}_{1,n-1}(\mathcal{O}_F)$, let

$$\gamma_{\lambda,\mu} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & \mu \\ \lambda' & 1_{n-1} & \mu' & 0 \\ \hline 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1_{n-1} \end{array}\right) \in G^J(\mathcal{O}_F).$$

(5.16)

Let $\iota_F(\gamma_{\lambda,\mu}) \in \widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ denote its image under the splitting (2.36); we choose $\widetilde{\gamma}_{\lambda,\mu} \in \widetilde{\Gamma}^J$ to be the archimedean part of a representative $\iota_F(\gamma_{\lambda,\mu}) = \widetilde{\gamma}_{\lambda,\mu} \cdot \widetilde{\gamma}_f$.

(iii) For $r \in \mathcal{O}_F$, let

$$\gamma_r = \left(\begin{array}{cc|cc} 1 & 0 & r & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{array} \right) \in G^J(\mathcal{O}_F), \tag{5.17}$$

and choose an element $\widetilde{\gamma}_r$ as the archimedean part of a representative of $\iota_F(\gamma_r)$, as before.

(iv) Finally, let

$$S = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{array} \right) \tag{5.18}$$

and take $\widetilde{S} \in \widetilde{\Gamma}^J$ to be the archimedean part of a representative of $\iota_F(S)$.

Now, rearranging the absolutely convergent sum (5.12), we have

$$\xi(\boldsymbol{\tau}) \;=\; \sum_{\mathbf{y} \in \Omega(T_2)} f_{\mathbf{y}}(\boldsymbol{\tau}) \wedge \mathfrak{g}(y a_2^*). \tag{5.19}$$

Using the aforementioned generators, a direct computation shows that $\mathbf{v}_2^* = \mathbf{v}_2 - \mathbf{v}_{12}' \mathbf{v}_{12}/\mathbf{v}_1$, viewed as a function on $\mathbb{H}_n^d$, is invariant under the action of $\widetilde{\Gamma}^J$; it therefore suffices to show that for a fixed $\mathbf{y}$, the $S(L^n)^{\vee}$-valued function $f_{\mathbf{y}}(\boldsymbol{\tau})$ transforms like a Jacobi form.

It is a straightforward verification to check that $f_{\mathbf{y}}(\boldsymbol{\tau})$ is invariant under the action of $\widetilde{\epsilon}(i)$, $\tilde{\gamma}_{\lambda,\mu}$, and $\tilde{\gamma}_r$. For example, the element $\widetilde{\gamma}_{\lambda,\mu}^{-1}$ acts on $S(L^n)$ by the formula

$$\rho(\widetilde{\gamma}_{\lambda,\mu}^{-1})(\varphi)(\mathbf{x}_1, \mathbf{y}) \;=\; e\big(2\langle \mathbf{x}_1, \mathbf{y}\rangle \mu' - \langle \mathbf{y}\lambda', \mathbf{y}'\rangle \mu'\big)\, \varphi(\mathbf{x}_1 - \mathbf{y}\lambda',\, \mathbf{y}) \tag{5.20}$$

and $\gamma_{\lambda,\mu}$ acts on $\mathbb{H}_n^d$ by the formula

$$\gamma_{\lambda,\mu} \cdot \boldsymbol{\tau} = \left(\begin{array}{cc} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} + \boldsymbol{\tau}_1 \lambda + \mu \\ \boldsymbol{\tau}_{12}' + \boldsymbol{\tau}_1 \lambda' + \mu' & \boldsymbol{\tau}_2 + (\lambda' \cdot \boldsymbol{\tau}_{12} + \boldsymbol{\tau}_{12}' \cdot \lambda) + \mu' \cdot \lambda \end{array} \right) \tag{5.21}$$

where $\boldsymbol{\tau} = \left(\begin{smallmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}_{12}' & \boldsymbol{\tau}_2 \end{smallmatrix} \right)$, and writing $\widetilde{\gamma}_{\lambda,\mu} = (\gamma_{\lambda,\mu}, (\phi_v))_v$ as in , we have $\prod \phi_v(\tau) = 1$. For $\mathbf{x}_1 \in V$ and $\mathbf{y} \in V^{n-1}$, a direct computation gives

$$\mathrm{tr}\Big(T(\mathbf{x}_1, \mathbf{y}) \cdot \mathrm{Re}(\gamma_{\lambda,\mu} \cdot \boldsymbol{\tau})\Big) = \mathrm{tr}\Big(T(\mathbf{x}_1 + \mathbf{y}\lambda', \mathbf{y})\mathbf{u}\Big) + 2\langle \mathbf{x}_1, \mathbf{y}\rangle \mu' + \langle \mathbf{y}\lambda', \mathbf{y}\rangle \mu'; \tag{5.22}$$

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therefore, applying the above identity and the change of variables $\mathbf{x}_1 \mapsto \mathbf{x}_1 - \mathbf{y} \cdot \lambda'$, we find

$$f_{\mathbf{y}}(\gamma_{\lambda,\mu} \cdot \boldsymbol{\tau})(\varphi) = \sum_{\mathbf{x}_1 \in V} \varphi(\mathbf{x}_1, \mathbf{y}) \varphi_{\mathrm{KM}} \left(\sqrt{v_1}(x_1 + y \cdot \lambda') + \frac{y \cdot v'_{12}}{\sqrt{v_1}} \right)$$

$$\times \left\{ \prod_{i=2}^{d} \varphi_{\infty_i} \left(\sigma_i(\mathbf{x}_1, \mathbf{y}) \left(\begin{smallmatrix} 1 \\ \lambda' \ 1 \end{smallmatrix} \right) a_i \right) \right\} e\Big(T(\mathbf{x}_1, \mathbf{y}) \mathrm{Re}(\gamma_{\lambda,\mu} \boldsymbol{\tau}) \Big) \qquad (5.23)$$

$$= \sum_{\mathbf{x}_1 \in V} \left\{ \varphi(\mathbf{x}_1 - \mathbf{y}\lambda') \, e(2\langle \mathbf{x}_1, \mathbf{y} \rangle \mu' - \langle \mathbf{y}\lambda', \mathbf{y} \rangle \mu') \right\} \varphi_{\mathrm{KM}} \left(\sqrt{v_1} x_1 + \frac{y \cdot v'_{12}}{\sqrt{v_1}} \right)$$

$$\times \prod_{i=2}^{d} \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1, \mathbf{y}) a_i) \, e(T(\mathbf{x}_1, \mathbf{y})\mathbf{u}) \qquad (5.24)$$

$$= f_{\mathbf{y}}(\boldsymbol{\tau}) \left(\rho(\widetilde{\gamma}_{\lambda,\mu}) \varphi \right). \qquad (5.25)$$

as required.

As for $\widetilde{S}$, recall that $\iota_F(S)$ acts on $S(V(\mathbb{A})^n)$ by the partial Fourier transform in the first variable; the desired invariance follows from Poisson summation on $\mathbf{x}_1$ and straightforward identities for the behaviour of the Fourier transform under translations and dilations. $\qquad \square$

6   Modularity II

In this section, we prove the modularity of the generating series $\widehat{\phi}_A(\boldsymbol{\tau})$. By Remark 4.3, we only need to consider the case that T_2 is totally positive semidefinite.

We begin by fixing an element $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}) \in \Omega(T_2)$, and setting $y = \sigma_1(\mathbf{y})$. Let

$$U_{\mathbf{y}} = \mathrm{span}(\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}) \subset V, \qquad (6.1)$$

so that $U_{\mathbf{y}}$ is totally positive definite. Let

$$\Lambda_{\mathbf{y}} := U_{\mathbf{y}} \cap L, \qquad \text{and} \qquad \Lambda_{\mathbf{y}}^{\perp} := U_{\mathbf{y}}^{\perp} \cap L \qquad (6.2)$$

and set

$$\Lambda := \Lambda_{\mathbf{y}} \oplus \Lambda_{\mathbf{y}}^{\perp} \subset L, \qquad (6.3)$$

so that

$$\Lambda \subset L \subset L' \subset \Lambda'. \qquad (6.4)$$

In light of the definition, cf. (1.3), we have a natural inclusion $S(L^n) \to S(\Lambda^n)$, and the composition

$$S(L^n) \to S(\Lambda^n) \xrightarrow{\sim} S(\Lambda_{\mathbf{y}}^n) \otimes S((\Lambda_{\mathbf{y}}^{\perp})^n). \qquad (6.5)$$

is equivariant for the action of $\widetilde{\Gamma}^J$, via ρ_L on the left hand side, and via $\rho_{\Lambda_{\mathbf{y}}} \otimes \rho_{\Lambda_{\mathbf{y}}^{\perp}}$ on the right; this fact can be deduced from explicit formulas for the Weil representation, cf. [Kud96, Proposition II.4.3].

Note that $U_{\mathbf{y}}^{\perp}$ is a quadratic space of signature $((p', 2), (p' + 2, 0), \ldots (p' + 2, 0))$ with $p' = p - \mathrm{rank}(T_2)$, so the constructions in Section 2 apply equally well in this case. In particular, let $X_{\mathbf{y}}(\mathbb{C}) = \Gamma_{\mathbf{y}} \backslash \mathbb{D}_y^+$. Then for $m \in F$ and $\mathbf{v}_1 \in (F_{\mathbb{R}})_{\gg 0}$, we have a special divisor

$$\widehat{Z}_{U_{\mathbf{y}}^{\perp}}(m, \mathbf{v}_1) \in \widehat{\mathrm{CH}}_{\mathbb{C}}^1(X_{\mathbf{y}}) \otimes S(\Lambda_{\mathbf{y}}^{\perp})^{\vee}, \qquad (6.6)$$

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where we introduce $U_{\mathbf{y}}^{\perp}$ in the notation to emphasize the underlying quadratic space.

Let

$$\pi_{\mathbf{y}} \colon X_{\mathbf{y}} \to X \tag{6.7}$$

denote the natural embedding, whose image is the cycle $Z(\mathbf{y})$ of codimension $\operatorname{rank}(T_2)$. We define a class

$$\widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1) \in \widehat{\mathrm{CH}}^{\mathrm{rk}(T_2)+1}(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S((\Lambda_{\mathbf{y}}^{\perp})^n) \tag{6.8}$$

as follows: suppose $\varphi \in S((\Lambda_{\mathbf{y}}^{\perp})^n)$ is of the form $\varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(\Lambda_{\mathbf{y}}^{\perp})$ and $\varphi_2 \in S((\Lambda_{\mathbf{y}}^{\perp})^{n-1})$. Then, using the pushforward $\pi_{\mathbf{y},*}$, we define the class

$$\widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1)(\varphi_1 \otimes \varphi_2) := \varphi_2(0) \cdot \pi_{\mathbf{y},*}\left(\widehat{Z}_{U_{\mathbf{y}}^{\perp}}(m, \mathbf{v}_1, \varphi_1)\right) \in \widehat{\mathrm{CH}}^{\mathrm{rk}(T_2)+1}(X, \mathcal{D}_{\mathrm{cur}}); \tag{6.9}$$

we extend this definition to arbitrary φ by linearity.

Finally, for $\boldsymbol{\tau} = \left(\begin{smallmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}_{12}' & \boldsymbol{\tau}_2 \end{smallmatrix}\right) \in \mathbb{H}_d^n$, consider the generating series

$$\widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}_1) := \sum_{m \in F} \widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1) \, q_1^m \tag{6.10}$$

where $\boldsymbol{\tau}_1 \in \mathbb{H}_1^d$ with $\mathbf{v}_1 = \mathrm{Im}(\boldsymbol{\tau}_1)$, and $q_1^m = e(m\boldsymbol{\tau}_1)$.

We also have a classical theta function attached to the totally positive definite space $U_{\mathbf{y}}$, defined as follows: let $\varphi \in S(\Lambda_{\mathbf{y}}^n)$ and suppose $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(\Lambda_{\mathbf{y}})$ and $\varphi_2 \in S(\Lambda_{\mathbf{y}}^{n-1})$. Then we set

$$\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi_1 \otimes \varphi_2) := \varphi_2(\mathbf{y}) \sum_{\lambda \in U_{\mathbf{y}}} \varphi_1(\lambda) \, e\left(\langle \lambda, \lambda \rangle \boldsymbol{\tau}_1 + 2\langle \lambda, \mathbf{y} \rangle \boldsymbol{\tau}_{12}'\right) e(T_2 \cdot \boldsymbol{\tau}_2). \tag{6.11}$$

It is well-known that $\theta_{\mathbf{y}}(\boldsymbol{\tau})$ is a holomorphic Jacobi modular form of weight $\dim U_{\mathbf{y}}/2 = \mathrm{rk}(T_2)/2$ and index T_2, see e.g. [EZ85, §II.7].

The Fourier expansion of $\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi)$ can be written, for $\varphi = \varphi_1 \otimes \varphi_2$ as above, as

$$\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi_1 \otimes \varphi_2) = \varphi_2(\mathbf{y}) \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} r_{\mathbf{y}}(T, \varphi_1) \, q^T, \tag{6.12}$$

where $r_{\mathbf{y}}(T) \in S(\Lambda_{\mathbf{y}})^{\vee}$ is given by the formula

$$r_{\mathbf{y}}\left(\left(\begin{smallmatrix} T_1 & T_{12} \\ T_{12}' & T_2 \end{smallmatrix}\right), \varphi_1\right) = \sum_{\substack{\lambda \in U_{\mathbf{y}} \\ \langle \lambda, \lambda \rangle = T_1 \\ \langle \lambda, \mathbf{y} \rangle = T_{12}}} \varphi_1(\lambda) \tag{6.13}$$

Finally, we note that given T as above, we must have either $\mathrm{rank}(T) = \mathrm{rank}(T_2)+1$, or $\mathrm{rank}(T) = \mathrm{rank}(T_2)$.

Lemma 6.1. *Suppose* $\mathrm{rank}(T) = \mathrm{rank}(T_2) + 1$. *Then for any* $\mathbf{y} \in \Omega(T_2)$, *we have* $r_{\mathbf{y}}(T) = 0$.

Proof. Suppose $r_{\mathbf{y}}(T) \neq 0$; then there exists a tuple $(\lambda, \mathbf{y}) \in \Omega(T)$ with $\mathrm{span}(\lambda, \mathbf{y}) = \mathrm{span}(\mathbf{y})$, which contradicts the assumption on $\mathrm{rank}(T)$.

$\square$

Proposition 6.2. *As formal generating series, we have*

$$\widehat{\phi}_A(\boldsymbol{\tau}) = \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{A}(T,\mathbf{v})\, q^T = \sum_{\substack{\mathbf{y}\in\Omega(T_2) \\ \mathrm{mod}\,\Gamma}} \widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}) \cdot \widehat{\omega}^{\,n-r(T_2)-1} \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau}), \tag{6.14}$$

where

$$\widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}) \cdot \widehat{\omega}^{\,n-r(T_2)-1} \;:=\; \sum_{m\in F} \widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1) \cdot \widehat{\omega}^{\,n-r(T_2)-1}\, q_1^m; \tag{6.15}$$

here, we view the right hand side of (6.14) *as valued in* $S(L^n)^\vee$ *by dualizing* (6.5).

Proof. By linearity, it suffices to evaluate both sides of the desired relation at a Schwartz function $\varphi \in S(L^n)$ of the form $\varphi = \varphi_1 \otimes \varphi_2$ for $\varphi_1 \in S(L)$ and $\varphi_2 \in S(L^{n-1})$.

Then we may write

$$Z(T)(\varphi_1 \otimes \varphi_2) \;=\; \sum_{\substack{\mathbf{x}\in\Omega(T) \\ \mathrm{mod}\,\Gamma}} (\varphi_1 \otimes \varphi_2)(\mathbf{x})\, Z(\mathbf{x}) \tag{6.16}$$

$$=\; \sum_{\substack{\mathbf{y}\in\Omega(T_2) \\ \mathrm{mod}\,\Gamma}} \varphi_2(\mathbf{y}) \sum_{\substack{\mathbf{x}_1\in\Omega(T_1) \\ \langle\mathbf{x}_1,\mathbf{y}\rangle=T_{12} \\ \mathrm{mod}\,\Gamma_{\mathbf{y}}}} \varphi_1(\mathbf{x}_1)\Gamma_{(\mathbf{x}_1,\mathbf{y})}\backslash\mathbb{D}_{(\mathbf{x}_1,\mathbf{y})}. \tag{6.17}$$

We may further assume that

$$\varphi_1 = \varphi_1' \otimes \varphi_1'' \in S(U_{\mathbf{y}}) \otimes S(U_{\mathbf{y}}^\perp) \qquad \text{and} \qquad \varphi_2 = \varphi_2' \otimes \varphi_2'' \in S(\Lambda_{\mathbf{y}}^{n-1}) \otimes S((\Lambda_{\mathbf{y}}^\perp)^{n-1}); \tag{6.18}$$

in this case, $\varphi_2(\mathbf{y}) = \varphi_2'(\mathbf{y})\varphi_2''(0)$.

For a vector $\mathbf{x}_1 \in V$ as above, write its orthogonal decomposition as

$$\mathbf{x}_1 = \mathbf{x}_1' \,+\, \mathbf{x}_1'' \;\in\; U_{\mathbf{y}} \oplus U_{\mathbf{y}}^\perp, \tag{6.19}$$

and note that $\mathbb{D}_{(x_1,y)}^+ = \mathbb{D}_{(x_1'',y)}^+$, where $x_1 = \sigma_1(\mathbf{x}_1)$, etc., and $\Gamma_{(\mathbf{x}_1,\mathbf{y})} = \Gamma_{(\mathbf{x}_1'',\mathbf{y})}$.

Thus, decomposing the sum on $\mathbf{x}_1$ as above and writing $T = \left(\begin{smallmatrix} T_1 & T_{12} \\ T_{12}' & T_2 \end{smallmatrix}\right)$, we have

$$Z(T)(\varphi_1 \otimes \varphi_2)$$

$$=\; \sum_{\substack{\mathbf{y}\in\Omega(T_2) \\ \mathrm{mod}\,\Gamma}} \varphi_2(\mathbf{y}) \sum_{m\in F} \left(\sum_{\substack{\mathbf{x}_1''\in U_{\mathbf{y}}^\perp \\ \langle\mathbf{x}_1'',\mathbf{x}_1''\rangle=m \\ \mathrm{mod}\,\Gamma_{\mathbf{y}}}} \varphi_1''(\mathbf{x}_1'')\Gamma_{(\mathbf{x}_1'',\mathbf{y})}\backslash\mathbb{D}_{(\mathbf{x}_1'',\mathbf{y})} \cdot \right) \cdot \left(\sum_{\substack{\mathbf{x}_1'\in U_{\mathbf{y}} \\ \langle\mathbf{x}_1',\mathbf{x}_1'\rangle=T_1-m \\ \langle\mathbf{x}_1',\mathbf{y}\rangle=T_{12}}} \varphi_1'(\mathbf{x}_1') \right)$$

$$=\; \sum_{\substack{\mathbf{y}\in\Omega(T_2) \\ \mathrm{mod}\,\Gamma}} \varphi_2(\mathbf{y}) \sum_{m\in F} \left(\sum_{\substack{\mathbf{x}_1''\in U_{\mathbf{y}}^\perp \\ \langle\mathbf{x}_1'',\mathbf{x}_1''\rangle=m \\ \mathrm{mod}\,\Gamma_{\mathbf{y}}}} \varphi_1''(\mathbf{x}_1'')\Gamma_{(\mathbf{x}_1'',\mathbf{y})}\backslash\mathbb{D}_{(\mathbf{x}_1'',\mathbf{y})} \cdot \right) \cdot r_{\mathbf{y}}\left(\left(\begin{smallmatrix} T_1-m & T_{12} \\ T_{12}' & T_2 \end{smallmatrix}\right), \varphi_1' \right). \tag{6.20}$$

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which we may rewrite as

$$Z(T)\left(\varphi_1 \otimes \varphi_2\right) = \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \mathrm{mod}\ \Gamma}} \varphi_2''(0)\varphi_2'(\mathbf{y})\sum_m \pi_{\mathbf{y},*}\left(Z_{U_{\mathbf{y}}^\perp}(m)(\varphi_1'')\right) \cdot r\left(\left(\begin{smallmatrix} T_1-m & T_{12} \\ T_{12'} & T_2 \end{smallmatrix}\right), \varphi_1'\right) \tag{6.21}$$

$$= \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \mathrm{mod}\ \Gamma}} \sum_m Z_{\mathbf{y}}(m)(\varphi_1'' \otimes \varphi_2'') \cdot \left\{\varphi_2'(\mathbf{y})\, r\left(\left(\begin{smallmatrix} T_1-m & T_{12} \\ T_{12'} & T_2 \end{smallmatrix}\right), \varphi_1'\right)\right\} \tag{6.22}$$

where in the second line, $Z_{\mathbf{y}}(m)$ denotes the $S((\Lambda_{\mathbf{y}}^\perp)^n)^\vee$-valued cycle

$$Z_{\mathbf{y}}(m)\colon \varphi'' \mapsto \varphi_2''(0)\,\pi_{\mathbf{y},*}Z_{U_{\mathbf{y}}^\perp}(m,\varphi_1''). \tag{6.23}$$

Now suppose that $\mathrm{rk}(T) = \mathrm{rk}(T_2)+1$. Then, by Lemma 6.1, the term $m = 0$ does not contribute to (6.21), and so all the terms $Z_{U_{\mathbf{y}}^\perp}(m)$ that do contribute are divisors. To incorporate Green currents in the discussion, note that, at the level of arithmetic Chow groups, the pushforward is given by the formula

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'') = \varphi_2''(0)\,\pi_{\mathbf{y},*}\widehat{Z}_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'') \tag{6.24}$$

$$= \left(\pi_{\mathbf{y},*}Z_{U_{\mathbf{y}}^\perp}(m,\varphi_1''),\ \left[\omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})}, \mathfrak{g}_{U_{\mathbf{y}}^\perp}^o(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})}\right]\right), \tag{6.25}$$

where, as before, we use the subscript $U_{\mathbf{y}}^\perp$ to denote objects defined with respect to that space.

This may be rewritten as

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'') = \widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}}(\varphi'')$$
$$+ \varphi_2''(0)\left(0, \left[\omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})} - \delta_{Z_{\mathbf{y}}(m)},\ \mathfrak{g}_{U_{\mathbf{y}}^\perp}^o(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})}\right]\right) \tag{6.26}$$

where $\widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}} = (Z_{\mathbf{y}}(m), [\delta_{Z_{\mathbf{y}}(m)}, 0])$ is the canonical class associated to $Z_{\mathbf{y}}(m)$. Thus,

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1) \cdot \widehat{\omega}^{n-\mathrm{rk}(T)} = \widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}} \cdot \widehat{\omega}^{n-\mathrm{rk}(T)} + (0, [\beta_{\mathbf{y}}(m,\mathbf{v}_1), \alpha_{\mathbf{y}}(m,\mathbf{v}_1)]) \tag{6.27}$$

where $\alpha_{\mathbf{y}}(m,\mathbf{v}_1)$ and $\beta_{\mathbf{y}}(m,\mathbf{v}_1)$ are $S((\Lambda_{\mathbf{y}}^\perp)^n)^\vee$-valued currents defined by

$$\alpha_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'') = \varphi_2''(0)\,\mathfrak{g}_{U_{\mathbf{y}}^\perp}^o(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})} \wedge \Omega^{n-\mathrm{rk}(T)} \tag{6.28}$$

and

$$\beta_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'') = \varphi_2''(0)\,\omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'') \wedge \delta_{Z(\mathbf{y})} \wedge \Omega^{n-\mathrm{rk}(T)} - \delta_{Z_{\mathbf{y}}(m)(\varphi'')} \wedge \Omega^{n-\mathrm{rk}(T)} \tag{6.29}$$

where $\varphi'' = \varphi_1'' \otimes \varphi_2''$ as before.

Turning to the class $\widehat{A}(T,\mathbf{v})$, it can be readily verified that

$$\widehat{A}(T,\mathbf{v}) = \widehat{Z(T)}^{\mathrm{can}} \cdot \widehat{\omega}^{n-\mathrm{rk}(T)}\ +\ \left(0, [\psi(T,\mathbf{v}) - \delta_{Z(T)} \wedge \Omega^{n-\mathrm{rk}(T)}, \mathfrak{a}(T,\mathbf{v})]\right) \tag{6.30}$$

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where the currents $\mathfrak{a}(T, \mathbf{v})$ and $\psi(T, \mathbf{v})$ are defined in (4.23) and (4.36), respectively. Now, by the same argument as in (6.21), and under the assumption $\mathrm{rank}(T) = \mathrm{rank}(T_2) + 1$, we have (as a Γ-invariant current on $\mathbb{D}$)

$$\mathfrak{a}(T, \mathbf{v})(\varphi_1 \otimes \varphi_2) = \sum_{\mathbf{y} \in \Omega(T_2)} \varphi_2(\mathbf{y}) \sum_{\substack{\mathbf{x}_1 \in \Omega(T_1) \\ \langle \mathbf{x}_1, \mathbf{y} \rangle = T_{12}}} \varphi_1(\mathbf{x}_1) \mathfrak{g}^o(\sqrt{v_1} x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-r(T)} \tag{6.31}$$

$$= \sum_{\mathbf{y} \in \Omega(T_2)} \varphi_2(\mathbf{y}) \cdot \sum_{m \in F} \left(\sum_{\substack{\mathbf{x}_1'' \in U_{\mathbf{y}}^\perp \\ \langle \mathbf{x}_1'', \mathbf{x}_1'' \rangle = m}} \varphi_1''(\mathbf{x}_1'') \, \mathfrak{g}^o(\sqrt{v_1} x_1'') \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-r(T)} \right) \tag{6.32}$$

$$\times r \left(\begin{pmatrix} T_1 - m & T_{12} \\ T_{12'} & T_2 \end{pmatrix}, \varphi_1' \right), \tag{6.33}$$

where we use the fact that $\mathfrak{g}^o(\sqrt{v_1} x_1) \wedge \delta_{\mathbb{D}_y^+}$ only depends on the orthogonal projection x_1'' of x_1 onto $U_y^\perp = \sigma_1(U_{\mathbf{y}}^\perp)$. Thus, as $S(L^n)^\vee$-valued currents on X, we obtain the identity

$$\mathfrak{a}(T, \mathbf{v})(\varphi_1 \otimes \varphi_2) = \sum_{\mathbf{y} \bmod \Gamma} \sum_{m \in F} \alpha_{\mathbf{y}}(m, \mathbf{v}_1)(\varphi_1'' \otimes \varphi_2'') \cdot \left\{ \varphi_2'(\mathbf{y}) \, r_{\mathbf{y}} \left(\begin{pmatrix} T_1 - m & T_{12} \\ T_{12'} & T_2 \end{pmatrix}, \varphi_1' \right) \right\} \tag{6.34}$$

with $\varphi_i = \varphi_i' \otimes \varphi_i''$ as above.

A similar argument gives

$$\psi(T, \mathbf{v})(\varphi) - \delta_{Z(T)(\varphi)} \wedge \Omega^{n-\mathrm{rk}(T)} = \sum_{\mathbf{y} \bmod \Gamma} \sum_{m \in F} \beta_{\mathbf{y}}(m, \mathbf{v}_1)(\varphi_1'' \otimes \varphi_2'') \cdot \left\{ \varphi_2'(\mathbf{y}) \, r_{\mathbf{y}} \left(\begin{pmatrix} T_1 - m & T_{12} \\ T_{12'} & T_2 \end{pmatrix}, \varphi_1' \right) \right\}, \tag{6.35}$$

and so in total, we have

$$\widehat{A}(T, \mathbf{v})(\varphi_1 \otimes \varphi_2) = \sum_{\mathbf{y} \bmod \Gamma} \sum_m \widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1)(\varphi_1'' \otimes \varphi_2'') \cdot \widehat{\omega}^{n-\mathrm{rk}(T_2)-1} \cdot \left\{ \varphi_2'(\mathbf{y}) \, r_{\mathbf{y}} \left(\begin{pmatrix} T_1 - m & T_{12} \\ T_{12'} & T_2 \end{pmatrix} \right) (\varphi_1') \right\} \tag{6.36}$$

whenever $\mathrm{rank}(T) = \mathrm{rank}(T_2) + 1$.

Now suppose $\mathrm{rank}(T) = \mathrm{rank}(T_2)$. Then for any tuple $(\mathbf{x}_1, \mathbf{y}) \in \Omega(T)$, we must have $\mathbf{x}_1 \in U_{\mathbf{y}}$, and in particular, the only terms contributing to the right hand side of (6.36) are those with $m = 0$. On the other hand, we have

$$\mathfrak{a}(T, \mathbf{v}) = 0, \qquad \psi(T, \mathbf{v}) = \delta_{Z(T)} \wedge \Omega^{n-\mathrm{rk}(T)}, \tag{6.37}$$

and hence

$$\widehat{A}(T, \mathbf{v}) = \widehat{Z(T)}^{\mathrm{can}} \cdot \widehat{\omega}^{n-\mathrm{rk}(T)}; \tag{6.38}$$

with these observations, it follows easily from unwinding definitions that (6.36) continues to hold in this case.

Finally, the statement in the proposition follows by observing that the T'th q coefficient on the right hand side of (6.14) is precisely the right hand side of (6.36). $\qquad \square$

Corollary 6.3. *The series $\widehat{\phi}_A(\boldsymbol{\tau})$ is a Jacobi modular form of weight $\kappa := (p+2)/2$ and index T_2, in the sense of Definition 2.7.*

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Proof. Fix $\mathbf{y} \in \Omega(T_2)$. By Corollary 3.3, there exist finitely many $\widehat{z}_{\mathbf{y},1}, \ldots, \widehat{z}_{\mathbf{y},r} \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X_{\mathbf{y}})$, finitely many (elliptic) forms $f_{\mathbf{y},1}, \ldots, f_{\mathbf{y},r} \in A_\kappa(\rho^\vee_{\Lambda^\perp_{\mathbf{y}}})$ and an element $g_{\mathbf{y}} \in A_\kappa(\rho^\vee_{\Lambda^\perp_{\mathbf{y}}}; D^*(X))$ such that the identity

$$\sum_{m \in F} \widehat{Z}_{U^\perp_{\mathbf{y}}}(m, \mathbf{v}_1) q^m = \sum_{i=1}^r f_{\mathbf{y},i}(\boldsymbol{\tau}_1) \widehat{z}_{\mathbf{y},i} \ + \ a(g_{\mathbf{y}}(\boldsymbol{\tau}_1)) \tag{6.39}$$

holds at the level of q-coefficients; here $\boldsymbol{\tau}_1 \in \mathbb{H}^d_1$ and $\mathbf{v}_1 = \mathrm{Im}(\boldsymbol{\tau}_1)$. Moreover, from the proof of Corollary 3.3, we see that $g_{\mathbf{y}}(\tau)$ is smooth on X.

Therefore, applying Proposition 6.2 and unwinding definitions, we obtain the identity

$$\widehat{\phi}_A(\boldsymbol{\tau}) \ = \ \sum_{\substack{\mathbf{y} \in \Omega(T_2) \\ \mathrm{mod} \ \Gamma}} \sum_{i=1}^r (F_{\mathbf{y},i}(\boldsymbol{\tau}) \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau})) \widehat{Z}_{\mathbf{y},i} \ + \ a\left((G_{\mathbf{y}}(\boldsymbol{\tau}) \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau})) \wedge \delta_{Z(\mathbf{y})} \wedge \Omega^{n - \mathrm{rank}(T_2) - 1}\right)$$

$$\tag{6.40}$$

of formal generating series, where

$$\widehat{Z}_{\mathbf{y},i} \ := \ \pi_{\mathbf{y},*}(\widehat{z}_{\mathbf{y},i}) \cdot \widehat{\omega}^{n - \mathrm{rk}(T_2) - 1} \ \in \ \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X, \mathcal{D}_{\mathrm{cur}}), \tag{6.41}$$

and we promote the elliptic forms $f_{\mathbf{y},i}$ and $g_{\mathbf{y}}$ to $S((\Lambda^\perp_{\mathbf{y}})^n)^\vee$-valued functions by setting

$$F_{\mathbf{y},i}(\boldsymbol{\tau})(\varphi) \ := \ \varphi_2(0) \cdot f_{\mathbf{y},i}(\boldsymbol{\tau}_1)(\varphi_1), \qquad G_{\mathbf{y}}(\boldsymbol{\tau}) = \varphi_2(0) \cdot g_{\mathbf{y}}(\boldsymbol{\tau}_1)(\varphi_1) \tag{6.42}$$

for $\varphi = \varphi_1 \otimes \varphi_2 \in S(\Lambda^\perp_{\mathbf{y}}) \otimes S((\Lambda^\perp_{\mathbf{y}})^{n-1})$ and $\boldsymbol{\tau} = \left(\begin{smallmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}'_{12} & \boldsymbol{\tau}_2 \end{smallmatrix}\right)$.

It remains to show that $F_{\mathbf{y},i}(\boldsymbol{\tau}) \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau})$ and $G_{\mathbf{y}}(\tau) \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau})$ are invariant under the slash operators (2.38) for elements of $\widetilde{\Gamma}^J$; this can be verified directly using the generators (5.15) – (5.18), the modularity in genus one of $f_{\mathbf{y},i}$ and $g_{\mathbf{y}}$, and explicit formulas for the Weil representation (as in e.g. [Kud96]). $\square$

References

[BGKK07] J. I. Burgos Gil, J. Kramer, and U. Kühn. Cohomological arithmetic Chow rings. *J. Inst. Math. Jussieu*, 6(1):1–172, 2007.

[Bor99] Richard E. Borcherds. The Gross-Kohnen-Zagier theorem in higher dimensions. *Duke Math. J.*, 97(2):219–233, 1999.

[Bru12] Jan Hendrik Bruinier. Regularized theta lifts for orthogonal groups over totally real fields. *J. Reine Angew. Math.*, 672:177–222, 2012.

[BWR15] Jan Hendrik Bruinier and Martin Westerholt-Raum. Kudla's modularity conjecture and formal Fourier-Jacobi series. *Forum Math. Pi*, 3:e7, 30, 2015.

[ES18] Stephan Ehlen and Siddarth Sankaran. On two arithmetic theta lifts. *Compos. Math.*, 154(10):2090–2149, 2018.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[GS90]   Henri Gillet and Christophe Soulé. Arithmetic intersection theory. *Inst. Hautes Études Sci. Publ. Math.*, (72):93–174 (1991), 1990.

[GS19]   Luis E. Garcia and Siddarth Sankaran. Green forms and the arithmetic Siegel-Weil formula. *Invent. Math.*, 215(3):863–975, 2019.

[JS07]   Dihua Jiang and David Soudry. On the genericity of cuspidal automorphic forms of SO$(2n + 1)$. II. *Compos. Math.*, 143(3):721–748, 2007.

[KM90]   Stephen S. Kudla and John J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, (71):121–172, 1990.

[KRY06]   Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006.

[Kud96]   Stephen Kudla. Notes on the local theta correspondence. *Lecture Notes of the European School on Group Theory, Schloß Hirschberg*, 1996. http://www.math.toronto.edu/skudla/castle.pdf.

[Kud97]   Stephen S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.

[Kud19]   Stephen Kudla. Remarks on generating series for special cycles. *arXiv e-prints*, page arXiv:1908.08390, Aug 2019.

[Sou92]   C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[YZZ09]   Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. The Gross-Kohnen-Zagier theorem over totally real fields. *Compos. Math.*, 145(5):1147–1162, 2009.

[Zha09]   Wei Zhang. *Modularity of generating functions of special cycles on Shimura varieties*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Columbia University.