GREEN FORMS AND THE ARITHMETIC SIEGEL-WEIL FORMULA

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Abstract. We construct natural Green forms for special cycles in orthogonal and unitary Shimura varieties, in all codimensions, and, for compact Shimura varieties of type $\mathrm{O}(p,2)$ and $\mathrm{U}(p,1)$, we show that the resulting local archimedean height pairings are related to special values of derivatives of Siegel Eisentein series. A conjecture put forward by Kudla relates these derivatives to arithmetic intersections of special cycles, and our results settle the part of his conjecture involving local archimedean heights.

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Note: this is an updated version of the article [12]; the published version contains an error (in the definition of $\kappa(T,\Phi_f)$, see Definition 5.2.11 below) that is corrected in the present document. With this correction, the results of the paper and their proofs remain unchanged.

1. Introduction

The Arakelov theory of Shimura varieties has been intensively studied since, about twenty years ago, Kudla [25] launched a program relating families of special cycles in their arithmetic Chow groups with derivatives of Eisenstein series and Rankin-Selberg L-functions.

This paper is a contribution to the archimedean aspects of this theory. Building upon previous work of the first author [11], we use Quillen's formalism of superconnections [41] as developed by Bismut-Gillet-Soulé [2, 3] to construct natural Green forms for special cycles, in all codimensions, on orthogonal and unitary Shimura varieties. We show that these forms have good functorial properties and are compatible with star products. Next, specializing to compact Shimura varieties of type $\mathrm{GSpin}(p,2)$ or $\mathrm{U}(p,1)$, we study these Green forms using the theta correspondence, relating them to Siegel Eisenstein series.

Our main theorem is an explicit formula for the local archimedean height of a special cycle in terms of a Fourier coefficient of a special derivative of an Eisenstein series. This result provides compelling evidence for Kudla's conjectural identity, termed the *arithmetic Siegel-Weil formula*, between special derivatives of Eisenstein series and generating series of arithmetic heights of special cycles. More precisely, we show that the non-holomorphic terms on both sides are equal for these Shimura varieties.

Our methods combine Quillen's extension of Chern-Weil theory with the theory of the theta correspondence. This allows us to use representation theoretic arguments and the Siegel-Weil formula when computing archimedean local heights. In this way, we avoid the highly involved computations that feature in prior work, and give a conceptual explanation for the equality of non-holomorphic terms in Kudla's conjectural identities.

1.1. Main results.

The remainder of the introduction outlines our results in more detail. Throughout the paper, we treat $\mathrm{GSpin}(p,2)$ and $\mathrm{U}(p,q)$ Shimura varieties in parallel; these are referred to as the *orthogonal* and *unitary* cases, respectively.

Let F denote a totally real field of degree $[F:\mathbb{Q}]=d$ and let E be a CM extension of F equipped with a fixed CM type. Suppose that $\mathbb{V}$ is a quadratic space over F in the orthogonal case (resp. a Hermitian space over E in the unitary case). We assume that there is one archimedean place σ_1 such that $\mathbb{V}_{\sigma_1}$ satisfies the signature condition

$$\mathrm{signature}(\mathbb{V}_{\sigma_1}) = \begin{cases} (p,2) \text{ with } p>0 & \text{orthogonal case,} \\ (p,q) \text{ with } p,q>0 & \text{unitary case,} \end{cases} \tag{1.1.1}$$

and $\mathbb{V}$ is positive definite at all other archimedean places. Let

$$\mathbf{H} = \begin{cases} \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(\mathbb{V}), & \text{orthogonal case,} \\ \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(\mathbb{V}), & \text{unitary case,} \end{cases} \tag{1.1.2}$$

and $\mathbb{D}$ be the hermitian symmetric domain attached to $\mathbf{H}(\mathbb{R})$; concretely, $\mathbb{D}$ parametrizes oriented negative-definite real planes (resp. negative definite complex q-dimensional subspaces) in $\mathbb{V}_{\sigma_1}$ in the orthogonal (resp. unitary) case. For a fixed compact open subgroup $K \subset \mathbf{H}(\mathbb{A}_f)$, let $X_{\mathbb{V},K}$ be the corresponding Shimura variety, which has a canonical model over $\sigma_1(F)$ (resp. $\sigma_1(E)$). Then $X_{\mathbb{V},K}$ is a finite disjoint union

$$X_{\mathbb{V},K} \;=\; \coprod \Gamma\backslash\mathbb{D} \tag{1.1.3}$$

of quotients of $\mathbb{D}$ by certain arithmetic subgroups $\Gamma \subset \mathbf{H}(\mathbb{Q})$. Let $\mathcal{X}_K$ denote the variety obtained by viewing the canonical model of $X_{\mathbb{V},K}$ as a variety over $\mathrm{Spec}(\mathbb{Q})$. The complex points

$$\mathcal{X}_K(\mathbb{C}) = \coprod X_{\mathbb{V}[k],K} \tag{1.1.4}$$

are a finite disjoint union of Shimura varieties attached to $\mathbb{V}[1] := \mathbb{V}$ and its nearby spaces $\mathbb{V}[k]$ (see Section 4).

The variety $\mathcal{X}_K$ is equipped with a family of rational *special cycles*

$$\{Z(T,\varphi_f)\}, \tag{1.1.5}$$

as defined by Kudla [24], that are parametrized by pairs (T,φ_f) consisting of a matrix $T \in \mathrm{Sym}_r(F)$ (resp. $T \in \mathrm{Her}_r(E)$) and a K-invariant Schwartz function $\varphi_f \in \mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r)^K$. These cycles generalize the construction of Heegner points on modular curves and Hirzebruch-Zagier cycles on Hilbert modular surfaces.

The irreducible components of the cycle $Z(T,\varphi_f)$ on $X_{\mathbb{V},K}$ (say) admit a complex uniformization by certain complex submanifolds $\mathbb{D}_{\mathbf{v}}$ of $\mathbb{D}$ defined as follows: for a collection of vectors $\mathbf{v} = (v_1,\dots,v_r) \in \mathbb{V}^r$ satisfying

$$T(\mathbf{v}) := \left(\tfrac{1}{2}Q(v_i,v_j)\right)_{i,j=1,\dots,r} = T, \tag{1.1.6}$$

let

$$\mathbb{D}_{\mathbf{v}} \;:=\; \{z \in \mathbb{D} \mid z \perp v_i \text{ for all } i = 1,\dots,r\}, \tag{1.1.7}$$

so that $\mathbb{D}_{\mathbf{v}}$, if non-empty, is a hermitian symmetric subdomain of $\mathbb{D}$ of codimension $\tilde{r} := \mathrm{rk}(T)$ in the orthogonal case (resp. $\tilde{r} := q \cdot \mathrm{rk}(T)$ in the unitary case).

As a first step towards defining a Green current for $Z(T,\varphi_f)$, we construct a current $\mathfrak{g}^\circ(\mathbf{v})$ on $\mathbb{D}$ satisfying the equation

$$dd^c\mathfrak{g}^\circ(\mathbf{v}) \;+\; \delta_{\mathbb{D}_{\mathbf{v}}} \wedge \Omega_{\mathcal{E}^\vee}^{r-\mathrm{rk}(T)} \;=\; [\varphi^\circ_{\mathrm{KM}}(\mathbf{v})]. \tag{1.1.8}$$

Here $\delta_{\mathbb{D}_{\mathbf{v}}}$ is the current defined by integration along $\mathbb{D}_{\mathbf{v}}$ and $\Omega_{\mathcal{E}^\vee}$ is the top Chern form of the dual of the tautological bundle $\mathcal{E}$, see Section 2.2 below; the form $\varphi^\circ_{\mathrm{KM}}(\mathbf{v}) := e^{2\pi \mathrm{tr}(T(\mathbf{v}))}\varphi_{\mathrm{KM}}(\mathbf{v})$ is, up to a normalizing factor, the Schwartz[1] form $\varphi_{\mathrm{KM}}(\mathbf{v}) \in A^{\tilde{r},\tilde{r}}(\mathbb{D})$ introduced by Kudla and Millson [29].

[1]This means that $\varphi_{\mathrm{KM}}(\mathbf{v})$ and all its derivatives are of exponential decay in $\mathbf{v}$.

In recent work [11], the first author introduced a superconnection $\nabla_{\mathbf{v}}$ on $\mathbb{D}$ and showed that the component of degree $(\tilde{r}, \tilde{r})$ of the corresponding Chern form agrees with $\varphi^{\circ}_{\mathrm{KM}}(\mathbf{v})$. This allows us to apply the general results of [2] to obtain an explicit natural form $\nu^{\circ}(\mathbf{v})$ satisfying the transgression formula

$$\mathrm{dd^c}\, \nu^{\circ}(\sqrt{t}\,\mathbf{v}) \;=\; -t\,\frac{d}{dt}\varphi^{\circ}_{\mathrm{KM}}(\sqrt{t}\,\mathbf{v}), \qquad\qquad t \in \mathbb{R}_{>0}. \tag{1.1.9}$$

Moreover, the forms $\nu^{\circ}(t\mathbf{v})$ and $\varphi^{\circ}(t\mathbf{v})$ are of exponential decay in t on $\mathbb{D}\backslash\mathbb{D}_{\mathbf{v}}$.

To define $\mathfrak{g}^{\circ}(\mathbf{v})$, assume first that T is a non-degenerate matrix (so that $r = \mathrm{rk}(T)$) and consider the integral

$$\mathfrak{g}^{\circ}(\mathbf{v}) := \int_1^{\infty} \nu^{\circ}(\sqrt{t}\mathbf{v})\frac{dt}{t}, \tag{1.1.10}$$

initially defined on $\mathbb{D}\backslash\mathbb{D}_{\mathbf{v}}$. The estimates in [3] show that $\mathfrak{g}^{\circ}(\mathbf{v})$ is smooth on $\mathbb{D}\backslash\mathbb{D}_{\mathbf{v}}$, locally integrable on $\mathbb{D}$, and satisfies (1.1.8), which in this case reduces to Green's equation; in other words, $\mathfrak{g}^{\circ}(\mathbf{v})$ is a Green form for $\mathbb{D}_{\mathbf{v}}$, in the terminology of [6, §1.1].

When T is degenerate, the expression (1.1.10) is not locally integrable on $\mathbb{D}$ in general. We will circumvent this problem by regularizing the integral and obtain a current $\mathfrak{g}^{\circ}(\mathbf{v})$ satisfying (1.1.8) for every $\mathbf{v}$, see Section 2.6.1.

Returning to the special cycles $Z(T, \varphi_f)$, we define currents $\mathfrak{g}(T, \mathbf{y}, \varphi_f)$ on $\mathcal{X}_K(\mathbb{C})$ as weighted sums of $\mathfrak{g}^{\circ}(\mathbf{v})$ over vectors $\mathbf{v}$ satisfying $T(\mathbf{v}) = T$; these currents also depend on a parameter $\mathbf{y} \in \mathrm{Sym}_r(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$ or $\mathbf{y} \in \mathrm{Her}_r(E \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$ in the orthogonal or unitary cases, respectively.

Theorem 1.1.1. *The current $\mathfrak{g}(T, \mathbf{y}, \varphi_f)$ satisfies*

$$\mathrm{dd^c}\mathfrak{g}(T, \mathbf{y}, \varphi_f) \;+\; \delta_{Z(T,\varphi_f)(\mathbb{C})} \wedge \Omega_{\mathcal{E}^{\vee}}^{r-\mathrm{rk}(T)} \;=\; \omega(T, \mathbf{y}, \varphi_f), \tag{1.1.11}$$

where $\omega(T, \mathbf{y}, \varphi_f)$ is the T'th coefficient in the q-expansion of the theta function attached to $\varphi_{\mathrm{KM}} \otimes \varphi_f$.

Moreover, if T_1 and T_2 are non-degenerate and $Z(T_1, \varphi_1)$ and $Z(T_2, \varphi_2)$ intersect properly, then

$$\mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1) * \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2) \;\equiv\; \sum_{T = \left(\begin{smallmatrix} T_1 & * \\ * & T_2 \end{smallmatrix}\right)} \mathfrak{g}\left(T, \left(\begin{smallmatrix} \mathbf{y}_1 & \\ & \mathbf{y}_2 \end{smallmatrix}\right), \varphi_1 \otimes \varphi_2\right) \pmod{\mathrm{im}\, d + \mathrm{im}\, d^{\mathrm{c}}}.$$

Note that our construction is valid for all T; for example, when $\mathbb{V}$ is anisotropic we obtain

$$\mathfrak{g}(\mathbf{0}, \mathbf{y}, \varphi_f)|_{\mathcal{X}_{K,\sigma_k}(\mathbb{C})} \;=\; -\varphi_f(0)\, \log(\det \sigma_k(\mathbf{y})) \cdot c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^{\vee}, \nabla)^* \wedge \Omega_{\mathcal{E}^{\vee}}^{r-1} \tag{1.1.12}$$

for each real embedding σ_k of F.

When T is non-degenerate, (1.1.11) is Green's equation for the cycle $Z(T, \varphi_f)$ and hence $\mathfrak{g}(T, \mathbf{y}, \varphi_f)$ is a Green current (in fact, a Green form) for $Z(T, \varphi_f)$. When T is degenerate, the cycle $Z(T, \varphi_f)$ appears in the "wrong" codimension; following [24], this deficiency can be rectified by intersecting with a power of the tautological bundle and, as we discuss in Section 5.4, solutions to (1.1.11) correspond naturally to Green currents for this modified cycle.

Our main result computes local archimedean heights of special cycles in terms of Siegel Eisenstein series. We restrict our attention to the case that $\mathbb{V}$ is anisotropic, so that the corresponding Shimura variety is compact; our assumption on the signature of $\mathbb{V}$ ensures that this is the case whenever $F \neq \mathbb{Q}$. We further assume that $q = 1$ in the unitary case. Fix an integer $r \leq p+1$ and let

$$s_0 \ := \ \frac{p+1-r}{2}, \tag{1.1.13}$$

and, for a Schwartz function $\varphi_f \in \mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r)^K$, consider the corresponding genus r Siegel Eisenstein series

$$E(\boldsymbol{\tau}, \Phi_f, s) \ = \ \sum_T E_T(\boldsymbol{\tau}, \Phi_f, s) \tag{1.1.14}$$

of parallel scalar weight $l = \dim_F(\mathbb{V})/2$ (resp. $l = \dim_E(\mathbb{V})$), see Section 5.2; here $\boldsymbol{\tau} = \mathbf{x} + i\mathbf{y} \in \mathbb{H}_r^d$, where $\mathbb{H}_r$ is the Siegel (resp. Hermitian) upper half-space of genus r. Let

$$E'_T(\boldsymbol{\tau}, \Phi_f, s_0) \ := \ \frac{d}{ds} E_T(\boldsymbol{\tau}, \Phi_f, s)\Big|_{s=s_0} \tag{1.1.15}$$

denote the derivative of its Fourier coefficient $E_T(\boldsymbol{\tau}, \Phi_f, s)$ at $s = s_0$.

Theorem 1.1.2. *Suppose that $\mathbb{V}$ is anisotropic and, in the unitary case, that $q = 1$. Then for any T, there is an explicit constant $\kappa(T, \Phi_f)$, given by Definition 5.2.11, such that*

$$\frac{(-1)^r \kappa_0}{2\mathrm{Vol}(X_{\mathbb{V},K}, \Omega_{\mathcal{E}})} \int_{[\mathcal{X}_K(\mathbb{C})]} \mathfrak{g}(T, \mathbf{y}, \varphi_f) \wedge \Omega_{\mathcal{E}}^{p+1-r}\, q^T \ = \ E'_T(\boldsymbol{\tau}, \Phi_f, s_0) \ - \ \kappa(T, \Phi_f)\, q^T. \tag{1.1.16}$$

Here $q^T = e^{2\pi i \mathrm{tr}(T\boldsymbol{\tau})}$, and $\kappa_0 = 1$ if $s_0 > 0$ and $\kappa_0 = 2$ if $s_0 = 0$.

As a special case, suppose that T is non-degenerate, so that there is a factorization

$$E_T(\boldsymbol{\tau}, \Phi_f, s) \ = \ W_{T,\infty}(\boldsymbol{\tau}, \Phi^l_\infty, s) \cdot W_{T,f}(e, \Phi_f, s), \tag{1.1.17}$$

where the factors on the right are the products of the archimedean and non-archimedean local Whittaker functionals, respectively. Let

$$E'_T(\boldsymbol{\tau}, \Phi_f, s_0)_\infty \ = \ W'_{T,\infty}(\boldsymbol{\tau}, \Phi^l_\infty, s_0) \cdot W_{T,f}(e, \Phi_f, s_0) \tag{1.1.18}$$

denote the archimedean contribution to the special derivative. Then Theorem 1.1.2 specializes to the identity

$$\frac{(-1)^r \kappa_0}{2\mathrm{Vol}(X_{\mathbb{V},K}, \Omega_{\mathcal{E}})} \int_{[\mathcal{X}_K(\mathbb{C})]} \mathfrak{g}(T, \mathbf{y}, \varphi_f) \wedge \Omega_{\mathcal{E}}^{p+1-r} q^T = E'_T(\boldsymbol{\tau}, \Phi_f, s_0)_\infty \tag{1.1.19}$$

if T is not totally positive definite, and to

$$\begin{aligned} &\frac{(-1)^r \kappa_0}{2\mathrm{Vol}(X_{\mathbb{V},K}, \Omega_{\mathcal{E}})} \int_{[\mathcal{X}_K(\mathbb{C})]} \mathfrak{g}(T, \mathbf{y}, \varphi_f) \wedge \Omega_{\mathcal{E}}^{p+1-r} q^T \\ &\quad = E'_T(\boldsymbol{\tau}, \Phi_f, s_0)_\infty - E_T(\boldsymbol{\tau}, \Phi_f, s_0) \left(\frac{\iota d}{2} \left(r \log \pi - \frac{\Gamma'_r(\iota m/2)}{\Gamma_r(\iota m/2)} \right) + \frac{\iota}{2} \log N_{F/\mathbb{Q}} \det T \right) \end{aligned} \tag{1.1.20}$$

if T is totally positive definite; here $\iota = 1$ (resp. $\iota = 2$) in the orthogonal (resp. unitary) case.

When T is non-degenerate, the proof of the theorem can be summarized as follows: the current $\mathfrak{g}(T, \mathbf{y}, \varphi_f)$ is given by a sum of integrals of the form (1.1.10), for vectors $\mathbf{v}$ with $T(\mathbf{v}) = T$. Interchanging the order of integration, the Siegel-Weil formula relates the left hand side of (1.1.16) to the Fourier coefficient $E_T(\boldsymbol{\tau}, \Phi(\nu), s_0)$ of an Eisenstein series attached to the Schwartz form

$$\nu(\mathbf{v}) = e^{-2\pi \mathrm{tr}(T(\mathbf{v}))} \nu^\circ(\mathbf{v}). \tag{1.1.21}$$

We then analyze the behaviour of $\nu(\mathbf{v})$ under the action of the metaplectic group $\mathrm{Mp}_{2r}(\mathbb{R})$ (resp. the unitary group $\mathrm{U}(r, r)$) via the Weil representation. A multiplicity one argument allows us to identify $\Phi(\nu)$ explicitly, and in turn relate $E_T(\boldsymbol{\tau}, \Phi(\nu), s_0)$ to $E'_T(\boldsymbol{\tau}, \Phi_f, s_0)$ via a lowering operator. To conclude the proof, we apply work of Shimura [45] to derive asymptotic estimates for the Fourier coefficients $E_T(\boldsymbol{\tau}, \Phi_f, s)$ as $\mathbf{y} \to \infty$.

When T is degenerate, the idea is roughly the same, though additional care is required in handling the regularization, as well as establishing the required asymptotics of $E_T(\boldsymbol{\tau}, \Phi_f, s)$.

Prior results of this form have appeared in only a few special cases in the literature. For divisors, the Green function we define specializes to the one defined by Kudla [25], and Theorem 1.1.2 was proved in [33] for Shimura curves; a related result for $\mathrm{U}(p, 1)$ Shimura varieties over imaginary quadratic fields was proved by Ehlen and the second author [9].

In higher codimension much less was known. For (arithmetic) codimension two cycles on Shimura curves, Kudla [25] defined Green currents using star products; this construction does not coincide with ours, but does agree modulo exact currents by Theorem 1.1.1, and Theorem 1.1.2 reduces to results proved by elaborate explicit computations in [25] and [34]. Similar methods were used by Liu [37] for arithmetic codimension $p + 1$ cycles on $\mathrm{U}(p, 1)$ to prove a star product version of the particular case of the theorem given by (1.1.19). Again in the non-degenerate case, a recent preprint of Bruinier and Yang [15] proves a star product version of (1.1.19) for arithmetic codimension $p+1$ cycles on $\mathrm{O}(p, 2)$ by a different argument involving induction on p.

Finally, we place our results in the context of Kudla's conjectures on special cycles in arithmetic Chow groups. Putting aside the difficult issues involved in constructing integral models, the integral appearing in Theorem 1.1.2 is the archimedean contribution to the height of an arithmetic cycle lifting $Z(T, \varphi_f)$; according to Kudla's conjectural arithmetic Siegel-Weil formula, this height should equal the Fourier coefficient of an appropriately normalized version of the Eisenstein series appearing above. The remaining contribution to the arithmetic height is purely algebro-geometric in nature, and in particular should be independent of $\mathbf{y}$; thus Theorem 1.1.2 asserts that the non-holomorphic terms in Kudla's conjectural identity coincide. Put another way, our theorem reduces Kudla's conjecture to a relatively explicit conjectural formula for the analogue of the Faltings height (as in [6]) of a special cycle $Z(T, \varphi_f)$ in terms of T and φ_f; we discuss this point in more detail in Section 5.5.

1.2. **Notation and conventions.** Let $\mathbb{K}$ be a field endowed with a (possibly trivial) involution $a \mapsto \overline{a}$. We write $\mathrm{Sym}_r(\mathbb{K})$ (resp. $\mathrm{Her}_r(\mathbb{K})$) for the group of symmetric (resp. hermitian) r-by-r matrices with coefficients in $\mathbb{K}$ under matrix addition. For $a \in \mathrm{GL}_r(\mathbb{K})$ and $b \in \mathrm{Her}_r(\mathbb{K})$, let

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\overline{a}^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1_r & b \\ 0 & 1_r \end{pmatrix}, \quad w_r = \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}. \tag{1.2.1}$$

For $x = (x_1, \ldots, x_r) \in \mathbb{K}^r$, we write

$$d(x) = \mathrm{diag}(x_1, \ldots, x_r) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{pmatrix}. \tag{1.2.2}$$

We fix the following standard choice of additive character $\psi = \psi_F : F \to \mathbb{C}^\times$ when F is a local field of characteristic zero. If $F = \mathbb{R}$ we set $\psi(x) = e^{2\pi i x}$; if $F = \mathbb{Q}_p$ we choose $\psi = \psi_{\mathbb{Q}_p}$ so that $\psi(p^{-1}) = e^{-2\pi i/p}$; if F is a finite extension of $\mathbb{Q}_v$ we set $\psi_F(x) = \psi_{\mathbb{Q}_v}(\mathrm{tr}_{F/\mathbb{Q}_v}(x))$. If F is a global field, we write $\mathbb{A}_F^\times$ for the ideles of F and set $\psi_F : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$, where $\psi_F = \otimes_v \psi_{F_v}$ and the product runs over all places v of F.

We denote the connected component of the identity of a Lie group G by G^0.

We denote by $A^*(X)$ (resp. $D^*(X)$) the space of differential forms (resp. currents) on a smooth manifold X. Given $\alpha \in A^*(X) = \oplus_{k \geq 0} A^k(X)$, we write $\alpha_{[k]}$ for its component of degree k. If α is closed, we write $[\alpha] \in \mathrm{H}^*(X)$ for the cohomology class defined by α.

If X is a complex manifold, we let $\mathrm{d}^c = (4\pi i)^{-1}(\partial - \overline{\partial})$, so that $\mathrm{dd}^c = (-2\pi i)^{-1}\partial\overline{\partial}$. We denote by * the operator on $\oplus_{k\geq 0} A^{k,k}(X)$ acting by multiplication by $(-2\pi i)^{-k}$ on $A^{k,k}(X)$. The canonical orientation on X induces an inclusion $A^{p,q}(X) \subset D^{p,q}(X)$ sending a differential form ω to the current given by integration against ω on X, which we will denote by $[\omega]$ or simply by ω.

If $f(s)$ is a meromorphic function of a complex variable s, we write $\underset{s=0}{\mathrm{CT}} f(s)$ for the constant term of its Laurent expansion at $s = 0$.

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2. Green forms on hermitian symmetric domains

Let X be a complex manifold and $Z \subset X$ be a closed irreducible analytic subset of codimension c. A *Green current* for Z is a current $\mathfrak{g}_Z \in D^{c-1,c-1}(X)$ such that

$$\mathrm{dd^c}\mathfrak{g}_Z + \delta_Z = [\omega_Z], \tag{2.0.1}$$

where δ_Z denotes the current of integration on Z and ω_Z is a smooth differential form on X. A *Green form* is a Green current given by a form that is locally integrable on X and smooth on $X - Z$ (see [6, §1.1]).

Here we will construct Green forms for certain complex submanifolds of the hermitian symmetric space $\mathbb{D}$ attached to $\mathrm{O}(p,2)$ or $\mathrm{U}(p,q)$, where $p,q > 0$. Throughout the paper we will refer to the case involving $\mathrm{O}(p,2)$ as case 1 (or as the orthogonal case) and to the case involving $\mathrm{U}(p,q)$ as case 2 (or as the unitary case). Our methods apply uniformly in both cases.

There is a natural hermitian holomorphic vector bundle $\mathcal{E}$ over $\mathbb{D}$, and the submanifolds of $\mathbb{D}$ that we consider are the zero loci $Z(s)$ of certain natural holomorphic sections $s \in \mathrm{H}^0((\mathcal{E}^\vee)^r)$ $(r \geq 1)$. In this setting, the results in [4, 2, 3] (reviewed in Section 2.1) can be applied to construct some natural differential forms on $\mathbb{D}$, as we show in Section 2.2. We give explicit examples in Section 2.3, showing that this construction recovers some differential forms considered in previous work on special cycles (cf. [25, 28]), and then (Sections 2.4 and 2.5) we establish the main properties of these forms. Using these results, in Section 2.6 we define some currents related to $Z(s)$, including a Green form for $Z(s)$. The final Section 2.7 considers star products and will be used in the proof of Theorem 4.5.1.

2.1. Superconnections and characteristic forms of Koszul complexes.

In this section we review the construction of some characteristic differential forms attached to a pair $(\mathcal{E}, u)$, where $\mathcal{E}$ is a holomorphic hermitian vector bundle and u is a holomorphic section of its dual. The results in this section are due to Quillen [41], Bismut [4] and Bismut-Gillet-Soulé [2, 3].

We will use Quillen's formalism of superconnections and related notions of superalgebra. For more details, the reader is referred to [41, 1]. We briefly recall that a super vector space V is just a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector space; we write $V = V_0 \oplus V_1$ and refer to V_0 and V_1 as the even and odd part of V respectively. We write τ for the endomorphism of V determined by $\tau(v) = (-1)^{\deg(v)}v$. The supertrace $\mathrm{tr_s}\colon \mathrm{End}(V) \to \mathbb{C}$ is the linear form defined by

$$\mathrm{tr_s}(u) = \mathrm{tr}(\tau u), \tag{2.1.1}$$

where tr denotes the usual trace. Thus if $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathrm{End}(V_0)$, $d \in \mathrm{End}(V_1)$, $b \in \mathrm{Hom}(V_1, V_0)$ and $c \in \mathrm{Hom}(V_0, V_1)$, then $\mathrm{tr_s}(u) = \mathrm{tr}(a) - \mathrm{tr}(d)$.

2.1.1. Let $\mathcal{E}$ be a holomorphic vector bundle on a complex manifold X and $u \in \mathrm{H}^0(\mathcal{E}^\vee)$ be a holomorphic section of its dual $\mathcal{E}^\vee$. Let $K(u)$ be the Koszul complex of u: its underlying vector bundle is the exterior algebra $\wedge\mathcal{E}$ and its differential $u : \wedge^k\mathcal{E} \to \wedge^{k-1}\mathcal{E}$ is defined by

$$u(e_1 \wedge \cdots \wedge e_k) = \sum_{1 \leq i \leq k} (-1)^{i+1} u(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k. \tag{2.1.2}$$

The grading on $K(u)$ is given by $K(u)^{-k} = \wedge^k\mathcal{E}$, so that $K(u)$ is supported in non-positive degrees. We identify $K(u)$ with the corresponding complex of sheaves of sections of $\wedge^k\mathcal{E}$, and say that u is a *regular section* if the cohomology of $K(u)$ vanishes in negative degrees. If u is regular with zero locus $Z(u)$, then $K(u)$ is quasi-isomorphic to $\mathcal{O}_{Z(u)}$ (regarded as a complex supported in degree zero).

2.1.2. Assume now that $\mathcal{E}$ is endowed with a hermitian metric $\|\cdot\|_{\mathcal{E}}$. This induces a hermitian metric on $\wedge\mathcal{E}$: for any $x \in X$, the subspace $\wedge^k\mathcal{E}_x$ is orthogonal to $\wedge^j\mathcal{E}_x$ if $j \neq k$, and an orthonormal basis of $\wedge^k\mathcal{E}_x$ is given by all elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq \mathrm{rk}\mathcal{E}$ and $\{e_1, \ldots, e_{\mathrm{rk}\mathcal{E}}\}$ is an orthonormal basis of $\mathcal{E}_x$. Let ∇ be the corresponding Chern connection on $\wedge\mathcal{E}$. We regard $\wedge\mathcal{E}$ as a super vector bundle, with even part $\wedge^{\mathrm{even}}\mathcal{E}$ and odd part $\wedge^{\mathrm{odd}}\mathcal{E}$, and u as an odd endomorphism of $\wedge\mathcal{E}$. Let u^* be the adjoint of u, define the superconnection

$$\nabla_u = \nabla + i\sqrt{2\pi}(u + u^*) \tag{2.1.3}$$

on $\wedge\mathcal{E}$, and consider Quillen's Chern form

$$\varphi^\circ(u) = \varphi^\circ(\mathcal{E}, \|\cdot\|_{\mathcal{E}}, u) = \mathrm{tr_s}(e^{\nabla_u^2}) \in \bigoplus_{k\geq 0} A^{k,k}(X). \tag{2.1.4}$$

We recall some properties of $\varphi^\circ(u)$ established (in greater generality) by Quillen [41]. The form $\varphi^\circ(u)$ is closed and functorial: given a holomorphic map of complex manifolds $f : X' \to X$, consider the pullback bundle $(f^*\mathcal{E}, f^*\|\cdot\|)$ and the pullback section $f^*u \in \mathrm{H}^0(f^*\mathcal{E}^\vee)$. Then

$$\varphi^\circ(f^*u) = f^*\varphi^\circ(u). \tag{2.1.5}$$

Let * be the operator on $\oplus_{k\geq 0}A^{k,k}(X)$ acting by multiplication by $(-2\pi i)^{-k}$ on $A^{k,k}(X)$. Writing $[\varphi^\circ(u)^*]$ for the cohomology class of $\varphi^\circ(u)^*$ and $\mathrm{ch}(\cdot)$ for the Chern character, we have

$$[\varphi^\circ(u)^*] = \mathrm{ch}(\wedge\mathcal{E}) = \mathrm{ch}(\wedge^{\mathrm{even}}\mathcal{E}) - \mathrm{ch}(\wedge^{\mathrm{odd}}\mathcal{E}). \tag{2.1.6}$$

2.1.3. In particular, $[\varphi^\circ(u)^*]$ depends on $\mathcal{E}$, but not on u. Thus the forms $\varphi^\circ(tu)$ for $t \in \mathbb{R}_{>0}$ all belong to the same cohomology class, but as $t \to +\infty$ they concentrate on the zero locus of u. More precisely, recall that ∇_u^2 is an even element of the (super)algebra

$$A(X, \mathrm{End}(\wedge\mathcal{E})) := A^*(X)\hat{\otimes}_{\mathcal{C}^\infty(X)}\Gamma(\mathrm{End}(\wedge\mathcal{E})). \tag{2.1.7}$$

Given a relatively compact open subset $U \subset X$ whose closure $\overline{U}$ is disjoint from $Z(u)$ and a non-negative integer k, consider an algebra seminorm $\|\cdot\|_{\overline{U},\mathcal{E},k}$ on $A(X, \mathrm{End}(\wedge\mathcal{E}))$ measuring uniform convergence on $\overline{U}$ of partial derivatives of order at most k. We will need an estimate of $\|e^{\nabla_{tu}^2}\|_{\overline{U},\mathcal{E},k}$ for large t. To obtain it, write $\nabla_{tu}^2 = (\nabla_{tu}^2)_{[0]} + R_{tu}$, where $(\nabla_{tu}^2)_{[0]}$ has form-degree zero and R_{tu} has form-degree ≥ 1. Note that R_{tu} is nilpotent and that

$$(\nabla_{tu}^2)_{[0]} = -2\pi(tu + tu^*)^2 = -2\pi t^2\|u\|_{\mathcal{E}^\vee}^2 \otimes \mathrm{id} \in A^0(X) \otimes \mathrm{End}(\wedge\mathcal{E}) \tag{2.1.8}$$

(here $\|\cdot\|_{\mathcal{E}^\vee}$ denotes the unique metric on $\mathcal{E}^\vee$ such that the isomorphism $\mathcal{E}^\vee \simeq \overline{\mathcal{E}}$ induced by $\|\cdot\|_{\mathcal{E}}$ is an isometry). In particular, $(\nabla^2_{tu})_{[0]}$ and R_{tu} commute. Hence we have

$$\begin{aligned} e^{\nabla^2_{tu}} &= e^{(\nabla^2_{tu})_{[0]}} e^{R_{tu}} \\ &= e^{-2\pi t^2 \|u\|^2} \sum_{k=0}^{N} \tfrac{1}{k!} R^k_{tu} \end{aligned} \tag{2.1.9}$$

with $N \leq \dim_{\mathbb{R}} X$. Let a be any positive real number strictly less than $\min_{x\in\overline{U}}\{\|u(x)\|^2_{\mathcal{E}^\vee}\}$. Since R_{tu} is polynomial in t, it follows from (2.1.9) that

$$\|e^{\nabla^2_{tu}}\|_{\overline{U},\mathcal{E},k} \leq C e^{-2\pi a t^2}, \quad t \in \mathbb{R}_{>0} \tag{2.1.10}$$

for some positive real number C. Thus a similar bound holds for $\varphi^\circ(tu)$.

2.1.4. It follows from (2.1.6) that the form $\frac{d}{dt}\varphi^\circ(t^{1/2}u)$ (for $t \in \mathbb{R}_{>0}$) is exact, and one can ask for a construction of a functorial transgression of this form. Bismut, Gillet and Soulé [2] construct such a transgression, and the resulting form is key to our results. To define it, let $N \in \mathrm{End}(\wedge\mathcal{E})$ be the number operator acting on $\wedge^k\mathcal{E}$ by multiplication by $-k$ and set

$$\nu^\circ(u) = \mathrm{tr_s}(N e^{\nabla^2_u}) \in \bigoplus_{k\geq 0} A^{k,k}(X). \tag{2.1.11}$$

Then $\nu^\circ(u)$ is functorial with respect to holomorphic maps $f : X' \to X$ and satisfies ([2, Thm. 1.15])

$$-\frac{1}{t}\partial\overline{\partial}\nu^\circ(t^{1/2}u) = \frac{d}{dt}\varphi^\circ(t^{1/2}u), \quad t > 0. \tag{2.1.12}$$

2.1.5. Assume that the section u has no zeroes on X. Then the Koszul complex $K(u)$ is acyclic and (2.1.6) shows that $\varphi^\circ(u)$ is exact. In this case one can define a characteristic form $\xi^\circ(u)$ giving a $\overline{\partial}\partial$-transgression of $\varphi^\circ(u)$ by setting

$$\xi^\circ(u) = \int_1^{+\infty} \nu^\circ(t^{1/2}u)\frac{dt}{t}. \tag{2.1.13}$$

The bound (2.1.10) implies that all partial derivatives of $\nu^\circ(t^{1/2}u)$ decrease rapidly as $t \to +\infty$. In particular, the integral converges and defines a form in $\oplus_{k\geq 0} A^{k,k}(X)$; moreover, one can differentiate under the integral sign, so that by (2.1.12) we have

$$\partial\overline{\partial}\xi^\circ(u) = \varphi^\circ(u). \tag{2.1.14}$$

The form ξ° is functorial with respect to holomorphic maps $X' \to X$. In addition, $\xi^\circ(tu)$ (for $t \in \mathbb{R}_{>0}$) is rapidly decreasing as $t \to \infty$. More precisely, given a relatively compact open subset U of X with closure $\overline{U}$ and a positive integer k, we denote by $\|\cdot\|_{\overline{U},k}$ any seminorm on $A^*(X)$ measuring uniform convergence on $\overline{U}$ of partial derivatives of order at most k. The bound (2.1.10) shows that

$$\|\xi^\circ(tu)\|_{\overline{U},k} \leq C e^{-2\pi a t^2} \quad \text{for all } t \geq 1, \tag{2.1.15}$$

for some positive constants a and C.

2.1.6. We now go back to the general case where $\mathcal{E}$ is a hermitian holomorphic vector bundle on X and $u \in \mathrm{H}^0(\mathcal{E}^\vee)$, and now make the assumption that u is regular (see Section 2.1.1) and the zero locus $Z(u)$ is smooth; thus $K(u)$ is a resolution of $\mathcal{O}_{Z(u)}$ and the codimension of $Z(u)$ (if non-empty) in X equals $\mathrm{rk}(\mathcal{E})$. In this setting, the convergence of $\varphi^\circ(tu)$ and $\nu^\circ(tu)$ as $t \to \infty$ was studied by Bismut in [4], whose main result we now state, for the reader's convenience, in a form sufficient for our purposes.

Denote by $N^*_{\mathbb{R}}$ the real conormal bundle to $Z(u)$ in X and let $D^*_{N^*_{\mathbb{R}}}(X) \subset D^*(X)$ be the subset of currents on X whose wave front set is contained in $N^*_{\mathbb{R}}$ (see [16, Chap. VIII] for the definition and properties of wave front sets). Hörmander [16, p. 262] defines a family of seminorms on $D^*_{N^*_{\mathbb{R}}}(X)$ as follows. Let $U \subset X$ be an open set that is holomorphically equivalent to an open ball of $\mathbb{C}^N$ ($N = \dim X$) and write $\omega \mapsto \hat{\omega}$ for the Fourier transform on $\mathbb{C}^N$. We identify the real cotangent bundle $T^*_{\mathbb{R}}U$ with $U \times \mathbb{R}^{2N}$. Given a closed cone $\Gamma \subset \mathbb{R}^{2N}$ such that $\Gamma \cap N^*_{\mathbb{R}} = \emptyset$, a compactly supported smooth differential form ϕ on U and a positive integer m, define

$$p_{U,\Gamma,\phi,m}(\omega) = \sup_{v \in \Gamma} \|v\|^m |\widehat{\phi\omega}(v)|, \quad \omega \in D^*_{N^*_{\mathbb{R}}}(X). \tag{2.1.16}$$

Then $p_{U,\Gamma,\phi,m}$ is a seminorm on $D^*_{N^*_{\mathbb{R}}}(X)$. For a sequence $\omega_n \in D^*_{N^*_{\mathbb{R}}}(X)$ and $\omega \in D^*_{N^*_{\mathbb{R}}}(X)$, we say that $\omega_n \to \omega$ in $D^*_{N^*_{\mathbb{R}}}(X)$ if ω_n converges weakly to ω in $D^*(X)$ and

$$p_{U,\Gamma,\phi,m}(\omega_n - \omega) \to 0 \text{ as } n \to \infty, \tag{2.1.17}$$

for every choice of U, Γ, ϕ and m.

For a compact subset $K \subseteq X$ and a positive integer k, let $\|\cdot\|_{C^k_K(X)}$ be a norm on the space of smooth differential forms on X with support contained in K such that $\|\omega_n\| \to 0$ if and only if ω_n and its partial derivatives up to order k converge uniformly to 0.

We can now state Bismut's main result; we note that the bundle $N^*_{\mathbb{R}}$ is canonically identified with $\mathcal{E}|_{Z(u)}$, and Assumption (A) in [4, p. 68] holds for the metric on $N^*_{\mathbb{R}}$ making this identification an isometry. Note that while the results of [4] are formulated for compact manifolds, the estimates cited below hold for non-compact manifolds, provided that one considers differential forms μ with support contained in a fixed compact set K and allows the constants C_k in loc. cit. to vary with K (cf. [3, p. 263]).

Theorem 2.1.7 ([4, Theorem 3.2, Theorem 4.3])**.** *As $t \to \infty$, we have*

$$\begin{aligned} \varphi^\circ(tu)^*_{[2\mathrm{rk}(\mathcal{E})]} &\to \delta_{Z(u)} \text{ in } D^*_{N^*_{\mathbb{R}}}(X), \\ \nu^\circ(tu)^*_{[2\mathrm{rk}(\mathcal{E})-2]} &\to 0 \text{ in } D^*_{N^*_{\mathbb{R}}}(X). \end{aligned} \tag{2.1.18}$$

More precisely, let $k \geq 1$, K be a compact subset of X and $i : Z(u) \to X$ be the inclusion map. There exists a constant $C_{K,k} > 0$ such that, if $\mu \in A^*(X)$ has support contained in

K, then for $t \geq 1$

$$\begin{aligned} \left| \int_X \varphi^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})]} \wedge \mu - \int_{Z(u)} i^*\mu \right| &< \frac{C_{K,k}}{\sqrt{t}} \|\mu\|_{C^k_K(X)}, \\ \left| \int_X \nu^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})-2]} \wedge \mu \right| &< \frac{C_{K,k}}{\sqrt{t}} \|\mu\|_{C^k_K(X)}. \end{aligned} \tag{2.1.19}$$

For fixed U, Γ, ϕ and m as above and $t \geq 1$, we have

$$\begin{aligned} p_{U,\Gamma,\phi,m} \left(\varphi^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})]} - \delta_{Z(u)} \right) &= O(t^{-1/2}), \\ p_{U,\Gamma,\phi,m} \left(\nu^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})-2]} \right) &= O(t^{-1/2}). \end{aligned} \tag{2.1.20}$$

Define

$$\begin{aligned} \mathfrak{g}^\circ(u) &= \xi^\circ(u)^*_{[2\mathrm{rk}(\mathcal{E})-2]} \\ &= \left(\tfrac{i}{2\pi}\right)^{\mathrm{rk}(\mathcal{E})-1} \int_1^\infty \nu^\circ(t^{1/2}u)_{[2\mathrm{rk}(\mathcal{E})-2]} \frac{dt}{t}. \end{aligned} \tag{2.1.21}$$

The following proposition is contained in [3]. It shows that $\mathfrak{g}^\circ(u)$ is a Green form for $Z(u)$.

Proposition 2.1.8. *(1) The integral (2.1.21) converges to a smooth differential form $\mathfrak{g}^\circ(u) \in A^{\mathrm{rk}(\mathcal{E})-1,\mathrm{rk}(\mathcal{E})-1}(X - Z(u))$.*

(2) The form $\mathfrak{g}^\circ(u)$ is locally integrable on X.

(3) As currents on X we have

$$\mathrm{dd^c}\mathfrak{g}^\circ(u) + \delta_{Z(u)} = \varphi^\circ(u)^*_{[2\mathrm{rk}(\mathcal{E})]},$$

where $\mathrm{dd^c} = \frac{i}{2\pi}\partial\overline{\partial}$. Moreover, the wave front set of $\mathfrak{g}^\circ(u)$ is contained in $N^*_{\mathbb{R}}$.

Proof. Part (1) has already been discussed in Section 2.1.5.

Part (2) is shown to hold in the course of the proof of [3, Thm. 3.3]. In that paper, one considers an immersion $i\colon M' \to M$ of complex manifolds, a vector bundle η on M' and a complex (ξ, v) of holomorphic hermitian vector bundles on M that gives a resolution of $i_*\mathcal{O}_{M'}(\eta)$. Assume that $Z(u)$ is non-empty. We set $M = X$, $M' = Z(u)$, $\eta = \mathcal{O}_{Z(u)}$, and for the complex (ξ, v) we take the Koszul complex $(\wedge\mathcal{E}, u)$; with these choices, the form $\nu^\circ(t^{1/2}u)$ agrees with the form α_t defined in [3, (3.12)].

Let $x \in Z(u)$, choose local coordinates $z_1, \ldots, z_N$ ($N = \dim X$) around x such that $Z(u)$ is defined by the equations $z_1 = \cdots = z_{\mathrm{rk}(\mathcal{E})} = 0$ and let $|y| = (|z_1|^2 + \cdots + |z_{\mathrm{rk}(\mathcal{E})}|^2)^{1/2}$. Since the complex conormal bundle to $Z(u)$ in X is canonically identified with $\mathcal{E}|_{Z(u)}$, the equations [3, (3.24), (3.26)] show that

$$|y|^{2\mathrm{rk}(\mathcal{E})-1} \int_1^{+\infty} \nu^\circ(t^{1/2}u) \frac{dt}{t} \tag{2.1.22}$$

is bounded in a neighborhood of x; part (2) follows since $|y|^{-(2\mathrm{rk}(\mathcal{E})-1)}$ is locally integrable around x.

To prove (3), let β be a compactly supported form on X. Then

$$\begin{aligned}\int_X \mathfrak{g}^\circ(u)\wedge \mathrm{dd}^c\beta &= \lim_{a\to+\infty}\int_1^a\int_X \nu^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})-2]}\wedge \mathrm{dd}^c\beta\frac{dt}{t}\\ &= \lim_{a\to+\infty}\int_1^a\int_X \mathrm{dd}^c\nu^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})-2]}\wedge \beta\frac{dt}{t}\\ &= \lim_{a\to+\infty}\int_1^a\int_X (-t\frac{d}{dt}\varphi^\circ(t^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})]})\wedge \beta\frac{dt}{t}\\ &= \int_X \varphi^\circ(u)^*_{[2\mathrm{rk}(\mathcal{E})]}\wedge\beta - \lim_{a\to+\infty}\int_X \varphi^\circ(a^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})]}\wedge\beta.\end{aligned} \tag{2.1.23}$$

Here the first equality follows from dominated convergence since the integrals

$$|y|^{2\mathrm{rk}(\mathcal{E})-1}\int_1^a \nu^\circ(t^{1/2}u)^*\frac{dt}{t} \tag{2.1.24}$$

for $a\geq 1$ are uniformly bounded in a neighborhood of x, as shown in the proof of [3, Thm. 3.3]. The third equality follows from (2.1.12). This establishes the identity of currents in (3) since $\varphi^\circ(a^{1/2}u)^*_{[2\mathrm{rk}(\mathcal{E})]}$ approaches $\delta_{Z(u)}$ as $a\to+\infty$ by Theorem 2.1.7. The statement regarding the wave front set of $\mathfrak{g}^\circ(u)$ follows from (2.1.20), as explained in [3, p. 266-267]. □

2.2. **Hermitian symmetric domains of orthogonal and unitary groups.**

2.2.1. Let

$$\mathbb{K} = \begin{cases}\mathbb{R} & \text{case 1}\\ \mathbb{C} & \text{case 2}\end{cases} \tag{2.2.1}$$

and

$$\sigma = \begin{cases}\text{id} & \text{case 1}\\ \text{complex conjugation} & \text{case 2.}\end{cases} \tag{2.2.2}$$

Let $m\geq 2$ be a positive integer and let V be a $\mathbb{K}$-vector space of dimension m endowed with a non-degenerate σ-Hermitian bilinear form Q of signature (p,q). We assume that $pq\neq 0$ and in case 1 we further assume that $q=2$. Let $\mathrm{U}(V):=\mathrm{Aut}(V,Q)$ denote its isometry group, and set

$$G=\mathrm{U}(V)^0\cong\begin{cases}\mathrm{SO}(p,2)^0 & \text{case 1}\\ \mathrm{U}(p,q) & \text{case 2.}\end{cases} \tag{2.2.3}$$

We fix an orthogonal decomposition $V=V^+\oplus V^-$ with V^+ and V^- positive and negative definite respectively; in case 1 we also fix an orientation of V^-. Let K be the centralizer in G of the isometry of V acting as the identity on V^+ and as -1 on V^-. Then K is a maximal compact subgroup of G, given by

$$K=\begin{cases}\mathrm{SO}(V^+)\times\mathrm{SO}(V^-), & \text{case 1,}\\ \mathrm{U}(V^+)\times\mathrm{U}(V^-), & \text{case 2.}\end{cases} \tag{2.2.4}$$

Let $\mathrm{Gr}(q,V)$ be the Grassmannian of oriented two-dimensional subspaces of V in case 1, which consists of two copies of the usual Grassmannian. In case 2, we take $\mathrm{Gr}(q,V)$ to be the space of q-dimensional (complex) subspaces of V. Let

$$\mathbb{D} = \{z \in \mathrm{Gr}(q,V) |\ Q|_z < 0\} \tag{2.2.5}$$

be the open subset of $\mathrm{Gr}(q,V)$ consisting of negative definite subspaces. Then $\mathbb{D}$ has two connected components in case 1 and is connected in case 2. Let $z_0 \in \mathbb{D}$ be the point corresponding to V^- and $\mathbb{D}^+$ be the connected component of $\mathbb{D}$ containing z_0. Then G acts transitively on $\mathbb{D}^+$ and the stabilizer of z_0 is K; thus $\mathbb{D}^+ \simeq G/K$ is the symmetric domain associated with G. In case 2 it is clear that $\mathbb{D}$ carries a $\mathrm{U}(V)$-invariant complex structure; to see this in case 1, one can use the model

$$\mathbb{D} \simeq \{[v] \in \mathbb{P}(V(\mathbb{C})) | Q(v,v) = 0,\ Q(v,\overline{v}) < 0\}. \tag{2.2.6}$$

The correspondence between both models sends $z \in \mathbb{D}$ to the line $[e_z + ie'_z] \in \mathbb{P}(V(\mathbb{C}))$, where e_z and e'_z form an oriented orthogonal basis of z satisfying $Q(e_z,e_z) = Q(e'_z,e'_z)$.

2.2.2. Let $\mathcal{E}$ be the tautological bundle on $\mathbb{D}$, whose fiber over $z \in \mathbb{D}$ is the subspace $z \subset V$. Thus $\mathcal{E}$ is a holomorphic line bundle in case 1 (for it corresponds to the pullback of $\mathcal{O}_{\mathbb{P}(V(\mathbb{C}))}(-1)$ under the isomorphism (2.2.6)), and a holomorphic vector bundle of rank q in case 2. It carries a natural hermitian metric $h_{\mathcal{E}}$ defined by

$$h_{\mathcal{E}}(v_z) = \begin{cases} -Q(v_z, \overline{v_z}) & \text{case 1} \\ -Q(v_z, v_z) & \text{case 2} \end{cases} \tag{2.2.7}$$

for $v_z \in \mathcal{E}_z = z$. This metric is equivariant for the natural $\mathrm{U}(V)$-equivariant structure on $\mathcal{E}$. We denote by $\nabla_{\mathcal{E}}$ the corresponding Chern connection on $\mathcal{E}$ and by

$$\Omega = \Omega_{\mathcal{E}} := c^{\mathrm{top}}(\mathcal{E}, \nabla_{\mathcal{E}})^* = \left(\tfrac{i}{2\pi}\right)^{\mathrm{rk}\mathcal{E}} \det(\nabla^2_{\mathcal{E}}) \in A^{\mathrm{rk}\mathcal{E},\mathrm{rk}\mathcal{E}}(\mathbb{D}) \tag{2.2.8}$$

its Chern-Weil form of top degree.

We denote by $\Omega_{\mathbb{D}}$ the Kähler form

$$\Omega_{\mathbb{D}} = \partial\overline{\partial} \log k_{\mathbb{D}}(z,z), \tag{2.2.9}$$

where $k_{\mathbb{D}}$ is the Bergmann kernel function of $\mathbb{D}$. As shown in [50, p. 219], the invariant form $-\frac{i}{2\pi}\Omega_{\mathbb{D}}$ agrees with the first Chern form $c_1(\Omega_X^{\mathrm{top}})$ of the canonical bundle Ω_X^{top} on any quotient $X = \Gamma\backslash\mathbb{D}^+$ by a discrete torsion free subgroup $\Gamma \subset G$. When $\mathcal{E}$ has rank one, the canonical bundle on $\mathbb{D}$ is naturally isomorphic to $\mathcal{E}^{\otimes p}$ in case 1 (as an application of the adjunction formula shows) and to $\mathcal{E}^{\otimes(p+1)}$ in case 2, and so in both cases $-\frac{i}{2\pi}\Omega_{\mathbb{D}}$ is a positive integral multiple of $\Omega_{\mathcal{E}}$.

An element $v \in V$ defines a global holomorphic section s_v of $\mathcal{E}^\vee$: for $v'_z \in \mathcal{E}_z$, we define

$$s_v(v'_z) = Q(v'_z, v). \tag{2.2.10}$$

Let $\mathbb{D}_v$ be the zero locus of s_v on $\mathbb{D}$ and set $\mathbb{D}_v^+ = \mathbb{D}_v \cap \mathbb{D}^+$. We have

$$\mathbb{D}_v = \{z \in \mathbb{D} | v \perp z\} \tag{2.2.11}$$

and so $\mathbb{D}_v$ is non-empty only if $Q(v,v) > 0$ or $v = 0$. Assume that $Q(v,v) > 0$, so that the orthogonal complement $v^\perp$ of v has signature $(p-1,q)$. Writing G_v for the stabilizer of v

in G, we find that $G_v^0 \simeq \mathrm{Aut}(v^\perp, Q)^0$ acts transitively on $\mathbb{D}_v^+$ with stabilizers isomorphic to $\mathrm{SO}(p-1)\times\mathrm{SO}(q)$ in case 1 and to $\mathrm{U}(p-1)\times\mathrm{U}(q)$ in case 2. Thus $\mathbb{D}_v^+$ is the symmetric domain attached to G_v^0. We conclude that $\mathrm{codim}_{\mathbb{D}}\mathbb{D}_v = \mathrm{rk}\mathcal{E}^\vee$ and hence that s_v is a regular section of $\mathcal{E}^\vee$. In fact, this shows that s_v is regular whenever $v \neq 0$, since in the remaining case we have $Q(v,v) \leq 0$ and so s_v does not vanish on $\mathbb{D}$.

More generally, given a positive integer r and a vector $\mathbf{v} = (v_1, \dots, v_r) \in V^r$, there is a holomorphic section $s_{\mathbf{v}} = (s_{v_1}, \dots, s_{v_r})$ of $(\mathcal{E}^\vee)^r$, with zero locus

$$\mathbb{D}_{\mathbf{v}} := Z(s_{\mathbf{v}}) = \cap_{1\leq i\leq r}\mathbb{D}_{v_i}. \tag{2.2.12}$$

Note that $\mathbb{D}_{\mathbf{v}}$ depends only on the span $\langle v_1, \dots, v_r\rangle$. It is non-empty if and only if $\langle v_1, \dots, v_r\rangle$ is a a positive definite subpace of (V, Q) of positive dimension; in that case, its (complex) codimension in $\mathbb{D}$ is $\mathrm{rk}(\mathcal{E})\cdot\dim_{\mathbb{K}}\langle v_1, \dots, v_r\rangle$. We set $\mathbb{D}_{\mathbf{v}}^+ = \mathbb{D}_{\mathbf{v}} \cap \mathbb{D}^+$.

Let $z \in \mathbb{D}_{\mathbf{v}}$ and write $z^\perp$ (resp. $\mathbf{v}^\perp$) for the orthogonal complement of the subspace z (resp. $\langle v_1, \dots, v_r\rangle$) of V. Then the tangent space $T_z\mathbb{D}$ to $\mathbb{D}$ at z can be canonically identified with $\mathrm{Hom}(z, z^\perp)$, and the subspace $T_z\mathbb{D}_{\mathbf{v}} \subseteq T_z\mathbb{D}$ corresponds to $\mathrm{Hom}(z, z^\perp \cap \mathbf{v}^\perp)$ (see [29, p. 131]).

2.2.3. We will now specialize the constructions in Section 2.1 to the setting of hermitian symmetric domains.

Let r be a positive integer and $\mathbf{v} = (v_1, \dots, v_r)$ be an r-tuple of vectors in V. We write $K(\mathbf{v}) := K(s_{\mathbf{v}})$ for the Koszul complex associated with the section $s_{\mathbf{v}} = (s_{v_1}, \dots, s_{v_r})$ of $(\mathcal{E}^r)^\vee$. On its underlying vector bundle $\wedge(\mathcal{E}^r)$, we consider the superconnection

$$\nabla_{\mathbf{v}} := \nabla_{s_{\mathbf{v}}} = \nabla + i\sqrt{2\pi}(s_{\mathbf{v}} + s_{\mathbf{v}}^*), \tag{2.2.13}$$

and we define forms

$$\begin{aligned} \varphi^\circ(\mathbf{v}) &:= \varphi^\circ(s_{\mathbf{v}})^* = \sum_{k\geq 0}\left(\tfrac{i}{2\pi}\right)^k \mathrm{tr_s}(e^{\nabla_{\mathbf{v}}^2})_{[2k]}, \\ \nu^\circ(\mathbf{v}) &:= \nu^\circ(s_{\mathbf{v}})^* = \sum_{k\geq 0}\left(\tfrac{i}{2\pi}\right)^k \mathrm{tr_s}(Ne^{\nabla_{\mathbf{v}}^2})_{[2k]}, \end{aligned} \tag{2.2.14}$$

where N is the number operator on $\wedge(\mathcal{E}^r)$ acting on $\wedge^k(\mathcal{E}^r)$ by multiplication by $-k$.

Definition 2.2.4. *For $\mathbf{v} = (v_1, \dots, v_r) \in V^r$, write $Q(\mathbf{v}, \mathbf{v}) = Q(v_1, v_1) + \cdots + Q(v_r, v_r)$ and define*

$$\begin{aligned} \varphi(\mathbf{v}) &= e^{-\pi Q(\mathbf{v},\mathbf{v})}\,\varphi^\circ(\mathbf{v}), \\ \nu(\mathbf{v}) &= e^{-\pi Q(\mathbf{v},\mathbf{v})}\,\nu^\circ(\mathbf{v}). \end{aligned}$$

Thus $\varphi(\mathbf{v})$ and $\nu(\mathbf{v})$ belong to $\oplus_{k\geq 0}A^{k,k}(\mathbb{D})$.

The forms $\varphi(\mathbf{v})$ and $\nu(\mathbf{v})$ were already defined and studied in the setting of general period domains in [11].

2.3. **Explicit formulas for** $\mathrm{O}(p,2)$ **and** $\mathrm{U}(p,1)$**.** Let us give some explicit formulas when the tautological bundle $\mathcal{E}$ is a line bundle. Thus V is either a real vector space of signature $(p,2)$ (case 1) or a complex vector space of signature $(p,1)$ (case 2). There is a unique hermitian metric on $\mathcal{E}^\vee$ making the isomorphism $\overline{\mathcal{E}} \cong \mathcal{E}^\vee$ induced by $h_{\mathcal{E}}$ an isometry; we denote this metric by h and its Chern connection by $\nabla_{\mathcal{E}^\vee}$. For $v \in V$ we have

$$\begin{aligned} \varphi(v)_{[2]} &= e^{-\pi(Q(v,v)+2h(s_v))}\left(i\frac{\partial h(s_v)\wedge\overline{\partial} h(s_v)}{h(s_v)} - \Omega_{\mathcal{E}}\right), \\ \varphi(v) &= \varphi(v)_{[2]} \wedge \mathrm{Td}^{-1}(\mathcal{E}^\vee, \nabla)^*, \end{aligned} \tag{2.3.1}$$

where

$$\mathrm{Td}^{-1}(\mathcal{E}^\vee, \nabla) = \det\left(\frac{1-e^{-\nabla^2_{\mathcal{E}^\vee}}}{\nabla^2_{\mathcal{E}^\vee}}\right) \tag{2.3.2}$$

denotes the inverse Todd form of $(\mathcal{E}^\vee, \nabla_{\mathcal{E}^\vee})$. This is a special case of the Mathai-Quillen formula [38, Thm. 8.5]; see also [11, §3] for a proof in our setting, where it is also shown that the form $\varphi(v)_{[2]}$ coincides with the form $\varphi_{\mathrm{KM}}(v)$ defined by Kudla and Millson in [28].

Let us now consider the form $\nu(v)$ for $v \in V$. Here the Koszul complex $K(v)$ has just two terms: $K(v) = (\mathcal{E} \xrightarrow{s_v} \mathcal{O}_{\mathbb{D}})$, and the operator N acts by zero on $\mathcal{O}_{\mathbb{D}}$ and by -1 on $\mathcal{E}$. For the component of degree zero of $\nu^\circ(v)$ we obtain

$$\begin{aligned} \nu^\circ(v)_{[0]} &= \mathrm{tr_s}(Ne^{\nabla_v^2})_{[0]} \\ &= \mathrm{tr_s}(Ne^{(\nabla_v^2)_{[0]}}) \\ &= \mathrm{tr_s}(Ne^{-2\pi h(s_v)}) \\ &= e^{-2\pi h(s_v)}. \end{aligned} \tag{2.3.3}$$

Given $z \in \mathbb{D}$, let $z^\perp$ be the orthogonal complement of z in V, so that $V = z \oplus z^\perp$; we write v_z and $v_{z^\perp}$ for the orthogonal projection of $v \in V$ to z and $z^\perp$ respectively. Let Q_z be the (positive definite) Siegel majorant of Q defined by

$$Q_z(v,v) = Q(v_{z^\perp}, v_{z^\perp}) - Q(v_z, v_z). \tag{2.3.4}$$

Then we have $Q_z(v,v) = Q(v,v) + 2h_z(s_v)$ and we conclude that $\nu(v)_{[0]}$ is just the Siegel gaussian:

$$\nu(v)_{[0]} = \varphi^{\mathrm{SG}}(v) := e^{-\pi Q_z(v,v)}. \tag{2.3.5}$$

If $v \neq 0$, then using (2.3.3) we also obtain an explicit formula for the Green function $\mathfrak{g}^\circ(s_v)$ defined in (2.1.21) (see Example 2.6.3 below).

2.4. **Basic properties of the forms** φ **and** ν**.**

2.4.1. We first give formulas for the restriction of φ and ν to special cycles. Let $w \in V$ with $Q(w,w) > 0$ and recall the group G_w and complex submanifold $\mathbb{D}_w^+ \subset \mathbb{D}^+$ defined in 2.2.2. Then G_w is identified with the isometry group of $(w^\perp, Q)$, and we may identify $\mathbb{D}_w^+$ with the hermitian symmetric domain attached to G_w^0. We write $\varphi_{\mathbb{D}_w}(v')$ and $\nu_{\mathbb{D}_w}(v')$ for the forms on $\mathbb{D}_w$ given in Definition 2.2.4.

Lemma 2.4.2. *Let $v, w \in V$ with $Q(w,w) > 0$ and write $v = v' + v''$ with $v' \in w^\perp$ and $v'' \in \langle w \rangle$. Then*

$$\nu^\circ(v)|_{\mathbb{D}_w} = \nu^\circ_{\mathbb{D}_w}(v'), \quad \nu(v)|_{\mathbb{D}_w} = e^{-\pi Q(v'',v'')}\nu_{\mathbb{D}_w}(v'),$$
$$\varphi^\circ(v)|_{\mathbb{D}_w} = \varphi^\circ_{\mathbb{D}_w}(v'), \quad \varphi(v)|_{\mathbb{D}_w} = e^{-\pi Q(v'',v'')}\varphi_{\mathbb{D}_w}(v').$$

Proof. Let $\mathcal{E}_w$ be the tautological bundle on $\mathbb{D}_w$ whose fiber over $z \in \mathbb{D}_w$ is $z \subset w^\perp$. The restriction of $\mathcal{E}$ to $\mathbb{D}_w$ is isometric to $\mathcal{E}_w$. For any $v \in V$, this isometry induces an isomorphism $K(v)|_{\mathbb{D}_w} \cong K(v')$, where $K(v')$ denotes the Koszul complex with underlying vector bundle $\wedge\mathcal{E}_w$ and differential $s_{v'}$. Thus $\nabla_v|_{\mathbb{D}_w} = \nabla_{v'}$ and the lemma follows. □

2.4.3. The proof of the next proposition is a straightforward consequence of general properties of Koszul complexes and Chern forms.

Proposition 2.4.4. *Let $r \geq 1$ and $\mathbf{v} = (v_1, \ldots, v_r) \in V^r$. Then:*
(a) $\varphi(\mathbf{v}) = \varphi(v_1) \wedge \ldots \wedge \varphi(v_r)$.
(b) $\varphi(\mathbf{v})$ *is closed.*
(c) $\varphi(\mathbf{v})_{[k]} = 0$ *if* $k < 2r \cdot \mathrm{rk}(\mathcal{E})$.
(d) For every $g \in \mathrm{Aut}(V,Q)$, *we have* $g^*\varphi(gv_1, \ldots, gv_r) = \varphi(v_1, \ldots, v_r)$.
(e) $\varphi(0) = c_{\mathrm{rk}(\mathcal{E})}(\mathcal{E}^\vee, \nabla)^* \wedge \mathrm{Td}^{-1}(\mathcal{E}^\vee, \nabla)^*$ *(here we assume $r = 1$).*
(f) Let

$$h \in \begin{cases} \mathrm{O}(r), & \textit{case 1} \\ \mathrm{U}(r), & \textit{case 2.} \end{cases}$$

Then $\varphi((v_1, \ldots, v_r) \cdot h) = \varphi(v_1, \ldots, v_r)$.

Proof. Except for (c), which follows from (a) and the Mathai-Quillen formula [38, Thm. 8.5], all statements are proved in [11, Prop. 2.3 and (2.17)] in case 1, and the proof there extends without modification to case 2. □

Consider now the form ν. We write

$$c(F, \nabla_F) = \det(t\nabla_F^2 + 1_{\mathrm{rk}(F)}) = 1 + c_1(F, \nabla_F)t + \ldots + c_{\mathrm{rk}(F)}(F, \nabla_F)t^{\mathrm{rk}(F)} \tag{2.4.1}$$

for the total Chern-Weil form of a vector bundle F with connection ∇_F, and recall that the cohomology class of $c_k(F, \nabla_F)^* = (-2\pi i)^{-k} c_k(F, \nabla_F)$ is the k-th Chern class $c_k(F)$.

Proposition 2.4.5. *Let $r \geq 1$ and $\mathbf{v} = (v_1, \ldots, v_r) \in V^r$.*
(a) We have $\nu(\mathbf{v}) = \sum_{1 \leq i \leq r} \nu_i(\mathbf{v})$, *where*

$$\nu_i(\mathbf{v}) := \nu(v_i) \wedge \varphi(v_1, \ldots, \hat{v}_i, \ldots, v_r).$$

(b) For $t > 0$:

$$\mathrm{dd}^c \nu^\circ(t^{1/2}\mathbf{v}) = -t\frac{d}{dt}\varphi^\circ(t^{1/2}\mathbf{v}).$$

(c) $\nu(\mathbf{v})_{[k]} = 0$ *if* $k < 2r \cdot \mathrm{rk}(\mathcal{E}) - 2$.
(d) For any $g \in \mathrm{Aut}(V,Q)$, *we have* $g^*\nu(gv_1, \ldots, gv_r) = \nu(v_1, \ldots, v_r)$.

(e) *For the zero vector* $\mathbf{0} \in V^r$, *we have*

$$\nu(\mathbf{0})_{[2r\cdot \mathrm{rk}(\mathcal{E})-2]} \;=\; r\cdot c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^* \wedge \left(c_{\mathrm{rk}(\mathcal{E})}(\mathcal{E}^\vee,\nabla)^*\right)^{r-1}.$$

In particular, $\nu(0)_{[0]} = 1$ *when* $r = \mathrm{rk}(\mathcal{E}) = 1$.

(f) *Let*

$$h \in \begin{cases} \mathrm{O}(r), & \textit{case 1} \\ \mathrm{U}(r), & \textit{case 2.} \end{cases}$$

Then $\nu((v_1,\ldots,v_r)\cdot h) = \nu(v_1,\ldots,v_r)$.

Proof. Recall that $\nu(\mathbf{v}) = e^{-\pi Q(\mathbf{v},\mathbf{v})}\mathrm{tr_s}(Ne^{\nabla^2_{\mathbf{v}}})$, where N is the number operator on the Koszul complex $K(\mathbf{v}) \simeq \otimes_{1\le i\le r} K(v_i)$. Letting N_i be the number operator on $K(v_i)$, we can write $N = N_1 + \cdots + N_r$ and $\nabla^2_{\mathbf{v}} = \nabla^2_{v_1} + \cdots + \nabla^2_{v_r}$, where $[N_i, \nabla^2_{v_j}] = [\nabla^2_{v_i}, \nabla^2_{v_j}] = 0$ for $i \neq j$; this proves (a).

Part (b) follows from (2.1.12). By part (a) and Proposition 2.4.4.(c), it suffices to prove (c) when $r = 1$. Then (c) is vacuously true if $\mathrm{rk}(\mathcal{E}) = 1$, and in general it follows from [5, (3.72),(3.35), Thm. 3.10].

For any $\mathbf{v}$ and any $g \in \mathrm{U}(V)$, the $\mathrm{U}(V)$-equivariant structure on $\mathcal{E}$ induces an isomorphism $g^*K(g\mathbf{v}) \simeq K(\mathbf{v})$ preserving the metric; this proves (d).

For part (e), first consider the case $r = 1$. When $\mathcal{E}$ also has rank one, the desired relation follows immediately from (2.3.5). For general $\mathcal{E}$, by taking $u = 0$ in [5, (3.35)] we find that

$$\begin{aligned} \mathrm{tr_s}(Ne^{\nabla^2})^*_{[2\mathrm{rk}(\mathcal{E})-2]} &= -\frac{d}{db}\det\left(\tfrac{i}{2\pi}\nabla^2_{\mathcal{E}^\vee} - b1_{\mathrm{rk}(\mathcal{E})}\right)\Big|_{b=0} \\ &= (-1)^{r-1}\frac{d}{db}\det\left(\tfrac{i}{2\pi}\nabla^2_{\mathcal{E}} + b1_{\mathrm{rk}(\mathcal{E})}\right)\Big|_{b=0} \\ &= (-1)^{r-1}c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E},\nabla)^* \\ &= c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*. \end{aligned} \tag{2.4.2}$$

Note that the number operator N in op. cit. has sign opposite from ours. The formula for general r follows from part (a) above together with Proposition 2.4.4(a),(e).

To prove (f), note that h induces an isometry

$$K(v_1,\ldots,v_r) \xrightarrow{i(h)} K((v_1,\ldots,v_r)\cdot h) \tag{2.4.3}$$

that commutes with N and such that $\nabla_{(v_1,\ldots,v_r)\cdot h} = i(h)^{-1}\nabla_{(v_1,\ldots,v_r)}i(h)$. Thus (f) follows from the conjugation invariance of $\mathrm{tr_s}$. □

The properties of $\varphi(\mathbf{v})$ and $\nu(\mathbf{v})$ established in Propositions 2.4.4 and 2.4.5 also hold for $\varphi^\circ(\mathbf{v})$ and $\nu^\circ(\mathbf{v})$. In contrast, the next result, showing that the forms φ and ν are rapidly decreasing as functions of $\mathbf{v}$, really requires the additional factor $e^{-\pi Q(\mathbf{v},\mathbf{v})}$ to hold (note for example that the restriction of $\varphi^\circ(t\mathbf{v})$ to $\mathbb{D}_{\mathbf{v}}$ is independent of $t \in \mathbb{R}$).

Let $\mathcal{S}(V^r)$ be the Schwartz space of complex-valued smooth functions on V^r all whose derivatives are rapidly decreasing.

Lemma 2.4.6. *For fixed $z \in \mathbb{D}$ and $r \geq 1$, we have*

$$\varphi(\cdot, z), \nu(\cdot, z) \in \mathcal{S}(V^r) \otimes \wedge T_z^*\mathbb{D}.$$

Proof. By Proposition 2.4.4.(a) and Proposition 2.4.5.(a), we may assume that $r = 1$. Recall that the quadratic form $Q_z(v) = \frac{1}{2}Q(v, v) + h_z(s_v)$ on V is positive definite. Write $\nabla_v^2(z) = (\nabla_v^2)_{[0]}(z) + S(v, z)$. By Duhamel's formula ([46, p. 144]) we have

$$\begin{aligned} & e^{-\pi Q(v,v)} e^{\nabla_v^2(z)} = e^{-2\pi Q_z(v)} \\ & + \sum_{k \geq 1} (-1)^k \int_{\Delta^k} e^{-2\pi(1-t_k)Q_z(v)} S(v, z) e^{-2\pi(t_k - t_{k-1})Q_z(v)} \cdots S(v, z) e^{-2\pi t_1 Q_z(v)} dt_1 \cdots dt_k. \end{aligned} \tag{2.4.4}$$

Here $\Delta^k = \{(t_1, \ldots, t_k) \in \mathbb{R}^k | 0 \leq t_1 \leq \ldots \leq t_k \leq 1\}$ is the k-simplex and the sum is finite since $S(v, z)$ has positive degree. Let $\| \cdot \|_{\overline{U},k}$ be an algebra seminorm as in Section 2.1.2 and let $Q_{\overline{U}}$ a positive definite quadratic form on V such that $Q_{\overline{U}} < Q_z$ for all $z \in \overline{U}$. Then

$$\| e^{-2\pi Q_z(v)} \|_{\overline{U},k} < C e^{-2\pi Q_{\overline{U}}(v)}, \quad v \in V, \tag{2.4.5}$$

for some positive constant C. Since $S(v, z)$ grows linearly with v, (2.4.4) implies that

$$\| e^{-\pi Q(v,v)} e^{\nabla_v^2(z)} \|_{\overline{U},k} < C e^{-2\pi Q_{\overline{U}}(v)}, \quad v \in V, \tag{2.4.6}$$

(with different C) and hence that the same bound holds for $\|\phi(v)\|_{\overline{U},k}$ where $\phi(v)$ is any derivative of φ or ν in the v variable. ☐

2.5. **Behaviour under the Weil representation.**

2.5.1. Let r be a positive integer. We write 0 and 1_r for the identically zero and identity r-by-r matrices respectively. Consider the vector space $W_r := \mathbb{K}^{2r}$ endowed with the σ-skew-Hermitian form determined by

$$\begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}. \tag{2.5.1}$$

Its isometry group is the symplectic group

$$\mathrm{Sp}_{2r}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2r}(\mathbb{R}) \;\middle|\; g \left(\begin{smallmatrix} 0 & 1_r \\ -1_r & 0 \end{smallmatrix} \right) {}^t g = \left(\begin{smallmatrix} 0 & 1_r \\ -1_r & 0 \end{smallmatrix} \right) \right\} \tag{2.5.2}$$

in case 1 and the quasi-split unitary group

$$\mathrm{U}(r, r) = \left\{ g \in \mathrm{GL}_{2r}(\mathbb{C}) \;\middle|\; g \left(\begin{smallmatrix} 0 & 1_r \\ -1_r & 0 \end{smallmatrix} \right) {}^t \overline{g} = \left(\begin{smallmatrix} 0 & 1_r \\ -1_r & 0 \end{smallmatrix} \right) \right\} \tag{2.5.3}$$

in case 2. Denote by $\mathrm{Mp}_{2r}(\mathbb{R})$ the metaplectic double cover of $\mathrm{Sp}_{2r}(\mathbb{R})$, and identify it (as a set) with $\mathrm{Sp}_{2r}(\mathbb{R}) \times \{\pm 1\}$ as in [42]. Define

$$G'_r = \begin{cases} \mathrm{Mp}_{2r}(\mathbb{R}), & \text{orthogonal case} \\ \mathrm{U}(r, r), & \text{unitary case.} \end{cases} \tag{2.5.4}$$

Let N_r and M_r be the subgroups of G'_r given by

$$N_r = \{(n(b),1) \mid b \in \mathrm{Sym}_r(\mathbb{R})\} \tag{2.5.5}$$
$$M_r = \{(m(a),\epsilon) \mid a \in \mathrm{GL}_r(\mathbb{R}), \epsilon = \pm 1\} \tag{2.5.6}$$

in case 1 and by

$$\begin{aligned} N_r &= \{n(b) | b \in \mathrm{Her}_r\}, \\ M_r &= \{m(a) | a \in \mathrm{GL}_r(\mathbb{C})\} \end{aligned} \tag{2.5.7}$$

in case 2.

We also fix a maximal compact subgroup K'_r of G'_r as follows. In the orthogonal case, let K'_r be the inverse image under the metaplectic cover of the standard maximal compact subgroup

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a + ib \in \mathrm{U}(r) \right\} \cong \mathrm{U}(r) \tag{2.5.8}$$

of $\mathrm{Sp}_{2r}(\mathbb{R})$. In the unitary case, we define $K'_r = G'_r \cap \mathrm{U}(2r)$. Thus in this case $K'_r \simeq \mathrm{U}(r) \times \mathrm{U}(r)$; an explicit isomorphism $\mathrm{U}(r) \times \mathrm{U}(r) \xrightarrow{\simeq} K'_r$ is given by

$$(k_1, k_2) \mapsto [k_1, k_2] := \frac{1}{2} \begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{pmatrix}. \tag{2.5.9}$$

2.5.2. Recall that we have fixed the additive character $\psi(x) = e^{2\pi i x}$; in the unitary case, we also fix a character $\chi = \chi_V$ of $\mathbb{C}^\times$ such that $\chi|_{\mathbb{R}^\times} = \mathrm{sgn}(\cdot)^m$, where $m = \dim V$. Then $G'_r \times \mathrm{U}(V)$ acts on $\mathcal{S}(V^r)$ via the Weil representation

$$\omega = \begin{cases} \omega_\psi, & \text{orthogonal case,} \\ \omega_{\psi,\chi}, & \text{unitary case.} \end{cases} \tag{2.5.10}$$

(see [14, §1]). Here the action of $\mathrm{U}(V)$ is in both cases given by

$$\omega(g)\phi(\mathbf{v}) = \phi(g^{-1}\mathbf{v}), \quad g \in \mathrm{U}(V),\ \phi \in \mathcal{S}(V^r). \tag{2.5.11}$$

To describe the action of G'_r, we write

$$\begin{aligned} \underline{m}(a) &= \begin{cases} (m(a),1), & \text{for } a \in \mathrm{GL}_r(\mathbb{R}) \text{ in case 1,} \\ m(a), & \text{for } a \in \mathrm{GL}_r(\mathbb{C}) \text{ in case 2,} \end{cases} \\ \underline{n}(b) &= \begin{cases} (n(b),1), & \text{for } b \in \mathrm{Sym}_r(\mathbb{R}) \text{ in case 1,} \\ n(b), & \text{for } b \in \mathrm{Her}_r \text{ in case 2,} \end{cases} \\ \underline{w}_r &= \begin{cases} (w_r,1), & \text{case 1,} \\ w_r, & \text{case 2.} \end{cases} \end{aligned} \tag{2.5.12}$$

Let $|\cdot|_{\mathbb{K}}$ denote the normalized absolute value on $\mathbb{K}$; thus $|z|_{\mathbb{R}} = |z|$ and $|z|_{\mathbb{C}} = z\overline{z}$. Then

$$\begin{aligned}
\omega(\underline{m}(a))\phi(\mathbf{v}) &= |\det a|_{\mathbb{K}}^{m/2}\phi(\mathbf{v}\cdot a)\cdot \begin{cases}\chi_\psi(\det a) & \text{case 1}\\ \chi(\det a) & \text{case 2,}\end{cases}\\
\omega(\underline{n}(b))\phi(\mathbf{v}) &= \psi\left(\operatorname{tr}(bT(\mathbf{v}))\right)\phi(\mathbf{v}),\\
\omega(\underline{w}_r)\phi(\mathbf{v}) &= \gamma_{V^r}\,\hat{\phi}(\mathbf{v});
\end{aligned} \tag{2.5.13}$$

see [23] and also [19, §4.2] for explicit formulas for χ_ψ and γ_{V^r}. Here for $\mathbf{v} = (v_1, \dots, v_r)$ we define $T(\mathbf{v}) = \frac{1}{2}(Q(v_i, v_j))$ and $\hat{\phi}(\mathbf{v})$ denotes the Fourier transform

$$\hat{\phi}(\mathbf{v}) = \int_{V^r} \phi(\mathbf{w})\,\psi\left(\tfrac{1}{2}\operatorname{tr}_{\mathbb{C}/\mathbb{R}}\operatorname{tr}(Q(\mathbf{v},\mathbf{w}))\right) d\mathbf{w}, \tag{2.5.14}$$

where $d\mathbf{w}$ is the self-dual Haar measure on V^r with respect to ψ.

2.5.3. For our purposes it is crucial to understand the action of K'_r on the Schwartz forms φ and ν. This was done for the form φ in [28], where it is shown that φ spans a one-dimensional representation of K'_r. We will show that ν also generates an irreducible representation of K'_r. To describe it we next recall the parametrization of irreducible representations of K'_r by highest weights; we denote by π_λ the (unique up to isomorphism) representation with highest weight λ.

Consider first the orthogonal case. Then K'_r is a double cover of $\mathrm{U}(r)$ and its irreducible representations are π_λ with

$$\lambda = (l_1, \dots, l_r) \in \mathbb{Z}^r \cup (\tfrac{1}{2} + \mathbb{Z})^r, \quad l_1 \geq \dots \geq l_r. \tag{2.5.15}$$

For an integer k, we write $\det^{k/2}$ for the character of K'_r whose square factors through $\mathrm{U}(r)$ and defines the k-th power of the usual determinant character $\det\colon \mathrm{U}(r) \to \mathbb{C}^\times$. Kudla and Millson [28, Thm. 3.1.(ii)] show that

$$\omega(k')\varphi = \det(k')^{m/2}\varphi, \quad k' \in K'_r, \tag{2.5.16}$$

and so φ affords the one-dimensional representation π_l of K'_r with highest weight

$$l := \tfrac{m}{2}(1, \dots, 1). \tag{2.5.17}$$

Now consider the unitary case. Using the isomorphism (2.5.9), any irreducible representation of K'_r is isomorphic to $\pi_{\lambda_1} \boxtimes \pi_{\lambda_2}$ for a unique pair $\lambda = (\lambda_1, \lambda_2)$ of dominant weights of $\mathrm{U}(r)$. Let $k(\chi)$ be the unique integer such that $\chi(z) = (z/|z|)^{k(\chi)}$, and note that $k(\chi)$ and m have the same parity. Kudla and Millson show that

$$\omega([k_1, k_2])\varphi = (\det k_1)^{(m+k(\chi))/2}(\det k_2)^{(-m+k(\chi))/2}\varphi, \quad k_1, k_2 \in \mathrm{U}(r), \tag{2.5.18}$$

and so φ generates the one-dimensional representation π_l of K'_r with highest weight

$$l := (\tfrac{m+k(\chi)}{2}(1, \dots, 1), \tfrac{-m+k(\chi)}{2}(1, \dots, 1)). \tag{2.5.19}$$

We will now determine the K_r' representation generated by the Schwartz form $\nu(\mathbf{v})_{[2r-2]}$. Recall that by Proposition 2.4.5 we can write

$$\nu(\mathbf{v}) = \sum_{1\le i\le r} \nu_i(\mathbf{v}), \tag{2.5.20}$$

where

$$\nu_i(\mathbf{v}) = e^{-\pi Q(v_i,v_i)} \mathrm{tr_s}(N_i e^{\nabla^2_{v_i}}) \wedge \varphi(v_1,\dots,\hat{v}_i,\dots,v_r). \tag{2.5.21}$$

Let $\epsilon_r = 1_r$ and, for $1 \le i < r$, set

$$\epsilon_i = \begin{pmatrix} 1_{i-1} & & & \\ & 0 & & 1 \\ & & 1_{r-i-1} & \\ & -1 & & 0 \end{pmatrix} \in \mathrm{SO}(r). \tag{2.5.22}$$

Then

$$\nu_i(\mathbf{v}) = \omega(\underline{m}(\epsilon_i))\nu_r(\mathbf{v}), \quad 1 \le i \le r; \tag{2.5.23}$$

thus $\nu(\mathbf{v})_{[2r-2]}$ belongs to the K_r'-representation generated by $\nu_r(\mathbf{v})_{[2r-2]}$.

Define a weight λ_0 of K_r' by setting

$$\lambda_0 := (\tfrac{m}{2},\dots,\tfrac{m}{2},\tfrac{m}{2}-2) \tag{2.5.24a}$$

in the orthogonal case, and

$$\lambda_0 := ((\tfrac{m+k(\chi)}{2},\dots,\tfrac{m+k(\chi)}{2},\tfrac{m+k(\chi)}{2}-1),(\tfrac{-m+k(\chi)}{2},\dots,\tfrac{-m+k(\chi)}{2},\tfrac{-m+k(\chi)}{2}+1)) \tag{2.5.24b}$$

in the unitary case.

Lemma 2.5.4. *Let $r \ge 1$ and assume that $\mathrm{rk}(\mathcal{E}) = 1$. Under the action of K_r' via the Weil representation, the form $\nu(\mathbf{v})_{[2r-2]}$ generates an irreducible representation π_{λ_0} with highest weight λ_0. The form $\nu_r(\mathbf{v})_{[2r-2]}$ is a highest weight vector in π_{λ_0}.*

Proof. Assume first that $r = 1$. By (2.3.5), $\nu(z)_{[0]}$ is the Siegel gaussian, which has weight

$$\lambda_0 = \tfrac{p-q}{2} = \tfrac{m}{2} - 2 \tag{2.5.25}$$

in case 1 and

$$\lambda_0 = (\tfrac{p-q+k(\chi)}{2}, \tfrac{q-p+k(\chi)}{2}) = (\tfrac{m+k(\chi)}{2}-1, \tfrac{-m+k(\chi)}{2}+1) \tag{2.5.26}$$

in case 2. Now assume that $r > 1$. By Proposition 2.4.4.(c), we have

$$\nu_r(\mathbf{v})_{[2r-2]} = \nu(v_r)_{[0]} \cdot \varphi(v_1,\dots,v_{r-1})_{[2r-2]} \tag{2.5.27}$$

and hence by (2.5.16), (2.5.18), (2.5.25) and (2.5.26), the form $\nu_r(\mathbf{v})_{[2r-2]}$ has weight λ_0. To show that $\nu_r(\mathbf{v})_{[2r-2]}$ is a highest weight vector, one needs to check that $\omega(\alpha)\nu_r(\mathbf{v})_{[2r-2]} = 0$ for every compact positive root $\alpha \in \Delta_c^+$ (see (3.1.4) and (3.2.6)). This can be done by a direct computation using (2.5.13) and the explicit formulas (2.3.5) and (2.3.1) for $\nu_{[0]}$ and φ, or alternatively as follows. Evaluating at z_0, (2.3.5) and (2.3.1) show that

$$\nu_r(\mathbf{v},z_0) \in (S(V^r)\otimes \wedge\mathfrak{p}^*)^K; \tag{2.5.28}$$

here $S(V^r) \subset \mathcal{S}(V^r)$ is the subspace spanned by functions of the form $e^{-\pi Q_{z_0}(\mathbf{v},\mathbf{v})}p(\mathbf{v})$, where $p(\mathbf{v})$ is a polynomial on V^r. For $\nu_r(\mathbf{v},z_0)$, the degree of these polynomials (called the Howe degree) is $2r-2$. Now it follows from the formulas in [21, §III.6] (see also [13, Prop. 4.2.1] in case 1) that the only K'_r-representation containing the weight λ_0 realized in $S(V^r)$ in Howe degree $2r-2$ is π_{λ_0}, and the statement follows. □

2.6. Green forms.

2.6.1. In this section we will construct certain currents on $\mathbb{D}$ depending on a parameter $\mathbf{v}=(v_1,\dots,v_r)\in V^r$ and having singularities at $\mathbb{D}_{\mathbf{v}}$. We begin with the following special case.

Definition 2.6.2. *A tuple $\mathbf{v}=(v_1,\dots,v_r)\in V^r$ is non-degenerate if $\{v_1,\dots,v_r\}$ is linearly independent. We say that $\mathbf{v}$ is regular if $\mathbb{D}_{\mathbf{v}}$ is either empty or of codimension r in $\mathbb{D}$. Note that the latter occurs if and only if $\mathbf{v}$ is non-degenerate and $v_1,\dots,v_r$ span a positive definite subspace of V.*

If $\mathbf{v}$ is regular, then $s_{\mathbf{v}}$ is a regular section of $(\mathcal{E}^r)^\vee$ in the sense of Section 2.1.1, and $\mathbb{D}_{\mathbf{v}}$ is smooth. Therefore, setting

$$\mathfrak{g}^\circ(\mathbf{v}) := \int_1^\infty \nu^\circ(\sqrt{t}\mathbf{v})_{[2r\cdot\mathrm{rk}(\mathcal{E})-2]}\frac{dt}{t} \in A^{r\cdot\mathrm{rk}(\mathcal{E})-1,r\cdot\mathrm{rk}(\mathcal{E})-1}(\mathbb{D}-\mathbb{D}_{\mathbf{v}}), \tag{2.6.1}$$

Proposition 2.1.8 shows that $\mathfrak{g}^\circ(\mathbf{v})=\mathfrak{g}^\circ(s_{\mathbf{v}})$ is smooth on $\mathbb{D}-\mathbb{D}_{\mathbf{v}}$, locally integrable on $\mathbb{D}$ and, as a current, satisfies

$$\mathrm{dd}^c\mathfrak{g}^\circ(\mathbf{v})+\delta_{\mathbb{D}_{\mathbf{v}}}=\varphi^\circ(\mathbf{v})_{[2r\,\mathrm{rk}(\mathcal{E})]}, \tag{2.6.2}$$

so that $\mathfrak{g}^\circ(\mathbf{v})$ is a Green form for $\mathbb{D}_{\mathbf{v}}$.

Example 2.6.3. Let $v\neq 0\in V$ and assume that $\mathrm{rk}(\mathcal{E})=1$. Then, using (2.3.3), we compute

$$\mathfrak{g}^\circ(v)=\int_1^{+\infty}\nu^\circ(t^{1/2}v)_{[0]}\frac{dt}{t}=\int_1^{+\infty}e^{-2\pi t h(s_v)}\frac{dt}{t}=-\mathrm{Ei}(-2\pi h(s_v)), \tag{2.6.3}$$

where $\mathrm{Ei}(-z)=-\int_1^\infty e^{-zt}\frac{dt}{t}$ denotes the exponential integral. Thus $\mathfrak{g}^\circ(v)$ coincides with the Green form defined in [25, (11.24)]. ◇

2.6.4. If $\mathbf{v}$ is no longer assumed regular, then the integral in (2.6.1) is often no longer convergent. We shall overcome this deficiency by regularization: for any $\mathbf{v}\in V^r$ and $\rho\in\mathbb{C}$ with $\mathrm{Re}(\rho)>0$, define

$$\mathfrak{g}^\circ(\mathbf{v};\rho)=\int_1^{+\infty}\nu^\circ(\sqrt{t}\mathbf{v})_{[2r\cdot\mathrm{rk}(\mathcal{E})-2]}\,\frac{dt}{t^{\rho+1}}. \tag{2.6.4}$$

As $\nu^\circ(t\mathbf{v})$ is bounded as $t\to\infty$, locally uniformly on $\mathbb{D}$, this integral defines a smooth form on $\mathbb{D}$. The integral is an incomplete Mellin transform, in contrast with the usual Mellin transform used by Bismut-Gillet-Soulé in their construction of Bott-Chern currents, see e.g. [3, Section 2].

We will show that (2.6.4) admits a meromorphic continuation (as a current) to a neighbourhood of $\rho=0$, beginning first with the case $r=1$.

Lemma 2.6.5. *For $v \in V$ with $v \neq 0$ and a complex parameter ρ, consider the integral on $\mathbb{D} - \mathbb{D}_v$:*

$$\mathfrak{g}^\circ(v;\rho) \ := \ \int_1^\infty \nu^\circ(\sqrt{t}v)_{[2\,\mathrm{rk}(\mathcal{E})-2]}\,\frac{dt}{t^{1+\rho}}.$$

For $\mathrm{Re}(\rho) > -1/2$, this integral converges to a locally integrable form on $\mathbb{D}$, and the convergence is locally uniform on $\mathbb{D}$ and in ρ.

Proof. Recalling that $\nu^\circ(v) = \mathrm{tr_s}(Ne^{\nabla_v^2})$, it follows by taking $s = 1$ in (2.1.9) that we may write

$$\nu^\circ(\sqrt{t}v)_{[2\mathrm{rk}(\mathcal{E})-2]} \ = \ \sum_{k=0}^{\mathrm{rk}(\mathcal{E})-1} t^k e^{-2\pi\, t\, h(s_v)}\,\eta_k(v) \tag{2.6.5}$$

for some differential forms $\eta_k(v)$ that are smooth on $\mathbb{D}$. For convenience, set $x = 2\pi h(s_v)$; then

$$\mathfrak{g}^\circ(v;\rho) = \sum_{k=0}^{\mathrm{rk}(\mathcal{E})-1} \int_1^\infty t^{k-\rho}e^{-tx}\frac{dt}{t}\cdot\eta_k(v) = \sum_{k=0}^{\mathrm{rk}(\mathcal{E})-1} x^{\rho-k}\left(\int_x^\infty t^{k-\rho}e^{-t}\frac{dt}{t}\right)\eta_k(v). \tag{2.6.6}$$

The forms $\eta_k(v)$ are locally bounded since they are smooth, and a straightforward computation in local coordinates, as in the proof of Proposition 2.1.8, implies that $x^{\rho-k}$ is locally integrable for $\mathrm{Re}(\rho) > -1/2$ and $k \leq \mathrm{rk}(\mathcal{E}) - 1$. As for the integrals, write

$$\begin{aligned}\int_x^\infty t^{k-\rho}e^{-t}\frac{dt}{t} &= \int_x^1 t^{k-\rho}\frac{dt}{t} - \int_x^1 t^{k-\rho}\left(1-e^{-t}\right)\frac{dt}{t} + \int_1^\infty t^{k-\rho}e^{-t}\frac{dt}{t}\\ &= \frac{1-x^{k-\rho}}{k-\rho} - \int_x^1 t^{k-\rho}\left(1-e^{-t}\right)\frac{dt}{t} + \int_1^\infty t^{k-\rho}e^{-t}\frac{dt}{t}.\end{aligned} \tag{2.6.7}$$

The latter two integrals are absolutely bounded uniformly in x and in ρ for $-\frac{1}{2} < \mathrm{Re}(\rho) < \frac{1}{2}$, say, while the first term is evidently holomorphic on this region. In particular, multiplying each of these terms by $x^{\rho-k}$ yields locally integrable functions for $-\frac{1}{2} < \mathrm{Re}(\rho) < \frac{1}{2}$; since the more direct estimate covers the case $\mathrm{Re}(\rho) > 0$, this proves the lemma. ☐

Proposition 2.6.6. *Let $\mathbf{v} = (v_1, \dots, v_r) \in V^r$ and let $r' = \dim\langle v_1, \dots, v_r\rangle$.*

(i) The integral (2.6.4) converges to a smooth form on $\mathbb{D}$ if $\mathrm{Re}(\rho) > 0$.

(ii) Let $k \in \mathrm{O}(r)$ (case 1) or $k \in \mathrm{U}(r)$ (case 2). Then

$$\mathfrak{g}^\circ(\mathbf{v}\cdot k;\rho) = \mathfrak{g}^\circ(\mathbf{v};\rho).$$

(iii) As a current, $\mathfrak{g}^\circ(\mathbf{v};\rho)$ extends meromorphically[2] to the right half plane $\mathrm{Re}(\rho) > -\frac{1}{2}$.

(iv) If $\mathbf{v}$ is regular, then the current $\mathfrak{g}^\circ(\mathbf{v};\rho)$ is regular at $\rho = 0$ and

$$\mathfrak{g}^\circ(\mathbf{v};0) = \mathfrak{g}^\circ(\mathbf{v}).$$

Similarly, for any $\mathbf{v}$, the identity $\mathfrak{g}^\circ(\mathbf{v};0) = \int_1^{+\infty}\nu^\circ(t^{1/2}(\mathbf{v}))\frac{dt}{t}$ holds on $\mathbb{D} - \mathbb{D}_{\mathbf{v}}$.

[2]More precisely, we are asserting that for any compactly supported form η, the expression $\int_{\mathbb{D}}\mathfrak{g}^\circ(\mathbf{v};\rho)\wedge\eta$ admits a meromorphic extension as a function of ρ, and is continuous in η in the sense of distributions.

(v) *The constant term of* $\mathfrak{g}^\circ(\mathbf{v};\rho)$ *at* $\rho=0$ *is given by*

$$\mathrm{CT}_{\rho=0}\ \mathfrak{g}^\circ(\mathbf{v};\rho) = \int_1^\infty \left(\nu^\circ(t^{1/2}\mathbf{v})_{[2r\cdot\mathrm{rk}(\mathcal{E})-2]} - (r-r')\delta_{\mathbb{D}_\mathbf{v}}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge \Omega_{\mathcal{E}^\vee}^{r-r'-1}\right)\frac{dt}{t}$$

(vi) *The constant term* $\underset{\rho=0}{\mathrm{CT}}\,\mathfrak{g}^\circ(\mathbf{v};\rho)$ *satisfies the equation*

$$\mathrm{dd^c}\underset{\rho=0}{\mathrm{CT}}\,\mathfrak{g}^\circ(\mathbf{v};\rho) + \delta_{\mathbb{D}_\mathbf{v}}\wedge\Omega_{\mathcal{E}^\vee}^{r-r'} = \varphi^\circ(\mathbf{v})_{[2r\cdot\mathrm{rk}(\mathcal{E})]}$$

of currents on $\mathbb{D}$, *where* $\Omega_{\mathcal{E}^\vee} = c^{\mathrm{top}}(\mathcal{E}^\vee,\nabla)^*$.

Proof. Part (i) follows immediately from the expression (2.1.9), which shows that (locally on $\mathbb{D}$) $\nu^\circ(\sqrt{t}\mathbf{v})$ and its partial derivatives stay bounded as $t\to+\infty$. Part (ii) follows from Proposition 2.4.5.(f).

To show (iii), let k be as in (ii) such that $\mathbf{v}\cdot k = (\mathbf{0}_{r-r'},\mathbf{v}')$ with $\mathbf{v}'$ non-degenerate. For convenience, set $q'=\mathrm{rk}(\mathcal{E})$. By Propositions 2.4.4.(c) and 2.4.5.(c),(f) we can write

$$\begin{aligned}\nu^\circ(\mathbf{v})_{[2rq'-2]} &= \nu^\circ(\mathbf{v}\cdot k)_{[2rq'-2]}\\ &= \nu^\circ(\mathbf{v}')_{[2r'q'-2]}\wedge\varphi^\circ(\mathbf{0}_{r-r'})_{[2(r-r')q']} + \nu^\circ(\mathbf{0}_{r-r'})_{[2(r-r')q'-2]}\wedge\varphi^\circ(\mathbf{v}')_{[2r'q']}.\end{aligned} \tag{2.6.8}$$

The same propositions also show that

$$\begin{aligned}\varphi^\circ(\mathbf{0}_{r-r'})_{[2(r-r')q']} &= \Omega_{\mathcal{E}^\vee}^{r-r'}\\ \nu^\circ(\mathbf{0}_{r-r'})_{[2(r-r')q'-2]} &= (r-r')c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-r'-1}\end{aligned} \tag{2.6.9}$$

and hence

$$\begin{aligned}\nu^\circ(\mathbf{v})_{[2rq'-2]} &= \nu^\circ(\mathbf{v}')_{[2r'q'-2]}\wedge\Omega_{\mathcal{E}^\vee}^{r-r'}\\ &\quad + \varphi^\circ(\mathbf{v}')_{[2r'q']}\wedge(r-r')c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-r'-1}.\end{aligned} \tag{2.6.10}$$

Consider the contribution of each term in the last expression to $\mathfrak{g}^\circ(\mathbf{v};\rho)$. Writing $\mathbf{v}_i' = (v_1',\dots,\widehat{v_i'},\dots,v_{r'}')$, the first term contributes

$$\sum_{1\le i\le r'}\int_1^{+\infty}\nu^\circ(\sqrt{t}v_i')_{[2q'-2]}\wedge\varphi^\circ(\sqrt{t}\mathbf{v}_i')_{[2(r'-1)q']}\frac{dt}{t^{\rho+1}}\wedge\Omega_{\mathcal{E}^\vee}^{r-r'}. \tag{2.6.11}$$

Since $\varphi^\circ(\sqrt{t}\mathbf{v}_i')$ stays bounded as $t\to+\infty$, Lemma 2.6.5 shows that (2.6.11) converges to a locally integrable form on $\mathbb{D}$ for $\mathrm{Re}(\rho)>-1/2$.

The contribution of the second term is

$$\int_1^{+\infty}\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']}\frac{dt}{t^{\rho+1}}\wedge(r-r')c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-r'-1}, \tag{2.6.12}$$

hence it suffices to prove meromorphic continuation of the integral in this expression. We rewrite this integral as

$$\int_1^{+\infty}\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']}\frac{dt}{t^{\rho+1}} = \int_1^{+\infty}(\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']}-\delta_{\mathbb{D}_\mathbf{v}})\frac{dt}{t^{\rho+1}} + \frac{1}{\rho}\delta_{\mathbb{D}_\mathbf{v}}; \tag{2.6.13}$$

then (2.1.19) implies that the integral on the right hand side converges, as a current, when $\mathrm{Re}(\rho) > -1/2$, and so the right hand side provides the desired meromorphic continuation to $\mathrm{Re}(\rho) > -1/2$, proving (iii).

When $\mathbf{v}$ is regular, or for general $\mathbf{v}$ upon restriction to $\mathbb{D} - \mathbb{D}_{\mathbf{v}}$, the proof of (iii) shows that the integral defining $\mathfrak{g}^\circ(\mathbf{v};\rho)$ converges when $\mathrm{Re}(\rho) > -1/2$; thus we can set $\rho = 0$ and obtain (iv).

To prove (v), we proceed as in (iii) and analyze the contribution to $\mathrm{CT}_{\rho=0}\ \mathfrak{g}^\circ(\mathbf{v};\rho)$ of each summand in the right hand side of (2.6.10). Observe that the constant term at $\rho = 0$ of (2.6.11) is simply

$$\sum_{1\leq i\leq r'} \int_1^{+\infty} \nu^\circ(\sqrt{t}v_i')_{[2q'-2]} \wedge \varphi^\circ(\sqrt{t}\mathbf{v}_i')_{[2(r'-1)q']}\frac{dt}{t} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'}, \tag{2.6.14}$$

whereas the constant term of (2.6.13) is

$$\int_1^{+\infty} (\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']} - \delta_{\mathbb{D}_{\mathbf{v}}})\frac{dt}{t}. \tag{2.6.15}$$

Substituting in (2.6.12) and adding these two contributions gives (v).

Finally, note that (by (ii) and Proposition 2.4.4.(f)) all terms in (vi) are invariant under replacing $\mathbf{v}$ with $\mathbf{v}\cdot k$ for any matrix $k \in \mathrm{O}(r)$ (case 1) or $k \in \mathrm{U}(r)$ (case 2). Thus we can assume that $\mathbf{v} = (\mathbf{0}_{r-r'}, \mathbf{v}')$, where $\mathbf{v}' \in V^{r'}$ is non-degenerate. Then $\nu^\circ(\mathbf{v})_{[2rq'-2]}$ is given by (2.6.10); since $\varphi^\circ(\mathbf{v}')$ is closed, we conclude that

$$\begin{aligned} \mathrm{dd^c}\nu^\circ(\sqrt{t}\mathbf{v})_{[2rq'-2]} &= \mathrm{dd^c}\nu^\circ(\sqrt{t}\mathbf{v}')_{[2r'q'-2]} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'} \\ &= -t\frac{d}{dt}\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'}. \end{aligned} \tag{2.6.16}$$

Using (v) gives

$$\begin{aligned} \mathrm{dd^c}\mathrm{CT}_{\rho=0}\ \mathfrak{g}^\circ(\mathbf{v};\rho) &= \int_1^\infty \mathrm{dd^c}\Big(\nu^\circ(t^{1/2}\mathbf{v})_{[2rq'-2]} \\ &\qquad -(r-r')\delta_{\mathbb{D}_{\mathbf{v}}} \wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^* \wedge \Omega_{\mathcal{E}^\vee}^{r-r'-1}\Big)\frac{dt}{t} \\ &= \int_1^\infty \mathrm{dd^c}\nu^\circ(t^{1/2}\mathbf{v})_{[2rq'-2]}\frac{dt}{t} \\ &= \int_1^\infty -t\frac{d}{dt}\varphi^\circ(t^{1/2}\mathbf{v}')_{[2r'q']}\frac{dt}{t} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'} \\ &= \varphi^\circ(\mathbf{v})_{[2rq']} - \lim_{t\to\infty}\varphi^\circ(t^{1/2}\mathbf{v}')_{[2r'q']} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'} \\ &= \varphi^\circ(\mathbf{v})_{[2rq']} - \delta_{\mathbb{D}_{\mathbf{v}}} \wedge \Omega_{\mathcal{E}^\vee}^{r-r'}, \end{aligned} \tag{2.6.17}$$

where the last equality follows from Theorem 2.1.7, proving (vi). □

Definition 2.6.7. *Let* $\mathbf{v} = (v_1, \ldots, v_r) \in V^r$*. Define*

$$\mathfrak{g}^\circ(\mathbf{v}) = \mathrm{CT}_{\rho=0}\ \mathfrak{g}^\circ(\mathbf{v};\rho) \in D^*(\mathbb{D}).$$

Example 2.6.8. Suppose that $\mathbf{v} \in V^r$ is degenerate and choose k in $\mathrm{O}(r)$ or $\mathrm{U}(r)$ such that $\mathbf{v}\cdot k = (\mathbf{0}_{r-r'}, \mathbf{v}')$, where $\mathbf{v}' \in V^{r'}$ is non-degenerate. Then the proof of Proposition 2.6.6.(iii) shows that

$$\mathfrak{g}^\circ(\mathbf{v}) = \mathfrak{g}^\circ(\mathbf{v}') \wedge \Omega_{\mathcal{E}^\vee}^{r-r'} + \mu(\mathbf{v}'), \tag{2.6.18}$$

where

$$\mu(\mathbf{v}') = (r-r') \int_1^{+\infty} \left(\varphi^\circ(\sqrt{t}\mathbf{v}')_{[2r'q']} - \delta_{\mathbb{D}_\mathbf{v}}\right) \frac{dt}{t} \cdot c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee, \nabla)^* \wedge \Omega_{\mathcal{E}^\vee}^{r-r'-1}. \tag{2.6.19}$$

2.7. **Star products.** Let k, l be positive integers with $(k+l)\mathrm{rk}(\mathcal{E}) \leq \dim(\mathbb{D}) + 1$. Fix regular tuples $\mathbf{v}' = (v_1', \ldots, v_k')$ and $\mathbf{v}'' = (v_1'', \ldots, v_l'') \in V^l$ such that the tuple

$$\mathbf{v} = (v_1, \ldots, v_{k+l}) := (v_1', \ldots, v_k', v_1'', \ldots, v_l'') \in V^{k+l} \tag{2.7.1}$$

is also regular. Then $\mathbb{D}_{\mathbf{v}'}$ and $\mathbb{D}_{\mathbf{v}''}$ intersect transversely (if both are non-empty), as follows from the description of the tangent spaces $T_z\mathbb{D}_{\mathbf{v}'}$ and $T_z\mathbb{D}_{\mathbf{v}''}$ in Section 2.2.2.

Define the star product of the Green forms $\mathfrak{g}^\circ(\mathbf{v}')$ and $\mathfrak{g}^\circ(\mathbf{v}'')$, which for the regular case are given by (2.6.1), by

$$\mathfrak{g}^\circ(\mathbf{v}') * \mathfrak{g}^\circ(\mathbf{v}'') = \mathfrak{g}^\circ(\mathbf{v}') \wedge \delta_{\mathbb{D}_{\mathbf{v}''}} + \varphi^\circ(\mathbf{v}')_{[2\mathrm{rk}(\mathcal{E})k]} \wedge \mathfrak{g}^\circ(\mathbf{v}'') \in D^{2(k+l)\mathrm{rk}(\mathcal{E})-2}(\mathbb{D}). \tag{2.7.2}$$

Our next goal is to compare the currents $\mathfrak{g}^\circ(\mathbf{v})$ and $\mathfrak{g}^\circ(\mathbf{v}') * \mathfrak{g}^\circ(\mathbf{v}'')$. For $t_1, t_2 \in \mathbb{R}_{>0}$, define

$$\begin{aligned} \alpha(t_1, t_2, \mathbf{v}', \mathbf{v}'') &= \frac{i}{2\pi}\, \nu^\circ(t_1^{1/2}\mathbf{v}')_{[2k\cdot\mathrm{rk}(\mathcal{E})-2]} \wedge \overline{\partial}(\nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2l\cdot\mathrm{rk}(\mathcal{E})-2]}) \wedge \frac{dt_1\,dt_2}{t_1\,t_2}, \\ \beta(t_1, t_2, \mathbf{v}', \mathbf{v}'') &= \frac{i}{2\pi}\, \partial(\nu^\circ(t_1^{1/2}\mathbf{v}')_{[2k\cdot\mathrm{rk}(\mathcal{E})-2]}) \wedge \nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2l\cdot\mathrm{rk}(\mathcal{E})-2]} \wedge \frac{dt_1\,dt_2}{t_1\,t_2}. \end{aligned} \tag{2.7.3}$$

We set

$$\begin{aligned} \alpha(\mathbf{v}', \mathbf{v}'') &= \int\limits_{1\leq t_1 \leq t_2 \leq +\infty} \alpha(t_1, t_2, \mathbf{v}', \mathbf{v}''), \\ \beta(\mathbf{v}', \mathbf{v}'') &= \int\limits_{1\leq t_1 \leq t_2 \leq +\infty} \beta(t_1, t_2, \mathbf{v}', \mathbf{v}''). \end{aligned} \tag{2.7.4}$$

The estimate (2.1.10) shows that the integrals converge to smooth forms on $\mathbb{D} - (\mathbb{D}_{\mathbf{v}'} \cup \mathbb{D}_{\mathbf{v}''})$.

Lemma 2.7.1. *The forms $\alpha(\mathbf{v}', \mathbf{v}'')$ and $\beta(\mathbf{v}', \mathbf{v}'')$ are locally integrable on $\mathbb{D}$.*

Proof. First, fix $t_1 > 1$ and let $l' := l \cdot \mathrm{rk}\,\mathcal{E}$, and consider the integral

$$\int_{t_1}^{\infty} \overline{\partial} \left(\nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2l'-2]}\right) \frac{dt_2}{t_2}, \tag{2.7.5}$$

which defines a smooth form on $\mathbb{D} - \mathbb{D}_{\mathbf{v}''}$. Fix $x \in \mathbb{D}_{\mathbf{v}''}$, choose local coordinates $z_1, \ldots, z_N$ ($N = \dim\,\mathbb{D}$) on a complex open ball U around x such that $\mathbb{D}_{\mathbf{v}''} \cap U$ is given by $z_1 = \cdots = z_{l'} = 0$ and let $|y| = (|z_1|^2 + \cdots + |z_{l'}|^2)^{1/2}$. Specializing [3, Theorem 1.4] to our situation, we may identify $\overline{\partial}(\nu^\circ(t^{1/2}\mathbf{v}'')_{[2l'-2]})$ as the component in complex bi-degree $(l'-1, l')$ of the form $-\operatorname{tr}_s\left(\sqrt{t}V\exp(-A_t^2)\right)$ as defined in [3, §1]; then [3, (3.32), (3.33)] imply that

$$|y|^{2l'-1}\int_{t_1}^{\infty}\overline{\partial}\left(\nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2l'-2]}\right)\frac{dt_2}{t_2}$$

is bounded on U, uniformly in t_1. The local integrability of $\alpha(\mathbf{v}',\mathbf{v}'')$ follows, using the fact that $\mathbb{D}_{\mathbf{v}'}$ and $\mathbb{D}_{\mathbf{v}''}$ intersect transversely and the bound for $\int_1^\infty \nu^\circ(t_1^{1/2}\mathbf{v}')_{[2k\cdot \mathrm{rk}(\mathcal{E})-2]}\frac{dt_1}{t_1}$ given by (2.1.22). A similar proof gives the result for $\beta(\mathbf{v}',\mathbf{v}'')$. □

Let $[\alpha(\mathbf{v}',\mathbf{v}'')]$ (resp. $[\beta(\mathbf{v}',\mathbf{v}'')]$) denote the current on $\mathbb{D}$ defined by integration against $\alpha(\mathbf{v}',\mathbf{v}'')$ (resp. $\beta(\mathbf{v}',\mathbf{v}'')$) on $\mathbb{D}$.

Theorem 2.7.2. *The currents $[\alpha(\mathbf{v}',\mathbf{v}'')]$ and $[\beta(\mathbf{v}',\mathbf{v}'')]$ satisfy $g_*\alpha[(\mathbf{v}',\mathbf{v}'')] = [\alpha(g\mathbf{v}',g\mathbf{v}'')]$ and $g_*[\beta(\mathbf{v}',\mathbf{v}'')] = [\beta(g\mathbf{v}',g\mathbf{v}'')]$ for $g\in G$ and*

$$\mathfrak{g}^\circ(\mathbf{v}') * \mathfrak{g}^\circ(\mathbf{v}'') - \mathfrak{g}^\circ(\mathbf{v}) = \partial[\alpha(\mathbf{v}',\mathbf{v}'')] + \overline{\partial}[\beta(\mathbf{v}',\mathbf{v}'')].$$

Proof. Set $q' = \mathrm{rk}(\mathcal{E})$. Denote by t_1, t_2 the coordinates on $\mathbb{R}^2_{>0}$ and consider the form $\widetilde{\nu}^\circ(\mathbf{v}) \in A^*(\mathbb{D}\times\mathbb{R}^2_{>0})$ defined by

$$\begin{aligned}\widetilde{\nu}^\circ(\mathbf{v}) \;=\; &\nu^\circ(\sqrt{t_1}\mathbf{v}')_{[2kq'-2]} \wedge \varphi^\circ(\sqrt{t_2}\mathbf{v}'')_{[2lq']}\frac{dt_1}{t_1}\\ &+ \varphi^\circ(\sqrt{t_1}\mathbf{v}')_{[2kq']} \wedge \nu^\circ(\sqrt{t_2}\mathbf{v}'')_{[2lq'-2]}\frac{dt_2}{t_2}.\end{aligned} \tag{2.7.6}$$

For a piecewise smooth path $\gamma : I \to \mathbb{R}^2_{>0}$ ($I\subset\mathbb{R}$ a closed interval) and $\alpha\in A^*(\mathbb{D}\times\mathbb{R}^2_{>0})$, let

$$\int_\gamma \alpha \in A^{*-1}(\mathbb{D}) \tag{2.7.7}$$

be the form obtained by integrating $(\mathrm{id}\times\gamma)^*\alpha$ along the fibers of the projection of $\mathbb{D}\times I\to\mathbb{D}$. Fix a real number $M>1$ and consider the paths

$$\gamma_{d,M} = (t,t), \quad \gamma'_M = (1,t), \quad \text{and } \gamma''_M = (t, M) \tag{2.7.8}$$

with $t\in[1,M]$. By Proposition 2.4.5.(a), we have

$$\lim_{M\to\infty}\int_{\gamma_{d,M}}\widetilde{\nu}^\circ(\mathbf{v}) \;=\; \lim_{M\to\infty}\int_1^M \nu^\circ(t^{1/2}\mathbf{v})_{[2(k+l)q'-2]}\frac{dt}{t} \;=\; \mathfrak{g}^\circ(\mathbf{v}). \tag{2.7.9}$$

Next, note that

$$(\mathrm{id}\times\gamma'_M)^*\,\widetilde{\nu}^\circ(\mathbf{v}) = \varphi^\circ(\mathbf{v}')_{[2kq']}\wedge\nu^\circ(t^{1/2}\mathbf{v}'')_{[2lq'-2]}\wedge\frac{dt}{t}, \tag{2.7.10}$$

so that

$$\lim_{M\to+\infty}\int_{\gamma_{M'}}\widetilde{\nu}^\circ(\mathbf{v}) \;=\; \varphi^\circ(\mathbf{v}')_{[2kq']}\wedge\mathfrak{g}^\circ(\mathbf{v}''). \tag{2.7.11}$$

Finally, we have

$$(\mathrm{id}\times\gamma''_M)^*\widetilde{\nu}^\circ(\mathbf{v}) = \nu^\circ(t^{1/2}\mathbf{v}')_{[2kq'-2]}\wedge\varphi^\circ(M^{1/2}\mathbf{v}'')_{[2lq']}\wedge\frac{dt}{t}. \tag{2.7.12}$$

Let $N^*_{\mathbf{v}'}$ and $N^*_{\mathbf{v}''}$ denote the real conormal bundles of $\mathbb{D}_{\mathbf{v}'}$ and $\mathbb{D}_{\mathbf{v}''}$ respectively; then $N^*_{\mathbf{v}'} \cap N^*_{\mathbf{v}''} = \{0\}$ since $\mathbb{D}_{\mathbf{v}'}$ and $\mathbb{D}_{\mathbf{v}''}$ intersect transversely. Recall that the wave front set of $\mathfrak{g}^\circ(\mathbf{v}')$ is contained in $N^*_{\mathbf{v}'}$, see Proposition 2.1.8, and that

$$\varphi^\circ(M^{1/2}\mathbf{v}'')_{[2lq']} \xrightarrow{M\to\infty} \delta_{\mathbb{D}''_{\mathbf{v}}} \quad \text{in} \quad D_{N^*_{\mathbf{v}''}}(\mathbb{D})$$

as in Theorem 2.1.7; by [16, Chap. VIII.2] we have

$$\lim_{M\to\infty} \int_{\gamma_{M''}} \widetilde{\nu}^\circ(\mathbf{v}) = \lim_{M\to\infty} \int_1^M \nu^\circ(t^{1/2}\mathbf{v}')_{[2kq'-2]} \frac{dt}{t} \wedge \varphi^\circ(M^{1/2}\mathbf{v}'')_{[2lq']} = \mathfrak{g}^\circ(\mathbf{v}') \wedge \delta_{\mathbb{D}_{\mathbf{v}''}} \tag{2.7.13}$$

and hence

$$\lim_{M\to\infty} \left(\int_{\gamma_{d,M}} \widetilde{\nu}^\circ(\mathbf{v}) - \int_{\gamma'_M+\gamma''_M} \widetilde{\nu}^\circ(\mathbf{v}) \right) \;=\; \mathfrak{g}^\circ(\mathbf{v}) - \mathfrak{g}^\circ(\mathbf{v}') * \mathfrak{g}^\circ(\mathbf{v}''). \tag{2.7.14}$$

Let

$$\Delta_M \;=\; \{(t_1,t_2) \mid 1 \le t_1 \le t_2 \le M\} \;\subset\; \mathbb{R}^2_{>0}, \tag{2.7.15}$$

oriented so that $\partial\Delta_M = \gamma_{d,M} - \gamma'_M - \gamma''_M$. Let $d = d_1 + d_2$ be the differential on $\mathbb{D} \times \mathbb{R}^2_{>0}$, where $d_1 = \partial + \overline{\partial}$ is the differential on $\mathbb{D}$ and d_2 is the differential on $\mathbb{R}^2_{>0}$. Then we have

$$\int_{\gamma_{d,M}} \widetilde{\nu}^\circ(\mathbf{v}) - \int_{\gamma'_M+\gamma''_M} \widetilde{\nu}^\circ(\mathbf{v}) \;=\; \int_{\partial\Delta_M} \widetilde{\nu}^\circ(\mathbf{v}) \;=\; \int_{\Delta_M} d_2\,\widetilde{\nu}^\circ(\mathbf{v}). \tag{2.7.16}$$

Applying Proposition 2.4.5.(b) we obtain

$$\begin{aligned} d_2\,\widetilde{\nu}^\circ(\mathbf{v}) &= \Big(t_1 \frac{d}{dt_1}\varphi^\circ(t_1^{1/2}\mathbf{v}')_{[2kq']} \wedge \nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2lq'-2]} \\ &\qquad - \nu^\circ(t_1^{1/2}\mathbf{v}')_{[2kq'-2]} \wedge t_2 \frac{d}{dt_2}\varphi^\circ(t_2^{1/2}\mathbf{v}'')_{[2lq']}\Big) \wedge \frac{dt_1 dt_2}{t_1 t_2} \\ &= (-2\pi i)^{-1} \Big(-(\partial\overline{\partial}\nu^\circ(t_1^{1/2}\mathbf{v}'))_{[2kq']} \wedge \nu^\circ(t_2^{1/2}\mathbf{v}'')_{[2lq'-2]} \\ &\qquad + \nu^\circ(t_1^{1/2}\mathbf{v}')_{[2kq'-2]} \wedge (\partial\overline{\partial}\nu^\circ(t_2^{1/2}\mathbf{v}''))_{[2lq']}\Big) \wedge \frac{dt_1 dt_2}{t_1 t_2} \\ &= \partial\alpha(t_1,t_2,\mathbf{v}',\mathbf{v}'') + \overline{\partial}\beta(t_1,t_2,\mathbf{v}',\mathbf{v}''). \end{aligned} \tag{2.7.17}$$

The statement follows by taking the limit as $M \to +\infty$ in (2.7.16), and the equivariance property under $g \in G$ follows from Proposition 2.4.5.(d). □

As a corollary, we obtain the following invariance property of star products.

Corollary 2.7.3. *Let $k \in \mathrm{O}(k+l)$ (case 1) or $k \in \mathrm{U}(k+l)$ (case 2) and suppose that $\mathbf{v} = (\mathbf{v}', \mathbf{v}'') \in V^{k+l}$ is non-degenerate. Let $\mathbf{v}'_k$, $\mathbf{v}''_k$ be defined by $\mathbf{v} \cdot k = (\mathbf{v}'_k, \mathbf{v}''_k)$ and set*

$$[\alpha(k;\mathbf{v}',\mathbf{v}'')] = [\alpha(\mathbf{v}',\mathbf{v}'')] - [\alpha(\mathbf{v}'_k,\mathbf{v}''_k)], \quad [\beta(k;\mathbf{v}',\mathbf{v}'')] = [\beta(\mathbf{v}',\mathbf{v}'')] - [\beta(\mathbf{v}'_k,\mathbf{v}''_k)],$$

with α and β as in (2.7.4). Then

$$\mathfrak{g}^\circ(\mathbf{v}') * \mathfrak{g}^\circ(\mathbf{v}'') - \mathfrak{g}^\circ(\mathbf{v}'_k) * \mathfrak{g}^\circ(\mathbf{v}''_k) = \partial[\alpha(k;\mathbf{v}',\mathbf{v}'')] + \overline{\partial}[\beta(k;\mathbf{v}',\mathbf{v}'')] \in D^{2(k+l)\mathrm{rk}(\mathcal{E})-2}(\mathbb{D}).$$

Proof. This is a consequence of the theorem and the invariance property $\mathfrak{g}^\circ(\mathbf{v}\cdot k) = \mathfrak{g}^\circ(\mathbf{v})$, which follows from Proposition 2.4.5.(f). □

When $G = \mathrm{SO}(1,2)^0$, this invariance property is one of the main results in [25], where it is shown to hold by a long explicit computation (see also [37] for a similar proof in arbitrary dimension in case 2 when $q = 1$ and $k + l = p + 1$). Our corollary, and its global counterpart (Corollary 5.3.3), generalizes these results to arbitrary hermitian symmetric spaces of orthogonal or unitary type and gives a conceptual proof.

3. Archimedean heights and derivatives of Whittaker functionals

Let $r \leq p+1$ and $\mathbf{v} = (v_1, \dots, v_r) \in V^r$ be non-degenerate (recall that by this we mean that $v_1, \dots, v_r$ are linearly independent) and denote by $\mathrm{Stab}_G\langle v_1, \dots, v_r\rangle$ the pointwise stabilizer of $\langle v_1, \dots, v_r\rangle$ in G. Let $\mathfrak{g}^\circ(\mathbf{v})$ be the form given in Section 2.6. Assuming that $\mathrm{rk}(\mathcal{E}) = 1$, we will compute the integral

$$\int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1}, \tag{3.0.1}$$

where $\Gamma_{\mathbf{v}}$ is a discrete subgroup of finite covolume in $\mathrm{Stab}_G\langle v_1, \dots, v_r\rangle$, under some additional conditions ensuring that the integral converges. The result is stated in Theorem 3.4.10 and relates this integral to the derivative of a Whittaker functional defined on a degenerate principal series representation of G'_r. Despite the length of this section, its proof is conceptually simple and follows easily from Lemma 2.5.4, which determines the weight of $\nu(\mathbf{v})_{[2r-2]}$, together with results in [31, 35] concerning reducibility of these representations and multiplicity one for their K-types (we review these results in Sections 3.1 and 3.2) and estimates of Shimura [45] for Whittaker functionals that we review in Section 3.3.

3.1. Degenerate principal series of $\mathrm{Mp}_{2r}(\mathbb{R})$.

In this section we fix $r \geq 1$ and let $G' = \mathrm{Mp}_{2r}(\mathbb{R})$. We abbreviate $N = N_r$, $M = M_r$ and $K' = K'_r$ (see 2.5.1).

3.1.1. We denote by $\mathfrak{g}'$ the complexified Lie algebra of G'. We have the Harish-Chandra decomposition

$$\mathfrak{g}' = \mathfrak{p}_+ \oplus \mathfrak{p}_- \oplus \mathfrak{k}', \tag{3.1.1}$$

with

$$\begin{aligned}
\mathfrak{k}' &= \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \middle| \, X_1^t = -X_1, X_2^t = X_2 \right\}, \\
\mathfrak{p}_+ &= \left\{ p_+(X) = \frac{1}{2}\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \middle| \, X^t = X \right\}, \\
\mathfrak{p}_- &= \left\{ p_-(X) = \overline{p_+(X)} = \frac{1}{2}\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix} \middle| \, X^t = X \right\}.
\end{aligned} \tag{3.1.2}$$

Note that $\mathfrak{k}' = \mathrm{Lie}(K')_{\mathbb{C}}$, where K' is the maximal compact subgroup of G' in Section 2.5.1. For $x = (x_1, \ldots, x_r) \in \mathbb{C}^r$, let $d(x)$ denote the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_r)$, write

$$h(x) = \begin{pmatrix} & -i \cdot d(x) \\ i \cdot d(x) & \end{pmatrix} \tag{3.1.3}$$

and define $e_j(h(x)) = x_j$. Then $\mathfrak{h}' = \{h(x) | x \in \mathbb{C}^r\}$ is a Cartan subalgebra of $\mathfrak{k}'$, and we choose the set of positive roots $\Delta^+ = \Delta_c^+ \sqcup \Delta_{nc}^+$ given by

$$\begin{aligned} \Delta_c^+ &= \{e_i - e_j | 1 \leq i < j \leq r\}, \\ \Delta_{nc}^+ &= \{e_i + e_j | 1 \leq i \leq j \leq r\}, \end{aligned} \tag{3.1.4}$$

where Δ_c^+ and Δ_{nc}^+ denote the compact and non-compact roots respectively.

3.1.2. The group $P = MN$ is a maximal parabolic subgroup of G', the inverse image under the covering map of the standard Siegel parabolic of $\mathrm{Sp}_{2r}(\mathbb{R})$. The group M has a character of order four given by

$$\chi(m(a), \epsilon) = \epsilon \cdot \begin{cases} i, & \text{if } \det a < 0, \\ 1, & \text{if } \det a > 0. \end{cases} \tag{3.1.5}$$

For $\alpha \in \mathbb{Z}/4\mathbb{Z}$ and $s \in \mathbb{C}$, consider the character

$$\chi^\alpha |\cdot|^s : M \to \mathbb{R}, \quad (m(a), \epsilon) \mapsto \chi(m(a), \epsilon)^\alpha |\det a|^s, \tag{3.1.6}$$

which extends to a character of P by declaring it trivial on N. Consider the smooth induced representation

$$I^\alpha(s) = \mathrm{Ind}_P^{G'} \chi^\alpha |\cdot|^s \tag{3.1.7}$$

with its $\mathcal{C}^\infty$ topology, where the induction is normalized so that $I^\alpha(s)$ is unitary when $\mathrm{Re}(s) = 0$. In concrete terms, $I^\alpha(s)$ consists of smooth functions $\Phi \colon G' \to \mathbb{C}$ satisfying

$$\Phi\left((m(a), \epsilon)\, \underline{n}(b)\, g'\right) = \chi(m(a), \epsilon)^\alpha\, |\det a|^{s+\rho_r}\, \Phi(g'), \quad \rho_r := \tfrac{r+1}{2}, \tag{3.1.8}$$

with the action of G' defined by $r(g')\Phi(x) = \Phi(xg')$. Note that, by the Cartan decomposition $G' = PK'$, any such function is determined by its restriction to K'; in particular, given $\Phi(s_0) \in I^\alpha(s_0)$, there is a unique family $(\Phi(s) \in I^\alpha(s))_{s \in \mathbb{C}}$ such that $\Phi(s)|_{K'} = \Phi(s_0)|_{K'}$ for all s. Such a family is called a standard section of $I^\alpha(s)$.

3.1.3. We denote by χ^α the character of K' whose differential restricted to $\mathfrak{h}'$ has weight $\frac{\alpha}{2}(1, \ldots, 1)$. The K'-types appearing in $I^\alpha(s)$ were determined by Kudla and Rallis [31] to be precisely those irreducible representations π_λ of K' with highest weight $\lambda = (l_1, \ldots, l_r)$ (here $l_1 \geq \cdots \geq l_r$) such that $\pi_\lambda \otimes (\chi^\alpha)^{-1}$ descends to an irreducible representation of $\mathrm{U}(r)$ and satisfies

$$l_i \in \frac{\alpha}{2} + 2\mathbb{Z}, \quad 1 \leq i \leq r. \tag{3.1.9}$$

Moreover, these K'-types appear with multiplicity one in $I^\alpha(s)$ (op. cit., p.31). If $\Phi^\lambda(\cdot, s) \in I^\alpha(s)$ is a non-zero highest weight vector of weight λ, then $\Phi^\lambda(e, s) \neq 0$ (op. cit., Prop. 1.1), hence from now on we normalize all such highest weight vectors so that

$\Phi^\lambda(e,s)=1$. Note that the restriction of Φ^λ to K' is independent of s. For scalar weights $\lambda = l(1,\dots,1)$ with $l \in \frac{\alpha}{2}+2\mathbb{Z}$, we have

$$\Phi^l(k',s) := \Phi^{l(1,\dots,1)}(k',s) = (\det k')^l, \quad k' \in K'. \tag{3.1.10}$$

3.1.4. Suppose that $X \in \mathfrak{g}'$ is a highest weight vector for K' of weight λ. Then $X\Phi^l(s)$ is a highest weight vector of weight $\lambda + l$ and hence multiplicity one of K'-types implies that

$$X\Phi^l(s) = c(X,l,s)\Phi^{\lambda+l}(s) \tag{3.1.11}$$

for some constant $c(X,l,s) \in \mathbb{C}$. We will need to determine this constant explicitly for certain choices of X; let

$$e_i = (0,\dots,0,\underset{i-\mathrm{th}}{1},0,\dots,0) \tag{3.1.12}$$

and define $X_i^- = p_-(d(e_i))$. When $i=r$, the vector X_r^- is a highest weight vector for K' of weight $-2e_r$. Let $\iota_r\colon \mathrm{Mp}_2(\mathbb{R}) \to G'$ be the embedding defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon\right) \mapsto \left(\begin{pmatrix} 1_{r-1} & & 0_{r-1} & \\ & a & & b \\ 0_{r-1} & & 1_{r-1} & \\ & c & & d \end{pmatrix}, \epsilon\right). \tag{3.1.13}$$

Then, for any $\Phi \in I^\alpha(s)$, the value of $X_r^-\Phi(e,s)$ only depends on the pullback function $\iota_r^*\Phi(s) : \mathrm{Mp}_2(\mathbb{R}) \to \mathbb{C}$. Note that every element g' of $\mathrm{Mp}_2(\mathbb{R})$ can be written uniquely in the form

$$g' \;=\; \Big(n(x)m(y^{1/2}),\, 1\Big)\,\tilde{k}_\theta, \tag{3.1.14}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$, $\theta \in \mathbb{R}/4\pi\mathbb{Z}$ and we define

$$\tilde{k}_\theta = \begin{cases} (k_\theta, 1), & \text{if } -\pi < \theta \le \pi, \\ (k_\theta, -1), & \text{if } \pi < \theta \le 3\pi, \end{cases} \qquad \text{where } k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.1.15}$$

We think of x, y and θ as coordinates on $\mathrm{Mp}_2(\mathbb{R})$. In terms of these coordinates, we have

$$X_r^-\Phi \;=\; \left(-2iy\frac{d}{d\overline{\tau}} + \frac{i}{2}\frac{d}{d\theta}\right)\iota_r^*\Phi, \tag{3.1.16}$$

for any $\Phi \in I^\alpha(s)$, where $\frac{d}{d\overline{\tau}} = \frac{1}{2}\left(\frac{d}{dx} + i\frac{d}{dy}\right)$. Taking $\Phi = \Phi^l$, the following lemma is a straightforward computation using the explicit expression

$$\iota_r^*\Phi^l(x,y,\theta,s) = y^{\frac{1}{2}\left(s+\frac{r+1}{2}\right)}e^{i\theta l}. \tag{3.1.17}$$

Lemma 3.1.5. *Let $l \in \frac{\alpha}{2}+2\mathbb{Z}$. Then*

$$X_r^-\Phi^{l(1,\dots,1)}(s) = \tfrac{1}{2}\left(s + \tfrac{r+1}{2} - l\right)\Phi^{l(1,\dots,1)-2e_r}(s). \quad \square$$

In other words, this shows that $c(X_r^-, l, s) = \frac{1}{2}\left(s + \frac{r+1}{2} - l\right)$.

3.1.6. We now return to the quadratic space V over $\mathbb{R}$ of signature $(p, 2)$. Define

$$I_r(V, s) = I^{\dim V}(s) \qquad \text{and} \qquad s_0 = \frac{p - r + 1}{2}; \tag{3.1.18}$$

we assume that $r \leq p + 1$, so that

$$s_0 \geq 0. \tag{3.1.19}$$

Consider the Weil representation $\omega = \omega_\psi$ of $G' \times \mathrm{O}(V)$ on $\mathcal{S}(V^r)$, as in Section 2.5.2, and let $R_r(V)$ be its maximal quotient on which $(\mathfrak{so}(V), \mathrm{O}(V^+) \times \mathrm{O}(V^-))$ acts trivially. The map

$$\lambda \colon \mathcal{S}(V^r) \ \to \ I_r(V, s_0), \qquad \lambda(\varphi)(g') := \big(\omega(g')\varphi\big)(0) \tag{3.1.20}$$

is G'-intertwining and factors through $R_r(V)$. By [31], it defines an embedding

$$R_r(V) \hookrightarrow I_r(V, s_0). \tag{3.1.21}$$

This embedding provides the link between the Schwartz forms defined in Section 2.5.2 and the Whittaker functionals and Eisenstein series that will figure in our main results, see Sections 3.4 and 5.3 below.

3.2. **Degenerate principal series of** $\mathrm{U}(r, r)$**.** In this section we fix $r \geq 1$ and let $G' = \mathrm{U}(r, r)$. We abbreviate $N = N_r$, $M = M_r$ and $K' = K'_r$ (see Section 2.5.1). Our setup follows [20].

3.2.1. We denote by $\mathfrak{g}'$ the complexified Lie algebra of G' and let $\mathfrak{g}'_{ss} = \{X \in \mathfrak{g}' | \mathrm{tr} X = 0\}$. Let

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_r & 1_r \\ i1_r & -i1_r \end{pmatrix} \in \mathrm{U}(2r), \tag{3.2.1}$$

and note that $u^{-1}G'u$ is the isometry group of the Hermitian form determined by $\left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_r \end{smallmatrix}\right)$. In addition, we have

$$K'_r = \left\{ [k_1, k_2] = u \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} u^{-1} \mid k_1, k_2 \in \mathrm{U}(r) \right\}, \tag{3.2.2}$$

see (2.5.9), and the Harish-Chandra decomposition

$$\mathfrak{g}' = \mathfrak{p}_+ \oplus \mathfrak{p}_- \oplus \mathfrak{k}', \tag{3.2.3}$$

with $\mathfrak{g}' = \mathrm{Mat}_{2r}(\mathbb{C})$ and

$$\begin{aligned} \mathfrak{k}' &= \mathrm{Lie}(K'_r)_{\mathbb{C}} = \left\{ u \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} u^{-1} \middle| X_1, X_2 \in \mathrm{Mat}_r(\mathbb{C}) \right\}, \\ \mathfrak{p}_+ &= \left\{ p_+(X) := u \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} u^{-1} \middle| X \in \mathrm{Mat}_r(\mathbb{C}) \right\}, \\ \mathfrak{p}_- &= \left\{ p_-(X) := u \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} u^{-1} \middle| X \in \mathrm{Mat}_r(\mathbb{C}) \right\}. \end{aligned} \tag{3.2.4}$$

Let $\mathfrak{k}'_{ss} = \mathfrak{k}' \cap \mathfrak{g}'_{ss}$ and

$$\mathfrak{h} = \left\{ u \cdot d(x) \cdot u^{-1} \mid x = (x_1, \ldots, x_{2r}) \in \mathbb{C}^{2r}, \ x_1 + \ldots + x_{2r} = 0 \right\}. \tag{3.2.5}$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}'_{ss}$. For $1 \leq i \leq 2r$, the assignment $u \cdot d(x) \cdot u^{-1} \mapsto x_i$ defines a functional $e_i \colon \mathfrak{h} \to \mathbb{C}$. We write Δ for the set of roots of $(\mathfrak{g}_{ss}, \mathfrak{h})$ and fix the set of positive roots $\Delta^+ = \Delta_c^+ \sqcup \Delta_{nc}^+$ given by

$$\begin{aligned} \Delta_c^+ &= \{e_i - e_j | 1 \leq i < j \leq r\} \cup \{-e_i + e_j | r < i < j \leq 2r\}, \\ \Delta_{nc}^+ &= \{e_i - e_j | 1 \leq i \leq r < j \leq 2r\}, \end{aligned} \tag{3.2.6}$$

where Δ_c^+ and Δ_{nc}^+ denote the compact and non-compact roots respectively.

3.2.2. The group $P = MN$ is the Siegel parabolic of G'. For a character χ of $\mathbb{C}^\times$ and $s \in \mathbb{C}$, define a character $\chi|\cdot|_{\mathbb{C}}^s : P \to \mathbb{C}^\times$ by

$$\chi|\cdot|_{\mathbb{C}}^s(m(a)n(b)) = \chi(\det a)|\det a|_{\mathbb{C}}^s \tag{3.2.7}$$

and let

$$I(\chi, s) = \mathrm{Ind}_P^{G'}(\chi|\cdot|_{\mathbb{C}}^s) \tag{3.2.8}$$

be the degenerate principal series representation of G'. Thus $I(\chi, s)$ is the space of smooth functions $\Phi : G' \to \mathbb{C}$ satisfying

$$\Phi(m(a)n(b)g') = \chi(\det a)|\det a|_{\mathbb{C}}^{s+\rho_r}\Phi(g'), \quad \rho_r := \tfrac{r}{2}, \tag{3.2.9}$$

and the action of G' is via right translation: $r(g')\Phi(x) = \Phi(xg')$. Here the induction is normalized so that the $I(\chi, s)$ is unitary when χ is a unitary character and $s = 0$. Note that, by the Cartan decomposition $G' = PK'$, any such function is determined by its restriction to K'; in particular, given $\Phi(s_0) \in I(\chi, s_0)$, there is a unique family $(\Phi(s) \in I(\chi, s))_{s\in\mathbb{C}}$ such that $\Phi(s)|_{K'} = \Phi(s_0)|_{K'}$ for all s. Such a family is called a standard section of $I(\chi, s)$.

3.2.3. Some useful facts on K'-types of $I(\chi, s)$ were proved by Lee [35], namely that $I(\chi, s)$ is multiplicity free as a representation of K', and if $\Phi^{(\lambda_1,\lambda_2)}(\cdot, s) \in I(\chi, s)$ is a highest weight vector of weight (λ_1, λ_2), then $\Phi^{(\lambda_1,\lambda_2)}(e, s) \neq 0$. Hence from now on we normalize all such highest weight vectors so that $\Phi^{(\lambda_1,\lambda_2)}(e, s) = 1$. Note that the restriction of $\Phi^{(\lambda_1,\lambda_2)}$ to K' is independent of s, and for scalar weights

$$l = (l_1(1, \ldots, 1), l_2(1, \ldots, 1)), \quad l_1, l_2 \in \mathbb{Z}, \tag{3.2.10}$$

we have

$$\Phi^{(l_1,l_2)}([k_1, k_2], s) := \Phi^{(l_1(1,\ldots,1),l_2(1,\ldots,1))}([k_1, k_2], s) = (\det k_1)^{l_1}(\det k_2)^{l_2}, \quad k_1, k_2 \in \mathrm{U}(r). \tag{3.2.11}$$

3.2.4. Let $X_r^- \in \mathfrak{p}_-$ be as in Section 3.1.4; that is, $X_r^- = p_-(d(e_r))$ with $e_r = (0, \ldots, 0, 1) \in \mathbb{C}^r$. Then X_r^- is a highest weight vector for K' of weight $(-e_r, e_r)$ and so

$$X_r^- \Phi^l(s) \ = \ c(X_r^-, l, s)\, \Phi^{l+(-e_r,e_r)}(s) \tag{3.2.12}$$

for some constant $c(X_r^-, l, s) \in \mathbb{C}$. To compute this constant, note that $\mathrm{U}(r,r) \cap \mathrm{GL}_{2r}(\mathbb{R}) = \mathrm{Sp}_{2r}(\mathbb{R})$ and $X_r^- \in \mathfrak{sp}_{2r,\mathbb{C}}$. Moreover, if $\Phi \in I(\chi, s)$, then the restriction $\Phi|_{\mathrm{Sp}_{2r}(\mathbb{R})}$ belongs to $I^\alpha(s')$, where $s' = 2s + (r-1)/2$ and $\alpha = 0$ if $\chi|_{\mathbb{R}^\times}$ is trivial and $\alpha = 2$ otherwise (this follows directly from a comparison of (3.1.8) and (3.2.9)). If $l = (l_1(1,\dots,1), l_2(1,\dots,1))$ is a scalar weight, then

$$\Phi^l|_{\mathrm{Sp}_{2r}(\mathbb{R})}(s) = \Phi^{(l_1-l_2)\cdot(1,\dots,1)}(s') \in I^\alpha(s') \tag{3.2.13}$$

and so Lemma 3.1.5 and the above remarks show that the constant in (3.2.12) is given by

$$c(X_r^-, l, s) = s + \rho_r - \frac{l_1 - l_2}{2}. \tag{3.2.14}$$

3.2.5. We now return to the m-dimensional hermitian space V over $\mathbb{C}$ of signature (p,q) with $pq \neq 0$. From now on we fix a character $\chi = \chi_V$ of $\mathbb{C}^\times$ such that $\chi|_{\mathbb{R}^\times} = \mathrm{sgn}(\cdot)^m$ and define

$$\begin{aligned} I_r(V,s) &= I(\chi,s), \\ s_0 &= \frac{m-r}{2}. \end{aligned} \tag{3.2.15}$$

We assume that $r \leq p+1$, so that

$$s_0 \geq \frac{q-1}{2} \geq 0, \tag{3.2.16}$$

with equality only when $q = 1$ and $r = p+1$.

Consider the Weil representation $\omega = \omega_{\psi,\chi}$ of $G' \times \mathrm{U}(V)$ on $\mathcal{S}(V^r)$, as in Section 2.5.2, and let $R_r(V)$ be its maximal quotient on which $(\mathfrak{u}(V), K)$ acts trivially. The map

$$\lambda \colon \mathcal{S}(V^r) \to I_r(V,s_0), \qquad \lambda(\varphi)(g') := \big(\omega(g')\varphi\big)(0) \tag{3.2.17}$$

is G'-intertwining and factors through $R_r(V)$; by [36, Thm. 4.1], it defines an embedding

$$R_r(V) \hookrightarrow I_r(V,s_0). \tag{3.2.18}$$

This embedding is the crucial link relating the Schwartz forms in Section 2.5.2 to the Whittaker functionals and Eisenstein series appearing in our main results, see Sections 3.4 and 5.3 below.

3.3. **Whittaker functionals.**

3.3.1. Let

$$T \in \begin{cases} \mathrm{Sym}_r(\mathbb{R}), & \text{case 1} \\ \mathrm{Her}_r, & \text{case 2} \end{cases} \tag{3.3.1}$$

be a non-singular matrix and let $\psi_T \colon N_r \to \mathbb{C}^\times$ denote the character defined by

$$\psi_T(\underline{n}(b)) = \psi(\mathrm{tr}(Tb)) = e^{2\pi i\, \mathrm{tr}(Tb)}. \tag{3.3.2}$$

Let $I_r(V,s)$ and s_0 be as in (3.1.18) (resp. (3.2.15)) in case 1 (resp. in case 2). A continuous functional $l \colon I_r(V,s_0) \to \mathbb{C}$ is a called a Whittaker functional if it satisfies

$$l(r(n)\Phi) = \psi_T(n) \cdot l(\Phi) \quad \text{for all } n \in N_r,\ \Phi \in I_r(V,s_0). \tag{3.3.3}$$

Such a functional can be constructed as follows. Let dn be the Haar measure on N_r that is self-dual with respect to the pairing $(\underline{n}(b), \underline{n}(b')) \mapsto \psi(\mathrm{tr}(bb'))$. Embed $\Phi \in I_r(V, s_0)$ in a (unique) standard section $(\Phi(s))_{s\in\mathbb{C}}$ and define

$$W_T(\Phi, s) = \int_{N_r} \Phi(\underline{w}_r^{-1} n, s)\, \psi_T(-n)\, dn \tag{3.3.4}$$

The integral converges for $\mathrm{Re}(s) \gg 0$ and admits holomorphic continuation to all s [49]; its value at s_0 defines a Whittaker functional $W_T(s_0)$. It is shown in op. cit. that $W_T(s_0)$ spans the space of Whittaker functionals. For $g' \in G'_r$ and $\Phi \in I_r(V, s_0)$, define

$$W_T(g', \Phi, s) = \int_{N_r} \Phi(\underline{w}_r^{-1} n g', s)\, \psi_T(-n)\, dn. \tag{3.3.5}$$

and

$$W'_T(g', \Phi, s_0) = \left.\frac{d}{ds} W_T(g', \Phi, s)\right|_{s=s_0}. \tag{3.3.6}$$

3.3.2. In this section, we apply results of [45] to extract some necessary asymptotic estimates for Whittaker functionals. Assume throughout this section that T is non-degenerate. As in Section 2.2.1, let

$$\mathbb{K} = \begin{cases} \mathbb{R}, & \text{orthogonal case} \\ \mathbb{C}, & \text{unitary case,} \end{cases} \qquad \text{and} \qquad \iota := [\mathbb{K} : \mathbb{R}]. \tag{3.3.7}$$

It will be useful for us to work in symmetric space coordinates, as follows. Let $\mathbb{H}_r$ denote the Siegel (resp. Hermitian) upper half space of genus r, so that in the orthogonal case,

$$\mathbb{H}_r = \{\tau = x + iy \in \mathrm{Sym}_r(\mathbb{C}) \mid y > 0\} \tag{3.3.8}$$

and in the unitary case

$$\mathbb{H}_r = \left\{\tau \in \mathrm{Mat}_r(\mathbb{C}) \mid \frac{1}{2i}(\tau - {}^t\overline{\tau}) > 0\right\}; \tag{3.3.9}$$

in the latter case, write $\tau = x + iy$ with $y = \frac{1}{2i}(\tau - {}^t\overline{\tau}) \in \mathrm{Her}_r(\mathbb{C})_{>0}$ and $x = \tau - iy$.

For a point $\tau = x + iy \in \mathbb{H}_r$, fix a matrix $\alpha \in \mathrm{GL}_r(\mathbb{K})$ with $\det \alpha \in \mathbb{R}_{>0}$ and such that $y = \alpha \cdot {}^t\overline{\alpha}$ and let

$$g'_\tau := \underline{n}(x)\underline{m}(\alpha) \in G'_r. \tag{3.3.10}$$

Let $\Phi^l(s)$ be the normalized highest weight vector of $I_r(V, s)$ of scalar weight

$$l := \begin{cases} \dfrac{m}{2}, & \text{orthogonal case} \\ \left(\dfrac{m + k(\chi)}{2}, \dfrac{-m + k(\chi)}{2}\right), & \text{unitary case} \end{cases} \tag{3.3.11}$$

(see (3.1.10) and (3.2.11)) and define

$$\mathcal{W}_T(y, s) := (\det y)^{-\iota m/4}\, W_T(g'_{iy}, \Phi^l, s)\, e^{2\pi \mathrm{tr}(Ty)}; \tag{3.3.12}$$

note that this is independent of the choice of α in (3.3.10).

Proposition 3.3.3. *Suppose T is positive definite of rank r.*

(i) For any integer $k \geq 0$ and any fixed $s \in \mathbb{C}$,

$$\lim_{\lambda \to \infty} \frac{\partial^k}{\partial s^k} \mathcal{W}_T(\lambda y, s) < \infty.$$

(ii) There exists a constant $C > 0$, depending on s, y, T and k, such that

$$\frac{\partial}{\partial \lambda} \left[\frac{\partial^k}{\partial s^k} \mathcal{W}_T(\lambda y, s) \right] = O(\lambda^{-1-C})$$

as $\lambda \to \infty$.

Let $\kappa = 1 + \frac{\iota}{2}(r-1)$ and

$$s_0(r) = \begin{cases} \frac{m-(r+1)}{2}, & \text{orthogonal case} \\ \frac{m-r}{2}, & \text{unitary case.} \end{cases}$$

Then we have the following more precise results for the value and derivative at $s_0(r)$:

(iii)

$$\mathcal{W}_T(y, s_0(r)) \ = \ \frac{(-2\pi i)^{\iota r m/2}}{2^{r(\kappa-1)/2}\, \Gamma_r(\iota m/2)} (\det T)^{\iota s_0(r)}$$

(iv) There is an asymptotic formula

$$\mathcal{W}'_T(\lambda y, s_0(r)) = \left(\frac{\iota}{2}\right) \cdot \frac{(-2\pi i)^{\iota r m/2}\, (\det T)^{\iota\, s_0(r)}}{2^{r(\kappa-1)/2}\, \Gamma_r(\iota m/2)} \left[\log \det \pi T - \frac{\Gamma'_r(\iota m/2)}{\Gamma_r(\iota m/2)} \right] + O(\lambda^{-1})$$

as $\lambda \to \infty$, where the implied constant depends on y and T.

Proof. Write $T = (T^{\frac{1}{2}}) \cdot {}^t\overline{(T^{\frac{1}{2}})}$, for some matrix $T^{\frac{1}{2}} \in \mathrm{GL}_r(\mathbb{K})$. If we define

$$g \ := \ 4\pi \ {}^t\overline{(T^{\frac{1}{2}})} \cdot y \cdot (T^{\frac{1}{2}}), \tag{3.3.13}$$

then applying [45, (1.29), (3.3), (3.6)] gives

$$\mathcal{W}_T(y, s) \ = \ \frac{(-2\pi i)^{\iota r m/2}\, \pi^{r\beta}}{2^{r(\kappa-1)/2}\, \Gamma_r(\beta + \iota m/2)}\, (\det T)^{\beta + \iota m/2 - \kappa}\, \omega(g; \beta + \iota m/2, \beta) \tag{3.3.14}$$

where

- $\beta = \frac{\iota}{2}(s - s_0(r))$,
- $\Gamma_r(\beta) = \pi^{\iota r(r-1)/4} \prod_{k=0}^{r-1} \Gamma(\beta - \iota k/2)$, and
- $\omega(g; \alpha, \beta)$ is an entire function in $(\alpha, \beta) \in \mathbb{C}^2$, initially defined by the formula

$$\omega(g; \alpha, \beta) = \Gamma_r(\beta)^{-1} \ \det(g)^{\beta} \int\limits_{N^+} e^{-\mathrm{tr}(gx)} \det(x+1)^{\alpha-\kappa} \det(x)^{\beta-\kappa} dx \tag{3.3.15}$$

on the region $Re(\beta) > \kappa - 1$, where the integral is absolutely convergent; here we abuse notation and write

$$N^+ = \begin{cases} \mathrm{Sym}_r(\mathbb{R})_{>0}, & \text{orthogonal case} \\ \mathrm{Her}_r(\mathbb{C})_{>0}, & \text{unitary case.} \end{cases} \tag{3.3.16}$$

and take the measure dx to be the standard Euclidean measure, following [45, Section 1]. The analytic continuation of $\omega(g;\alpha,\beta)$ to $(\alpha,\beta)\in\mathbb{C}^2$ is proven in [45, Theorem 3.1].

Replacing y by λy corresponds to replacing g by λg; thus, in light of (3.3.14), in order to prove parts (i) and (ii) of the proposition, it suffices to prove the corresponding estimates for $\omega(g;\alpha,\beta)$. More precisely, we shall show that for fixed g, integers $k,k'\geq 0$, and $(\alpha,\beta)\in\mathbb{C}^2$,

$$\lim_{\lambda\to\infty}\left[\frac{\partial^{k+k'}}{\partial\alpha^k\,\partial\beta^{k'}}\omega(\lambda g;\alpha,\beta)\right] < \infty \tag{3.3.17}$$

and

$$\frac{\partial}{\partial\lambda}\left[\frac{\partial^{k+k'}}{\partial\alpha^k\,\partial\beta^{k'}}\omega(\lambda g;\alpha,\beta)\right] = O(\lambda^{-2}); \tag{3.3.18}$$

the implied constants may depend on k,k',α,β and g.

To prove these estimates, we would like to use (3.3.15), but our choice of parameters (α,β) may place us outside the range of absolute convergence of the integral; to circumvent this, consider the differential operator

$$\Delta = \begin{cases} \det\left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial g_{ij}}\right), & \text{orthogonal case} \\ \det(\frac{\partial}{\partial g_{ij}}), & \text{unitary case} \end{cases} \tag{3.3.19}$$

as in [45, (3.10.I-II)], where g_{ij} are the coordinates for the entries of $g\in N^+$; it satisfies the identity

$$\Delta\, e^{-\mathrm{tr}(g)} \;=\; (-1)^r\, e^{-\mathrm{tr}(g)}. \tag{3.3.20}$$

Then (3.12) and (3.7) of [45] together imply that for $N\in\mathbb{Z}_{>0}$,

$$\begin{aligned} \omega(g;\alpha,\beta) \;&= (-1)^{rN} e^{\mathrm{tr}(g)}\det(g)^{\beta}\cdot\Delta^N\left(e^{-\mathrm{tr}(g)}\det(g)^{-\beta}\omega(z;\alpha-N,\beta)\right) \\ &=(-1)^{rN} e^{\mathrm{tr}(g)}\det(g)^{\beta}\cdot\Delta^N\left[e^{-\mathrm{tr}(g)}\det(g)^{-\beta}\omega\left(g;\kappa-\beta,\kappa-\alpha+N\right)\right]. \end{aligned} \tag{3.3.21}$$

Fixing $N > \mathrm{Re}(\alpha)-1$, the final incarnation of $\omega(...)$ in (3.3.21) is now in the range of convergence of (3.3.15); for convenience, rewrite (3.3.15) for these parameters as

$$\widetilde{\omega}(g;\alpha,\beta) \;:=\; \omega\left(g;\kappa-\beta,\kappa-\alpha+N\right). \tag{3.3.22}$$

By passing partial derivatives in the coordinates of g under the integral (3.3.15) defining $\widetilde{\omega}(g;\alpha,\beta)$, it follows that $\omega(g;\alpha,\beta)$ can be written as a finite sum of terms of the form

$$f(\alpha,\beta)\,(\det g)^{\kappa-\alpha+N-m}\cdot F_1(g)\cdot\int\limits_{N^+} e^{-\mathrm{tr}(gx)}\,F_2(x)\,\det(x+1)^{-\beta}\det(x)^{-\alpha+N}dx \tag{3.3.23}$$

where $f(\alpha,\beta)$ is a holomorphic function independent of g, $m \geq 0$ is an integer, and $F_1(g)$ and $F_2(x)$ are homogeneous polynomials with $\deg F_1 < mr$; here F_1 and F_2 arise as products of iterated partial derivatives, of $\det(g)$ and $e^{-\mathrm{tr}(gx)}$ respectively, with respect to the entries of g.

For a parameter $\lambda > 0$ and fixed g, replacing g by λg and applying a change of variables in the previous display implies that $\omega(\lambda g;\alpha,\beta)$ can be written as a sum of terms of the form

$$\lambda^{-M} f(\alpha,\beta)\,(\det g)^{\kappa-\alpha+N-m}\cdot F_1(g)\cdot \int\limits_{N^+} e^{-\mathrm{tr}(gx)}\,F_2(x)\,\det(\lambda^{-1}x+1)^{-\beta}\det(x)^{-\alpha+N}dx \tag{3.3.24}$$

where $M = mr - \deg F_1 + \deg F_2 \geq 0$. It follows (again, for fixed g, etc.) that $\frac{\partial^{k+k'}}{\partial\alpha^k\partial\beta^{k'}}\omega(\lambda g;\alpha,\beta)$ can be written as a finite sum of terms of the form

$$\begin{aligned}&\lambda^{-M} f(\alpha,\beta)\,(\log\det g)^A(\det g)^{\kappa-\alpha+N-m}\,F_1(g)\\ &\qquad\times\int\limits_{N^+} e^{-\mathrm{tr}(gx)}\,F_2(x)\,\log(\det(\lambda^{-1}x+1))^B\,\det(\lambda^{-1}x+1)^{-\beta}\log(\det x)^C\det(x)^{-\alpha+N}dx\end{aligned} \tag{3.3.25}$$

where $f(\alpha,\beta)$ is independent of λ, and A,B,C are integers.

For $\lambda > 1$, we have the (rather crude) estimates

$$\begin{aligned}|\log(\det(\lambda^{-1}x+1))^B\,\det(\lambda^{-1}x+1)^{-\beta}| &\leq |\det(\lambda^{-1}x+1)^{-\beta+B}|\\ &\leq \begin{cases}\det(x+1)^{-Re(\beta)+B}, & \text{if } B \geq -Re(\beta)\\ 1, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.3.26}$$

Hence, by dominated convergence, we may pass the limit $\lambda \to \infty$ inside the integral in (3.3.25), and, by comparison with (3.3.15), conclude that the limit of this integral exists as $\lambda \to \infty$. Since $\frac{\partial^{k+k'}}{\partial\alpha^k\partial\beta^{k'}}\omega(\lambda g;\alpha,\beta)$ is a sum of terms as in (3.3.25), this shows the desired existence of the limit (3.3.17). To prove the second estimate (3.3.18), differentiate (3.3.25) with respect to λ; dominated convergence again allows us to pass the derivative under the integral sign, and the estimate in the same way. These two estimates in turn imply statements (i) and (ii) of the proposition.

We now turn to the more precise versions at $s = s_0(r)$. To prove (iii), recall that $\beta = 0$ when $s = s_0(r)$; since $\omega(g;\iota m/2,0) \equiv 1$ by [45, (3.15)], evaluating (3.3.14) at $s = s_0(r)$ yields the desired formula.

Finally to prove (*iv*), take the logarithmic derivative with respect to s in (3.3.14) to obtain

$$\frac{2^{r(\kappa-1)/2}\,\Gamma_r(\iota m/2)}{(-2\pi i)^{\iota r m/2}(\det T)^{\iota s_0(r)}}\cdot \mathcal{W}'_T(\lambda y, s_0(r)) \tag{3.3.27}$$
$$= \frac{\mathcal{W}'_T(\lambda y, s_0(r))}{\mathcal{W}_T(\lambda y, s_0(r))} = \frac{\iota}{2}\left(\log\det \pi T - \frac{\Gamma'_r(\iota m/2)}{\Gamma_r(\iota m/2)} + \frac{\partial}{\partial\beta}\omega(\lambda\cdot g;\beta+\iota m/2,\beta)\Big|_{\beta=0}\right).$$

It will therefore suffice to show that $\frac{\partial}{\partial\beta}\omega(\lambda\cdot g;\beta+\iota m/2,\beta)|_{\beta=0} = O(\lambda^{-1})$.

To this end, for sufficiently large integer N, we have

$$\omega(g;\beta+\iota m/2,\beta) \;=\; (-1)^{rN}e^{\mathrm{tr}(g)}\det(g)^{\beta}\cdot\Delta^N\left[e^{-\mathrm{tr}(g)}\det(g)^{-\beta}\widetilde{\omega}\,(g;\beta+\iota m/2,\beta)\right] \tag{3.3.28}$$

with

$$\widetilde{\omega}(g;\beta+\iota m/2,\beta) \;:=\; \omega\,(g;\kappa-\beta,\kappa-\beta-\iota m/2+N)\,. \tag{3.3.29}$$

Consider expanding $\frac{\partial}{\partial\beta}\omega(\lambda g,\beta+\iota m/2,\beta)$ as a sum of terms as in (3.3.25), and note that any terms with either F_1 or F_2 non-constant are $O(\lambda^{-1})$. As these terms arise from the partial derivatives of $\det g$ or $\widetilde{\omega}(g,\beta)$ respectively, it follows that

$$\frac{\partial}{\partial\beta}\omega(\lambda g;\beta+\iota m/2,\beta) = \frac{\partial}{\partial\beta}\widetilde{\omega}(\lambda g;\beta+\iota m/2,\beta) + O(\lambda^{-1}). \tag{3.3.30}$$

We are reduced to proving that $\frac{\partial}{\partial\beta}\widetilde{\omega}(\lambda g;\beta+\iota m/2,\beta)|_{\beta=0}$ is itself $O(\lambda^{-1})$; taking N large enough to ensure the convergence of (3.3.15), and applying a change of variables in the integral, gives

$$\widetilde{\omega}(\lambda\, g;\beta+\iota m/2,\beta) = \frac{(\det g)^{\kappa-\beta-\iota m/2+N}}{\Gamma_r(\kappa-\beta-\iota m/2+N)}\int\limits_{N^+} e^{-\mathrm{tr}(gx)}\det(\lambda^{-1}x+1)^{-\beta}\det(x)^{-\beta-\iota m/2+N}dx. \tag{3.3.31}$$

Now substitute the Taylor expansion

$$\det(\lambda^{-1}x+1)^{-\beta} \;=\; 1 \;-\; \log(\det(\lambda^{-1}x+1))\,\beta \;+\; \dots \tag{3.3.32}$$

at $\beta = 0$ into the previous expression, and note that the integral in (3.3.31) is uniformly convergent for β in a neighbourhood of $\beta = 0$. Moreover, there is a useful integral representation [45, (1.16)]:

$$\int\limits_{N^+} e^{-\mathrm{tr}(gx)}\,\det(x)^{-\beta-\iota m/2+N}\,dx \;=\; \Gamma_r(\kappa-\beta-\iota m/2+N)\,\det(g)^{-\kappa+\beta+\iota m/2-N} \tag{3.3.33}$$

Combining these observations, the Taylor expansion of $\widetilde{\omega}(\lambda g;\beta+\iota m/2,\beta)$ around $\beta=0$ is of the form

$$\widetilde{\omega}(\lambda g;\beta+\iota m/2,\beta) = 1 - \left[\frac{(\det g)^{\kappa-\iota m/2+N}}{\Gamma_r(\kappa-\iota m/2+N)}\int\limits_{N^+} e^{-\mathrm{tr}(gx)}\log(\det(\lambda^{-1}x+1))\det(x)^{-\iota m/2+N}dx\right]\beta$$
$$+\;\text{higher order terms in }\beta. \tag{3.3.34}$$

For fixed g, the coefficient of β is $O(\lambda^{-1})$ as $\lambda \to \infty$, as can be easily deduced from the estimate

$$\log(\det(\lambda^{-1}x + 1)) < \lambda^{-1}\mathrm{tr}(x) \qquad \text{for } x \in N^+. \tag{3.3.35}$$

This proves $\frac{\partial}{\partial\beta}\widetilde{\omega}(\lambda g; \beta + \iota m/2, \beta)|_{\beta=0} = O(\lambda^{-1})$ as required. □

Proposition 3.3.4. *Suppose T is non-degenerate of signature (p,q) with $q > 0$.*

(i) Let $s_0(r)$ be as in Proposition 3.3.3. Then $\mathcal{W}_T(y, s_0(r)) = 0$ for all y.

(ii) For any fixed $s_0 \in \mathbb{C}$, $k \in \mathbb{N}$ and $y > 0$, there are positive constants C and C' such that

$$\left[\frac{\partial^k}{\partial s^k}\mathcal{W}_T(\lambda y, s)\right]_{s=s_0} = O(e^{-C\lambda})$$

and

$$\frac{\partial}{\partial\lambda}\left(\left[\frac{\partial^k}{\partial s^k}\mathcal{W}_T(\lambda y, s)\right]_{s=s_0}\right) = O(e^{-C'\lambda});$$

here C, C' and the implied constants also depend on T, s_0, k and y.

Proof. Following the notation of [45], choose a symmetric positive matrix $y^{\frac{1}{2}}$ such that $y = (y^{\frac{1}{2}})^2$, consider the collection

$$(\mu_1, \dots \mu_r) \tag{3.3.36}$$

of eigenvalues (repeated with multiplicity) of the matrix $y^{\frac{1}{2}}Ty^{\frac{1}{2}}$, and define

$$\delta_+(y,T) \ := \ \prod_{\mu_i>0}\mu_i, \qquad \text{and} \qquad \delta_-(y,T) \ := \ \prod_{\mu_i<0}|\mu_i|. \tag{3.3.37}$$

Then, by [45, (4.34K)] there is a function $\omega(y, T; \alpha, \beta)$ that is holomorphic in $(\alpha, \beta) \in \mathbb{C}^2$ such that

$$\begin{aligned}\mathcal{W}_T(y,s) \ &= \ C_{p,q}(\beta)\left(\frac{\det(y)^{-\beta-\iota m/2+\kappa}}{\Gamma_p(\beta+\iota m/2)\cdot\Gamma_q(\beta)}\right)\delta_+(y,T)^{-\kappa+\beta+\iota m/2+\iota q/4}\delta_-(y,T)^{-\kappa+\beta-\iota p/4} \\ &\quad \times \omega(2\pi y, T; \beta+\iota m/2, \beta)\cdot e^{2\pi\,\mathrm{tr}(yT)}\end{aligned} \tag{3.3.38}$$

where $C_{p,q}(\beta)$ is an entire, nowhere vanishing, function depending only on p and q; here $\beta = \frac{\iota}{2}(s - s_0(r))$. Note that $\Gamma_q(\beta)^{-1}$ vanishes at $\beta = 0$, while the remaining terms are holomorphic; this proves (i).

Moreover, for fixed y and $\lambda \in \mathbb{R}_{>0}$, we may write

$$\mathcal{W}_T(\lambda y, s) = f(y,\beta)\,\frac{\lambda^{-\iota mq/2}}{\Gamma_p(\beta+\iota m/2)\Gamma_q(\beta)}\,\omega(2\pi\lambda y, T; \beta+\iota m/2, \beta)\,e^{2\pi\lambda\,\mathrm{tr}(yT)} \tag{3.3.39}$$

for some function $f(y,\beta)$ that is entire and nowhere-vanishing in β.

By [45, Theorem 4.2], for any compact subset U of $\mathbb{C}$, there are positive constants A and B (depending only on T and U) such that the estimate

$$|\omega(2\pi y, T; \beta+\iota m/2, \beta)\,e^{2\pi\,\mathrm{tr}(yT)}| \ < \ Ae^{-(\tau_-(y,T))}\left(1+\mu(y,T)^{-B}\right) \tag{3.3.40}$$

holds for all $y \in$ and $\beta \in U$; here $\tau_-(y,T) = \sum_{\mu_i<0} |\mu_i|$ and $\mu(y,T) = \min(|\mu_i|)$. Since T is not positive definite, at least one eigenvalue μ_i is negative, so $\tau_-(y,T) > 0$. Replacing y by λy and noting that

$$\tau_-(\lambda y, T) \;=\; \lambda\, \tau_-(y,T) \qquad \text{and} \qquad \mu(\lambda y, T) \;=\; \lambda\, \mu(y,T) \tag{3.3.41}$$

it follows easily that for fixed y, there is a constant C such that

$$\mathcal{W}_T(\lambda y, s) \;=\; O(e^{-C\lambda}) \tag{3.3.42}$$

uniformly for s in some neighbourhood of $s = s_0$, say. This proves (ii) for the case $k = 0$. The estimates for $k \geq 1$ follow immediately from Cauchy's integral formula

$$\left[\frac{\partial^k}{\partial s^k} \mathcal{W}_T(\lambda y, s)\right]_{s=s_0} \;=\; \frac{k!}{2\pi i} \oint \frac{\mathcal{W}_T(\lambda y, s)}{(s-s_0)^{k+1}}\, ds. \tag{3.3.43}$$

Finally, we turn to the derivative with respect to λ. The results of [45, Section 5], see especially Lemma 5.7, imply that for fixed y, the function

$$F(\lambda, s) \;:=\; \mathcal{W}_T(\lambda y, s) \tag{3.3.44}$$

is entire in s and extends to a complex function in λ that is holomorphic on $Re(\lambda) > 0$ and satisfies, in the same manner as before, the asymptotic $\frac{\partial^k}{\partial s^k} F(\lambda, s)|_{s=s_0} = O(e^{-C'\, Re(\lambda)})$ for some constant C'. The proposition follows from another application of Cauchy's integral formula: for a point $\lambda_0 > 1$,

$$\frac{\partial}{\partial \lambda}\left(\left[\frac{\partial^k}{\partial s^k}\mathcal{W}_T(\lambda y, s)\right]_{s=s_0}\right)_{\lambda=\lambda_0} = \frac{1}{2\pi i}\oint \frac{1}{(\lambda-\lambda_0)^2}\frac{\partial^k}{\partial s^k}F(\lambda,s)|_{s=s_0}\, d\lambda \;= O(e^{-C'\lambda_0}), \tag{3.3.45}$$

where the integral is taken around a circle of radius one (say) around λ_0. □

3.4. **The integral of $\mathfrak{g}^\circ$.** For the rest of Section 3 we assume that $\mathrm{rk}(\mathcal{E}) = 1$. Recall that we had constructed a Schwartz form

$$\nu(\mathbf{v}) \;=\; \sum_{i=1}^{r} \nu_i(\mathbf{v}) \tag{3.4.1}$$

for $\mathbf{v} \in V^r$, as in Proposition 2.4.5; note that for the permutation matrices ϵ_i $(1 \leq i \leq r)$ as in (2.5.22), we have $\nu_i(\mathbf{v}) = \omega(\underline{m}(\epsilon_i))\nu_r(\mathbf{v})$.

3.4.1. Let $z_0 \in \mathbb{D}$ be the base point fixed in Section 2.2.1, and consider the Schwartz functions $\tilde{\nu}_i, \tilde{\nu} \in \mathcal{S}(V^r)$ defined by

$$\nu_i(\mathbf{v}, z_0) \wedge \Omega^{p-r+1}(z_0) = \tilde{\nu}_i(\mathbf{v})\, \Omega^p(z_0), \qquad\qquad 1 \leq i \leq r \tag{3.4.2}$$

and

$$\tilde{\nu} := \tilde{\nu}_1 + \ldots, + \tilde{\nu}_r; \tag{3.4.3}$$

here $\Omega = \Omega_{\mathcal{E}} = \frac{i}{2\pi} c_1(\mathcal{E}, \nabla)$.

The next lemma computes the images of $\tilde{\nu}_1, \ldots, \tilde{\nu}_r$ under the map $\lambda\colon \mathcal{S}(V^r) \to I_r(V, s_0)$ described in Sections 3.1.6 and 3.2.5.

Lemma 3.4.2. *Let λ_0 be the weight of K' defined as follows: in the orthogonal case, we take*

$$\lambda_0 := (\tfrac{m}{2}, \dots, \tfrac{m}{2}, \tfrac{m}{2} - 2)$$

and in the unitary case,

$$\lambda_0 := ((\tfrac{m+k(\chi)}{2}, \dots, \tfrac{m+k(\chi)}{2}, \tfrac{m+k(\chi)}{2} - 1), (\tfrac{-m+k(\chi)}{2}, \dots, \tfrac{-m+k(\chi)}{2}, \tfrac{-m+k(\chi)}{2} + 1))$$

where $k(\chi)$ is given in Section 2.5.3. Let $\Phi^{\lambda_0}(s)$ be the unique vector in $I_r(V,s)$ of weight λ_0 with $\Phi^{\lambda_0}(e,s) = 1$. For $1 \le i \le r$, let $\tilde{\Phi}_i(s) = r(\underline{m}(\epsilon_i))\Phi^{\lambda_0}(s)$. Then

$$\lambda\left(\tilde{\nu}_i\right)(g') \;=\; \omega(g')(\tilde{\nu}_i)(0) = \;(-1)^{r-1}\tilde{\Phi}_i(g', s_0), \quad g' \in G'_r.$$

Proof. Since the map $\lambda \colon \mathcal{S}(V^r) \to I_r(V, s_0)$ is G'_r-intertwining, it suffices to consider the case $i = r$. By Proposition 2.4.5 we have $\nu_r(0)_{[2r-2]} = (-\Omega)^{r-1}$, and hence

$$\omega(g')\tilde{\nu}_r(0) = (-1)^{r-1}\Phi^{\lambda_0}(g', s_0), \quad g' \in G'_r, \tag{3.4.4}$$

since both sides define highest weight vectors of weight λ_0 in $I_r(V, s_0)$ (see Lemma 2.5.4) that moreover agree for $g' = 1$, and the K'-types in this representation appear with multiplicity one. □

Thus, setting

$$\tilde{\Phi}(s) = \sum_{1 \le i \le r} r(\underline{m}(\epsilon_i))\Phi^{\lambda_0}(s) \in I_r(V, s), \tag{3.4.5}$$

we have

$$\omega(g')\tilde{\nu}(0) = (-1)^{r-1}\tilde{\Phi}(g', s_0). \tag{3.4.6}$$

The following lemma relating the Whittaker functional $W_T(\cdot, s_0)$ evaluated at Φ^{λ_0} with the derivative $W'_T(\Phi^l, s_0)$ is the main ingredient in the proof of Theorem 3.4.10. It will be convenient to work in classical coordinates: for any $\Phi \in I_r(V,s)$ and $y \in \mathrm{Sym}_r(\mathbb{R})_{>0}$ (resp. $y \in \mathrm{Her}_r(\mathbb{C})_{>0}$) in the orthogonal (resp. unitary) case, let

$$\mathcal{W}_T(y, \Phi, s) \;:=\; (\det y)^{-\iota m/4}\, W_T(g'_{iy}, \tilde{\Phi}, s)\, e^{2\pi \mathrm{tr}(Ty)} \tag{3.4.7}$$

where $g'_{iy} = \underline{m}(\alpha)$ for any matrix $\alpha \in \mathrm{GL}_r(\mathbb{K})$ with $\det\alpha \in \mathbb{R}_{>0}$ and $y = \alpha \cdot {}^t\overline{\alpha}$.

Lemma 3.4.3. *Let $\mathcal{W}_T(y,s) = \mathcal{W}_T(y, \Phi^l, s)$ be given by (3.3.12) and write $\mathcal{W}'_T(y,s) = \frac{d}{ds}\mathcal{W}_T(y,s)$. For any $t \in \mathbb{R}_{>0}$ we have*

$$\mathcal{W}_T(ty, \tilde{\Phi}, s_0) = \frac{2}{\iota} \cdot t\frac{d}{dt}\mathcal{W}'_T(ty, s_0).$$

Proof. We begin with the orthogonal case. If $k \in \mathrm{O}(r)$, a change of variables in (3.3.5) shows that

$$W_T(\underline{m}(k)g, \Phi, s) = W_{{}^t\overline{k}Tk}(g, \Phi, s), \quad \Phi \in I_r(V, s). \tag{3.4.8}$$

Hence it suffices to consider the case where y is diagonal, i.e. we may assume

$$y = d(y_1, \dots, y_r). \tag{3.4.9}$$

The statement is now a direct computation using (3.1.16), as follows. For $1 \leq j \leq r$, let $\tilde{\Phi}_j(s) = r(\underline{m}(\epsilon_j))\Phi^{\lambda_0}(s)$. By Lemma 3.1.5, we can write

$$\tilde{\Phi}_j(s) = 2\,(s-s_0)^{-1}\,\mathrm{Ad}(\underline{m}(\epsilon_j))X_r^-\,\Phi^l(s). \tag{3.4.10}$$

Let $F(g) = W_T(g, \Phi^l, s)$ and $F_j(g) = W_T(g, \tilde{\Phi}_j, s)$ and set $y^{1/2} := d(y_1^{1/2}, \ldots, y_r^{1/2})$. Using coordinates x_j, y_j and θ_j for the embedding

$$\iota_j := \underline{m}(\epsilon_j)\iota_r\underline{m}(\epsilon_j)^{-1} \colon \mathrm{Mp}_2(\mathbb{R}) \to G'_r = \mathrm{Mp}_{2r}(\mathbb{R}) \tag{3.4.11}$$

as in (3.1.13)-(3.1.16), we compute

$$\begin{aligned} 2^{-1}(s-s_0)F_j(\underline{m}(y^{1/2})) &= \mathrm{Ad}(\underline{m}(\epsilon_j))X_r^-\,F(\underline{m}(y^{1/2})) \\ &= \left(-iy_j\frac{d}{dx_j} + y_j\frac{d}{dy_j} + \frac{i}{2}\frac{d}{d\theta_j}\right)F(\underline{m}(y^{1/2})). \end{aligned} \tag{3.4.12}$$

Note that for $\underline{n}(x) \in N_r$ and $k' \in K'_r$ we have

$$W_T\left(\underline{n}(x)\underline{m}(y^{\frac{1}{2}})k', \Phi^l, s\right) \;=\; e^{2\pi i\,\mathrm{tr}(Tx)}\,(\det k')^l\,W_T\left(\underline{m}(y^{\frac{1}{2}}), \Phi^l, s\right) \tag{3.4.13}$$

and hence, in coordinates (x_j, y_j, θ_j) as above, we find

$$\left(-iy_j\frac{d}{dx_j} + y_j\frac{d}{dy_j} + \frac{i}{2}\frac{d}{d\theta_j}\right)F(\underline{m}(y^{1/2})) \;=\; \left(2\pi y_j T_{jj} + y_j\frac{d}{dy_j} - \frac{m}{4}\right)F(\underline{m}(y^{1/2})). \tag{3.4.14}$$

Substituting this expression in (3.4.12) we conclude that

$$\begin{aligned} 2^{-1}\,(s-s_0)\,\mathcal{W}_T(y, \tilde{\Phi}_j, s) &= \prod_{i=1}^r y_i^{-m/4}\cdot\left[(2\pi y_j T_{jj} + y_j\frac{d}{dy_j} - \frac{m}{4})F(\underline{m}(\mathbf{y}^{1/2}))\right]\cdot e^{2\pi \mathrm{tr}(Ty))} \\ &= y_j\frac{d}{dy_j}\mathcal{W}_T(y,s). \end{aligned} \tag{3.4.15}$$

Adding these equations for $j = 1, \ldots, r$, the lemma follows in the orthogonal case by comparing the Taylor expansions around $s = s_0$. The unitary case follows from analogous considerations, using (3.2.14) instead of Lemma 3.1.5. □

3.4.4. Supppose $\det T \neq 0$ and let

$$\Omega_T(V) = \{(v_1, \ldots, v_r) \in V^r \mid (Q(v_i, v_j))_{i,j} = 2T\}. \tag{3.4.16}$$

Thus $\Omega_T(V) \neq \emptyset$ if and only if (V, Q) represents T, and in this case $\mathrm{U}(V) = \mathrm{Aut}(V, Q)$ acts transitively on $\Omega_T(V)$; assuming this, let $d\mu(\mathbf{v})$ be an $\mathrm{U}(V)$-invariant measure on $\Omega_T(V)$ and consider the functional $\mathcal{S}(V^r) \to \mathbb{C}$ defined by

$$\phi \;\mapsto\; \int_{\Omega_T(V)} \phi(\mathbf{v})\,d\mu(\mathbf{v}). \tag{3.4.17}$$

This functional is obviously $\mathrm{U}(V)$-invariant and non-zero, and hence defines a Whittaker functional on $R_r(V)$. We denote by $d\mu(\mathbf{v})^{\mathrm{SW}}$ the unique $\mathrm{U}(V)$-invariant measure on $\Omega_T(V)$ such that, for any $\phi \in \mathcal{S}(V^r)$,

$$\int_{\Omega_T(V)} \phi(\mathbf{v})\, d\mu(\mathbf{v})^{\mathrm{SW}} = \gamma_{V^r}^{-1} \cdot W_T(e, \Phi, s_0), \qquad \text{where } \Phi(g) := \omega(g)\phi(0). \tag{3.4.18}$$

We denote by $dg^{(2)}$ the invariant measure on $\mathrm{U}(V)$ defined as $dg^{(2)} = dp\, dk$, where dk is the unique Haar measure on $\mathrm{O}(V^+) \times \mathrm{O}(V^-)$ (resp. $\mathrm{U}(V^+) \times \mathrm{U}(V^-)$) with total volume one in case 1 (resp. case 2), and dp is the left Haar measure on P induced by the invariant volume form Ω^p on $\mathbb{D}^+ \cong G/K$.

Lemma 3.4.5. *Let λ_0 be given by (2.5.24) and let $\mathbf{v} = (v_1, \dots, v_r) \in \Omega_T(V)$, where $\det T \neq 0$. Let $G_{\mathbf{v}} \subset \mathrm{U}(V)$ be the pointwise stabilizer of $\langle v_1, \dots, v_r\rangle$ and $\Gamma_{\mathbf{v}} \subset G_{\mathbf{v}}^0$ be a torsion free subgroup of finite covolume. Then, for any $g' \in G_r'$, we have*

$$\int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+} \omega(g')\nu(\mathbf{v}) \wedge \Omega^{p-r+1} = -\frac{\iota}{2} C_{T,\Gamma_{\mathbf{v}}} W_T(g', \tilde{\Phi}, s_0),$$

where $\tilde{\Phi}$ is as in (3.4.5) and $C_{T,\Gamma_{\mathbf{v}}}$ is the non-zero constant given by

$$C_{T,\Gamma_{\mathbf{v}}} = \frac{2}{\iota} (-1)^r \operatorname{Vol}(\Gamma_{\mathbf{v}}\backslash G_{\mathbf{v}}, dg_{\mathbf{v}}) \frac{dg^{(2)}/dg_{\mathbf{v}}}{d\mu(\mathbf{v})^{\mathrm{SW}}} \gamma_{V^r}^{-1}.$$

Here $dg_{\mathbf{v}}$ is an arbitrary Haar measure on $G_{\mathbf{v}}$ and $dg^{(2)}/dg_{\mathbf{v}}$ denotes the quotient measure on $\Omega_T(V) \cong \mathrm{U}(V)/G_{\mathbf{v}}$ induced by $dg^{(2)}$ and $dg_{\mathbf{v}}$.

Proof. The hypothesis that $\det T \neq 0$ implies that $G_{\mathbf{v}}^0$ is reductive, hence the invariant measure $dg^{(2)}/dg_{\mathbf{v}}$ exists. By (3.4.2) and Proposition 2.4.5.(d), we have $\tilde{\nu}(gv) = \tilde{\nu}(v)$ for any $g \in \mathrm{U}(V)$ stabilizing V^-. We compute

$$\begin{aligned}
\int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+} \omega(g')\nu(\mathbf{v}) \wedge \Omega^{p-r+1} &= \int_{\Gamma_{\mathbf{v}}\backslash \mathrm{U}(V)} \omega(g')\tilde{\nu}(g^{-1}\mathbf{v})\, dg^{(2)} \\
&= \operatorname{Vol}(\Gamma_{\mathbf{v}}\backslash G_{\mathbf{v}}, dg_{\mathbf{v}}) \int_{G_{\mathbf{v}}\backslash \mathrm{U}(V)} \omega(g')\tilde{\nu}(g^{-1}\mathbf{v}) \frac{dg^{(2)}}{dg_{\mathbf{v}}} \\
&= \operatorname{Vol}(\Gamma_{\mathbf{v}}\backslash G_{\mathbf{v}}, dg_{\mathbf{v}}) \frac{dg^{(2)}/dg_{\mathbf{v}}}{d\mu(\mathbf{v})^{\mathrm{SW}}} \int_{\Omega_T(V)} \omega(g')\tilde{\nu}(\mathbf{v})\, d\mu(\mathbf{v})^{\mathrm{SW}} \\
&= (-1)^{r-1} \operatorname{Vol}(\Gamma_{\mathbf{v}}\backslash G_{\mathbf{v}}, dg_{\mathbf{v}}) \frac{dg^{(2)}/dg_{\mathbf{v}}}{d\mu(\mathbf{v})^{\mathrm{SW}}} \gamma_{V^r}^{-1} W_T(g', \tilde{\Phi}, s_0),
\end{aligned} \tag{3.4.19}$$

where the last equality follows from (3.4.18) and (3.4.6). ∎

Remark 3.4.6. In Remark 5.3.5 we will compute the constant $C_{T,\Gamma_{\mathbf{v}}}$ for certain (arithmetic) cocompact groups $\Gamma \subset G$.

3.4.7. We will now apply the results obtained in this section to compute the integral (3.0.1) under our standing assumption that $\mathrm{rk}(\mathcal{E}) = 1$. Rather than aiming for the most general result, we restrict to cocompact arithmetic subgroups Γ and integral vectors $\mathbf{v}$ (see below); this suffices for the application to compact Shimura varieties in Section 5.

Fix a lattice $L \subset V$ and a torsion free cocompact arithmetic subgroup $\Gamma \subset G$ stabilizing L, and let $\Gamma_{\mathbf{v}} = \mathrm{Stab}_\Gamma\langle v_1, \dots, v_r\rangle$ and $X_\Gamma = \Gamma\backslash\mathbb{D}^+$.

Lemma 3.4.8. *Assume that $\mathbf{v} \in L^r$ is non-degenerate and let $Z(\mathbf{v})_\Gamma$ be the image of the natural map $\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+_{\mathbf{v}} \to X_\Gamma$.*

(a) The sum

$$\mathfrak{g}^\circ(\mathbf{v})_\Gamma = \sum_{\gamma\in\Gamma_{\mathbf{v}}\backslash\Gamma} \mathfrak{g}^\circ(\gamma^{-1}\mathbf{v})$$

converges absolutely to a smooth form on $X_\Gamma - Z(\mathbf{v})_\Gamma$ that is locally integrable on X_Γ.

(b) As currents on X_Γ we have

$$\sum_{\gamma\in\Gamma_{\mathbf{v}}\backslash\Gamma} \int_1^M \nu^\circ(t^{1/2}\gamma^{-1}\mathbf{v})_{[2r-2]}\frac{dt}{t} \xrightarrow{M\to+\infty} \mathfrak{g}^\circ(\mathbf{v})_\Gamma.$$

Proof. Let us first show that $\mathfrak{g}^\circ(\mathbf{v})_\Gamma$ converges as claimed. Let $z \in \mathbb{D}^+$ and pick a relatively compact neighborhood U of z. Recall that there is a positive definite form Q_z defined by (2.3.4) that varies continuously with z. Write $\Gamma\mathbf{v} = S_1 \sqcup S_2$ (disjoint union), with

$$S_1 = \{\mathbf{v}' \in \Gamma\mathbf{v} \,|\, \min_{z\in U} h_z(s_{\mathbf{v}'}) < 1\}; \tag{3.4.20}$$

then S_1 is finite since $\Gamma\mathbf{v} \subset L^r$. We can split the sum defining $\mathfrak{g}^\circ(\mathbf{v})_\Gamma$ accordingly; the sum over S_1 converges to a locally integrable form on $\mathbb{D}^+$ that is smooth outside $\cup_{\mathbf{v}'\in S_1}\mathbb{D}^+_{\mathbf{v}'}$ by Proposition 2.1.8. The sum over S_2 converges to a smooth form on U by the estimate (2.1.10) and the standard argument of convergence of theta series. By Proposition 2.4.5.(d), the current defined by $\mathfrak{g}^\circ(\mathbf{v})_\Gamma$ is invariant under Γ, and this shows (a). For part (b), note that for each $\mathbf{v}' \in S_1$ we have $\int_1^M \nu^\circ(t^{1/2}\mathbf{v}')_{[2r-2]}\frac{dt}{t} \to \mathfrak{g}^\circ(\mathbf{v}')$ as $M \to +\infty$ (as currents on $\mathbb{D}^+$) by dominated convergence, as remarked in the proof of Proposition 2.1.8. For the sum over S_2 we apply the bound (2.4.6), and (b) follows. □

Lemma 3.4.9. *Assume that $\mathbf{v} \in L^r$ is non-degenerate. Then the integral* (3.0.1) *converges and*

$$\int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1} = \lim_{M\to+\infty}\int_1^M \int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+} \nu^\circ(t^{1/2}\mathbf{v}) \wedge \Omega^{p-r+1}\frac{dt}{t}.$$

Proof. By Lemma 3.4.8.(a), the integral (3.0.1) equals $\int_{\Gamma\backslash\mathbb{D}^+}\mathfrak{g}^\circ(\mathbf{v})_\Gamma \wedge \Omega^{p-r+1}$ and so it converges since X_Γ is compact. Applying Lemma 3.4.8.(b) and unfolding the sum and the integral proves the claim. □

The following theorem can be viewed as a local analogue of our main result, Theorem 5.3.1. While the strategy of proof for both theorems is similar, the assumption that T is non-singular simplifies the argument considerably and allows for a local proof that does not use the technical estimates of Section 5.2.

Theorem 3.4.10. *Suppose* $\det T \neq 0$ *and* $\mathbf{v} \in \Omega_T(V) \cap L^r$. *Let* $C_{T,\Gamma_{\mathbf{v}}}$ *be as in Lemma 3.4.5 and* l *be the weight in* (3.3.11). *Recall that we assume that* $\mathrm{rk}(\mathcal{E}) = 1$.

(a) If T *is not positive definite (so that* $\mathbb{D}_{\mathbf{v}} = \emptyset$*), then*

$$e^{-2\pi \mathrm{tr}(T)} \int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1} = C_{T,\Gamma_{\mathbf{v}}} \, W_T'(e, \Phi^l, s_0).$$

(b) If T *is positive definite (so that* $\mathbb{D}_{\mathbf{v}} \neq \emptyset$*), then*

$$\begin{aligned} e^{-2\pi \mathrm{tr}(T)} \int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1} &= C_{T,\Gamma_{\mathbf{v}}} \, W_T'(e, \Phi^l, s_0) \\ &\quad - C_{T,\Gamma_{\mathbf{v}}} \, W_T(e, \Phi^l, s_0) \, \frac{\iota}{2} \left(\log \det(\pi T) - \frac{\Gamma_r'(\iota m/2)}{\Gamma_r(\iota m/2)} \right). \end{aligned}$$

Proof. The integral converges by Lemma 3.4.9. We compute

$$\begin{aligned} \int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1} &= \int_1^\infty \left(\int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \nu^\circ(t^{1/2}\mathbf{v}) \wedge \Omega^{p-r+1} \right) \frac{dt}{t} \\ &= \int_1^\infty e^{2\pi \mathrm{tr}(tT)} \left(\int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \nu(t^{1/2}\mathbf{v}) \wedge \Omega^{p-r+1} \right) \frac{dt}{t} \\ &= \int_1^\infty e^{2\pi \mathrm{tr}(tT)} \left(\int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \omega(\underline{m}(t^{1/2} \cdot 1_r)) \nu(\mathbf{v}) \wedge \Omega^{p-r+1} \right) t^{-\frac{\iota r m}{4}} \, \frac{dt}{t} \\ &= -\frac{\iota}{2} C_{T,\Gamma_{\mathbf{v}}} \int_1^\infty e^{2\pi \mathrm{tr}(tT)} \, W_T(\underline{m}(t^{1/2} \cdot 1_r), \tilde{\Phi}, s_0) \, t^{-\frac{\iota r m}{4}} \, \frac{dt}{t}, \end{aligned} \tag{3.4.21}$$

where we have used Lemma 3.4.9 and Lemma 3.4.5 for the first and last equality respectively. The last integrand equals $\mathcal{W}_T(t \cdot 1_r, \tilde{\Phi}, s_0)$, and so applying Lemma 3.4.3 we conclude that

$$\int_{\Gamma_{\mathbf{v}} \backslash \mathbb{D}^+} \mathfrak{g}^\circ(\mathbf{v}) \wedge \Omega^{p-r+1} = C_{T,\Gamma_{\mathbf{v}}} (\mathcal{W}_T'(e, s_0) - \lim_{t \to \infty} \mathcal{W}_T'(t \cdot 1_r, s_0)). \tag{3.4.22}$$

If T is not positive definite, then the limit in the above expression vanishes by part (ii) of Proposition 3.3.4; this proves (a). If T is positive definite, then (b) follows from Proposition 3.3.3.(iii)-(iv). ∎

4. Green forms for special cycles on Shimura varieties

We now shift focus from the Hermitian symmetric domain $\mathbb{D}$ to its quotients $\Gamma \backslash \mathbb{D}$ by arithmetic subgroups, and apply the results of the previous sections to construct Green forms for the special cycles $Z(T, \varphi_f)$ on orthogonal and unitary Shimura varieties introduced by Kudla in [24].

4.1. Orthogonal Shimura varieties.

Let us briefly recall the definition and basic properties of orthogonal Shimura varieties attached to quadratic spaces over F; see [24] for more detail.

4.1.1. Let F be a totally real field with real embeddings $\sigma_1, \ldots, \sigma_d$.

Let $p \geq 1$ and $\mathbb{V}$ be a quadratic space over F of dimension $p+2$, with corresponding bilinear form $Q(\cdot,\cdot)$. Assume that

$$\operatorname{signature}(\mathbb{V}_{\sigma_i}) \;=\; \begin{cases} (p,2) & \text{if } i=1 \\ (p+2,0) & \text{if } i \neq 1, \end{cases} \tag{4.1.1}$$

where we abbreviate $\mathbb{V}_{\sigma_i} := \mathbb{V} \otimes_{F,\sigma_i} \mathbb{R}$ for the real quadratic spaces at each place.

Attached to $\mathbb{V}_{\sigma_1}$ is the symmetric space $\mathbb{D}(\mathbb{V}) = \mathbb{D}(\mathbb{V}_{\sigma_1})$ defined in (2.2.6); recall that it is defined to be

$$\mathbb{D}(\mathbb{V}) = \mathbb{D}(\mathbb{V}_{\sigma_1}) = \{[v] \in \mathbb{P}(\mathbb{V}_{\sigma_1}(\mathbb{C})) \mid Q(v,v)=0,\, Q(v,\overline{v})<0\}. \tag{4.1.2}$$

Here $Q(\cdot,\cdot)$ is the $\mathbb{C}-$bilinear extension of the bilinear form on $\mathbb{V}_{\sigma_1}$. Let

$$\mathbf{H}_{\mathbb{V}} := \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(\mathbb{V}). \tag{4.1.3}$$

Then

$$\mathbf{H}_{\mathbb{V}}(\mathbb{R}) \simeq \operatorname{GSpin}(\mathbb{V}_{\sigma_1}) \times \cdots \times \operatorname{GSpin}(\mathbb{V}_{\sigma_d}) \tag{4.1.4}$$

acts transitively on $\mathbb{D}(\mathbb{V})$ via the first factor.

Definition 4.1.2. *For a compact open subgroup $K \subset \mathbf{H}(\mathbb{A}_f)$, consider the* Shimura variety

$$X_{\mathbb{V},K} := \mathbf{H}_{\mathbb{V}}(\mathbb{Q}) \backslash \mathbb{D}(\mathbb{V}) \times \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)/K.$$

If K is neat, then $X_{\mathbb{V},K}$ is a complex quasi-projective algebraic variety. If $\mathbb{V}$ is moreover anisotropic, then $X_{\mathbb{V},K}$ is projective.[3]

The space $X_{\mathbb{V},K}$ may be written in a perhaps more familiar fashion: fix a connected component $\mathbb{D}^+ \subset \mathbb{D}(\mathbb{V})$ and let $\mathbf{H}_{\mathbb{V}}(\mathbb{R})^+$ denote its stabilizer in $\mathbf{H}_{\mathbb{V}}(\mathbb{R})$. Setting $\mathbf{H}_{\mathbb{V}}(\mathbb{Q})^+ = \mathbf{H}_{\mathbb{V}}(\mathbb{R})^+ \cap \mathbf{H}_{\mathbb{V}}(\mathbb{Q})$, there exist finitely many elements $h_1, \ldots h_t \in \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)$ such that

$$\mathbf{H}_{\mathbb{V}}(\mathbb{A}_f) = \coprod_j \mathbf{H}_{\mathbb{V}}(\mathbb{Q})^+ h_j K; \tag{4.1.5}$$

then we may write

$$X_{\mathbb{V},K} \simeq \coprod_j \Gamma_j \backslash \mathbb{D}^+ =: \coprod_j \; X_{\mathbb{V},j} \tag{4.1.6}$$

as a disjoint union of quotients of $\mathbb{D}^+$ by discrete subgroups

$$\Gamma_j = \mathbf{H}_{\mathbb{V}}(\mathbb{Q})^+ \cap \left(h_j K h_j^{-1} \right). \tag{4.1.7}$$

In general, we regard the quotients $X_{\mathbb{V},K}$ and $\Gamma_j \backslash \mathbb{D}^+$ as orbifolds. In particular, for a Γ_j-invariant differential form η of top degree on $\mathbb{D}^+$, we define

$$\int\limits_{[\Gamma_j \backslash \mathbb{D}^+]} \eta = [\Gamma_j : \Gamma']^{-1} \int\limits_{\Gamma' \backslash \mathbb{D}^+} \eta, \tag{4.1.8}$$

where $\Gamma' \subset \Gamma_j$ is any neat subgroup of finite index, and set $\int_{[X_K]} = \sum_j \int_{[\Gamma_j \backslash \mathbb{D}^+]}$.

[3]Our assumptions on the signature of $\mathbb{V}$ imply that this is always the case when $F \neq \mathbb{Q}$.

4.1.3. The theory of canonical models of Shimura varieties (see [44]) implies the existence of a quasi-projective model $\mathcal{X}_K$ over $\mathrm{Spec}(F)$, which is projective when $\mathbb{V}$ is anisotropic, such that

$$\mathcal{X}_K \otimes_{F,\sigma_1} \mathbb{C} \simeq X_{\mathbb{V},K}.$$

From the point of view of arithmetic intersection theory, it will be important to work with all the complex fibres of $\mathcal{X}_K$ simultaneously; the remaining fibres have the following concrete description.

For each $k = 2, \ldots, d$, let $\mathbb{V}[k]$ denote a quadratic space over F such that

(i) $\mathbb{V}[k]_{\sigma_k} \simeq \mathbb{V}_{\sigma_1}$, i.e. the signature of $\mathbb{V}[k]_{\sigma_k}$ is $(p, 2)$;
(ii) $\mathbb{V}[k]_{\sigma_1} \simeq \mathbb{V}_{\sigma_k}$;
(iii) and $(\mathbb{V}[k])_w \simeq \mathbb{V}_w$ at all other places.

The space $\mathbb{V}[k]$ is unique up to isometry, and we have $\mathbb{V}[1] \simeq \mathbb{V}$.

Fix, once and for all, identifications

$$\mathbb{V}[k] \otimes_F \mathbb{A}_f \simeq \mathbb{V} \otimes_F \mathbb{A}_f \tag{4.1.9}$$

inducing identifications

$$\mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f) \ \simeq \ \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)$$

for all k, and so in particular we may view $K \subset \mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f)$. Then, setting $\mathbb{D}(\mathbb{V}[k]) = \mathbb{D}(\mathbb{V}[k]_{\sigma_k}) \simeq \mathbb{D}(\mathbb{V})$, the theory of conjugation of Shimura varieties (see [39, 40], as well as [8, Section 7] for our particular situation) gives identifications

$$\mathcal{X}_K \times_{F,\sigma_k} \mathbb{C} \ \simeq \ X_{\mathbb{V}[k],K} \ = \ \mathbf{H}_{\mathbb{V}[k]}(\mathbb{Q}) \Big\backslash \mathbb{D}(\mathbb{V}[k]_{\sigma_k}) \times \mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f) \Big/ K. \tag{4.1.10}$$

In particular, viewing $\mathcal{X}_K$ as a scheme over $\mathbb{Q}$ via the map $\mathrm{Spec}(F) \to \mathrm{Spec}(\mathbb{Q})$, we have

$$\mathcal{X}_K(\mathbb{C}) \ = \ \coprod_{k=1}^{d} X_{\mathbb{V}[k],K}. \tag{4.1.11}$$

4.2. **Special cycles.** Recall that in Section 2.2.2 we have defined the tautological bundle $\mathcal{E}$ over $\mathbb{D}(\mathbb{V}[k])$ and a global section $s_{\mathbf{v}}$ of $(\mathcal{E}^r)^\vee$ for any $\mathbf{v} \in (\mathbb{V}[k]_{\sigma_k})^r$, whose zero locus $Z(s_{\mathbf{v}})$ we denote by $\mathbb{D}_{\mathbf{v}}$. Given a rational vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{V}[k]^r$, we set

$$\mathbb{D}_{\mathbf{v}} = Z(s_{\sigma_k(\mathbf{v})}) \tag{4.2.1}$$

and $\mathbb{D}_{\mathbf{v}}^+ = \mathbb{D}_{\mathbf{v}} \cap \mathbb{D}(\mathbb{V}[k])^+$.

Let $\mathbf{H}_{\mathbf{v}}(\mathbb{Q})$ be the pointwise stabilizer of $\mathrm{span}_F\,\{v_1, \ldots, v_r\}$ in $\mathbf{H}_{\mathbb{V}[k]}(\mathbb{Q})$. Given a component $X_j = \Gamma_j \backslash \mathbb{D}^+ \subset X_{\mathbb{V}[k],K}$ associated to h_j as in (4.1.6), let $\Gamma_j(\mathbf{v}) = \Gamma_j \cap H_{\mathbf{v}}(\mathbb{Q})$; then the natural map

$$\Gamma_j(\mathbf{v}) \backslash \mathbb{D}_{\mathbf{v}}^+ \ \to \ \Gamma_j \backslash \mathbb{D}^+ = X_j \tag{4.2.2}$$

defines a (complex algebraic) cycle on X_j that we denote by $c(\mathbf{v}, X_j)$. In addition, recall that for a matrix $T \in \mathrm{Sym}_r(F)$, we had defined

$$\Omega_T(\mathbb{V}[k]) \ = \ \{\mathbf{v} \in \mathbb{V}[k]^r \mid T(\mathbf{v}) = T\}; \tag{4.2.3}$$

where $T(\mathbf{v}) = (\frac{1}{2} Q(v_i, v_j))_{i,j} \in \mathrm{Sym}_r(F)$ is the moment matrix of $\mathbf{v}$.

Definition 4.2.1 ([24]). *Let $\varphi_f \in \mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r)^K$ be a K-invariant Schwartz function which, for each $k = 1, \ldots, d$, may be viewed as a Schwartz function on $\mathbb{V}[k](\mathbb{A}_f)^r$ via (4.1.9). For $T \in \mathrm{Sym}_r(F)$, define the weighted special cycle $Z(T, \varphi_f, K)$ on $\mathcal{X}_K(\mathbb{C}) = \coprod X_{\mathbb{V}[k],K}$ by*

$$Z(T, \varphi_f, K) \;=\; \sum_{k=1}^{d} \sum_{X_j \subset X_{\mathbb{V}[k],K}} \sum_{\substack{\mathbf{v} \in \Omega_T(\mathbb{V}[k]) \\ \bmod \Gamma_j}} \varphi_f(h_j^{-1}\mathbf{v})\, c(\mathbf{v}, X_j).$$

This is a complex algebraic cycle on $\mathcal{X}_K(\mathbb{C})$ that is in fact defined over F.

It follows from the discussion after (2.2.12) that if $Z(T, \varphi_f, K)$ is non-empty, then T is totally positive semi-definite and the codimension of $Z(T, \varphi_f, K)$ is equal to the rank of T.

Note that this definition is independent of all choices. Moreover, if $K' \subset K$ is an open subgroup of finite index and $\pi\colon \mathcal{X}_{K'} \to \mathcal{X}_K$ is the natural covering map, then $\pi^* Z(T, \varphi_f, K) = Z(T, \varphi_f, K')$. See [24, §5] for a proof of this and further properties of these cycles.

To lighten notation, we will once and for all fix a compact open subgroup $K \subset \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)$, and write, for example, $Z(T, \varphi_f) = Z(T, \varphi_f, K)$, $X_{\mathbb{V}[k]} = X_{\mathbb{V}[k],K}$ and $\mathcal{X} = \mathcal{X}_K$, etc.

4.3. **Green forms for special cycles.**

4.3.1. For the moment, fix a real embedding $\sigma_k\colon F \to \mathbb{R}$ and a component

$$X_j = \Gamma_j \backslash \mathbb{D}^+ \subset X_{\mathbb{V}[k]} \simeq \mathcal{X}_{\sigma_k}(\mathbb{C})$$

with $\mathbb{D}^+ = \mathbb{D}(\mathbb{V}[k]_{\sigma_k})^+$; here Γ_j is attached to $h_j \in \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)$ as in (4.1.7).

Let $T \in \mathrm{Sym}_r(F)$ with $\det T \neq 0$. Any collection of vectors $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{V}[k]^r$ with $T(\mathbf{v}) = T$ is necessarily linearly independent. For such $\mathbf{v}$, we defined in Section 2.6.1 a form satisfying the equation

$$\mathrm{dd^c}\mathfrak{g}^\circ(\sigma_k(\mathbf{v})) + \delta_{\mathbb{D}_\mathbf{v}} \;= \varphi^\circ(\sigma_k(\mathbf{v}))_{[2r]} \tag{4.3.1}$$

of currents on $\mathbb{D}^+$.

Next, we introduce Green forms that, like the Fourier coefficients of derivatives of Siegel Eisenstein series in the next section, depend on an auxiliary parameter $\mathbf{y} \in \mathrm{Sym}_r(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$; here $\mathbf{y}$ should be thought of as the imaginary part of an "automorphic" variable $\boldsymbol{\tau} = \mathbf{x} + i\mathbf{y} \in (\mathbb{H}_r)^d$, where $\mathbb{H}_r$ is the Siegel upper half-plane of genus r.

To this end, fix some element $\boldsymbol{\alpha} \in \mathrm{GL}_r(F_{\mathbb{R}})$ with totally positive determinant such that $\mathbf{y} = \boldsymbol{\alpha} \cdot {}^t\boldsymbol{\alpha}$ and, for a Schwartz function $\varphi_f \in S(\mathbb{V}(\mathbb{A}_f)^r)^K$, define a form $\mathfrak{g}_j(T, \mathbf{y}, \varphi_f)_{\sigma_k}$ on $\mathbb{D}^+$ by setting

$$\mathfrak{g}_j(T, \mathbf{y}, \varphi_f)_{\sigma_k} \;:=\; \sum_{\mathbf{v} \in \Omega_T(\mathbb{V}[k])} \varphi_f(h_j^{-1}\mathbf{v})\; \mathfrak{g}^\circ(\sigma_k(\mathbf{v}) \cdot \sigma_k(\boldsymbol{\alpha})); \tag{4.3.2}$$

here we view $\sigma_k(\boldsymbol{\alpha}) \in \mathrm{GL}_r(\mathbb{R})$ via the $\mathbb{R}$-linear map $\sigma_k\colon F \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ induced by σ_k. The convergence of this sum to a locally integrable form on $\mathbb{D}^+$ follows from Lemma 3.4.8.(a) and the fact that the number of orbits of Γ_j on $\mathrm{Supp}(\varphi_f) \cap \Omega_T(\mathbb{V}[k])$ is finite.

Note that the definition is independent of the choice of $\boldsymbol{\alpha}$, by Proposition 2.4.5.(f). Moreover, the form $\mathfrak{g}_j(T, \mathbf{y}, \varphi_f)_{\sigma_k}$ is invariant under the action of Γ_j by Proposition 2.4.5.(d), and so it descends to a form on the connected component X_j that, abusing notation, we also denote by $\mathfrak{g}_j(T, y, \varphi_f)_{\sigma_k}$.

Finally, let

$$\mathfrak{g}(T, \mathbf{y}, \varphi_f)_{\sigma_k} \tag{4.3.3}$$

denote the form on $X_{\mathbb{V}[k]} \simeq \mathcal{X}_{\sigma_k}(\mathbb{C})$ whose restriction to X_j is $\mathfrak{g}_j(T, \mathbf{y}, \varphi_f)_{\sigma_k}$. Essentially by construction, it is a Green form for the cycle $Z(T, \varphi_f)_{\sigma_k}$; more precisely, let $\omega(T, \mathbf{y}, \varphi_f)_{\sigma_k}$ be the differential form on $X_{\mathbb{V}[k]}$ whose restriction to the component X_j is

$$\omega(T, \mathbf{y}, \varphi_f)_{\sigma_k}\big|_{X_j} \;=\; \sum_{\mathbf{v} \in \Omega_T(\mathbb{V}[k])} \varphi_f(h_j^{-1}\mathbf{v}) \; \varphi^{\circ}(\sigma_k(\mathbf{v}) \cdot \sigma_k(\boldsymbol{\alpha}))_{[2r]}, \tag{4.3.4}$$

Then the form $\omega(T, \mathbf{y}, \varphi_f)_{\sigma_k}$ is the T'th coefficient of the theta function

$$\Theta_{\mathrm{KM}}(\boldsymbol{\tau}; \varphi_f)_{\sigma_k} \;:=\; \sum_{T \in \mathrm{Sym}_r(F)} \omega(T, \mathbf{y}, \varphi_f)_{\sigma_k}\, q^T, \tag{4.3.5}$$

where $\boldsymbol{\tau} \in (\mathbb{H}_r)^d$ and $q^T := \prod_{i=1}^d e^{2\pi i \operatorname{tr}(\tau_i \sigma_i(T))}$. This theta function was considered (in much greater generality) by Kudla and Millson [29].

Applying the identity (4.3.1) of currents on $\mathbb{D}$, summing over $\mathbf{x}$ with $T(\mathbf{x}) = T$ and descending to the Shimura variety $X_{\mathbb{V}[k]}$ yields the equation

$$\mathrm{dd^c}\mathfrak{g}(T, \mathbf{y}, \varphi_f)_{\sigma_k} + \delta_{Z(T, \varphi_f)_{\sigma_k}} = \omega(T, \mathbf{y}, \varphi_f)_{\sigma_k} \tag{4.3.6}$$

of currents on $X_{\mathbb{V}[k]} \simeq \mathcal{X}_{\sigma_k}(\mathbb{C})$. The collection

$$\{\mathfrak{g}(T, \mathbf{y}, \varphi_f)_{\sigma_k} \mid k = 1, \ldots, d\} \tag{4.3.7}$$

defines a Green form $\mathfrak{g}(T, \mathbf{y}, \varphi_f)$ for the cycle $Z(T, \varphi_f)$ on $\mathcal{X}$, in the sense of [46, Chap. II].

4.3.2. We will next construct a current $\mathfrak{g}(T, \mathbf{y}; \varphi_f) = \{\mathfrak{g}(T, \mathbf{y}, \varphi_f)_{\sigma_k}\}$ for an arbitrary matrix $T \in \mathrm{Sym}_r(F)$ and $\mathbf{y} \in \mathrm{Sym}_r(F_{\mathbb{R}})_{\gg 0}$. For the moment, choose an embedding σ_k and a component $X_j = \Gamma_j \backslash \mathbb{D} \subset \mathcal{X}_{\sigma_k}(\mathbb{C})$. Recall that for any $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{V}[k]^r$, we had defined the current

$$\mathfrak{g}^{\circ}(\sigma_k(\mathbf{v}); \rho) \;:=\; \int_1^{\infty} \nu^{\circ}(\sqrt{t}\mathbf{v})_{[2r \cdot \mathrm{rk}(\mathcal{E}) - 2]} \, \frac{dt}{t^{\rho+1}} \tag{4.3.8}$$

on $\mathbb{D}$, see (2.6.4). here ρ is a complex parameter.

Let $\mathbf{y} \in \mathrm{Sym}_r(F_{\mathbb{R}})_{\gg 0}$ and let $\boldsymbol{\alpha} \in \mathrm{GL}_r(F_{\mathbb{R}})_{\gg 0}$ such that $\mathbf{y} = \boldsymbol{\alpha} \cdot {}^t\boldsymbol{\alpha}$. Given a Schwartz function $\varphi_f \in S(\mathbb{V}(\mathbb{A}_f)^r)^K$, consider the sum

$$\mathfrak{g}_j^{\circ}(T, \mathbf{y}, \varphi_f; \rho)_{\sigma_k} := \sum_{\Omega_T(\mathbb{V}[k])} \varphi_f(h_j^{-1}\mathbf{v}) \; \mathfrak{g}^{\circ}\left(\sigma_k(\mathbf{v}) \cdot \sigma_k(\boldsymbol{\alpha}); \rho\right), \tag{4.3.9}$$

viewed as a current on $\mathbb{D}^+ = \mathbb{D}(\mathbb{V}[k]_{\sigma_k})^+$. Note that the right hand side is independent of the choice of $\boldsymbol{\alpha}$ by Proposition 2.6.6.(ii).

Proposition 4.3.3. *The sum* (4.3.9) *converges for* $\mathrm{Re}(\rho) \gg 0$ *to a* Γ_j*-invariant current on* $\mathbb{D}^+$ *that has a meromorphic continuation to* $\mathrm{Re}(\rho) > -1/2$. *In particular, the constant term in the Laurent expansion*

$$\mathfrak{g}_j^{\circ}(T, \mathbf{y}, \varphi_f)_{\sigma_k} := \mathrm{CT}_{\rho=0}\ \mathfrak{g}_j^{\circ}(T, \mathbf{y}, \varphi_f;\, \rho)_{\sigma_k}$$

descends to a current on $X_j = \Gamma_j \backslash \mathbb{D}^+$.

Proof. For convenience, we take $\sigma_k = \sigma_1$ and suppress this index from the notation, writing $y = y_1$ etc. The proof for the other embeddings σ_k is identical.

It suffices to work on a fixed compact subset $K \subset \mathbb{D}^+$. Let

$$S_1 \ := \ \{\mathbf{v} \in \mathbb{V}^r \mid T(\mathbf{v}) = T,\ \varphi_f(h_j^{-1}\mathbf{v}) \neq 0,\ \text{and } \mathbb{D}_{\mathbf{v}} \cap K \neq \emptyset\}, \tag{4.3.10}$$

and

$$S_2 \ := \ \{\mathbf{v} \in \mathbb{V}^r \mid T(\mathbf{v}) = T,\ \varphi_f(h_j^{-1}\mathbf{v}) \neq 0,\ \text{and } \mathbb{D}_{\mathbf{v}} \cap K = \emptyset\}, \tag{4.3.11}$$

so that $S_1 \sqcup S_2$ indexes the non-zero terms appearing on the right hand side of (4.3.9). Note also that S_1 is finite, while there exists a bound $C > 0$ such that

$$\min_{z \in K} \sum_{i=1}^{r} h_z(\sigma(v_i), \sigma(v_i)) > C \tag{4.3.12}$$

for all $\mathbf{v} = (v_1, \dots, v_r) \in S_2$.

By Proposition 2.6.6, the finite sum

$$\sum_{\mathbf{v} \in S_1} \varphi_f(h_j^{-1}\mathbf{v})\ \mathfrak{g}^{\circ}\left(\sigma(\mathbf{v}) \cdot \sigma(\boldsymbol{\alpha}); \rho\right) \tag{4.3.13}$$

converges for ρ large, and has meromorphic continuation, as a current on K, to $\mathrm{Re}(\rho) > -1/2$. For the same sum where $\mathbf{v}$ now runs over S_2, let V be a sufficiently small, relatively compact open neighbourhood of K such that $\mathbb{D}_{\mathbf{v}} \cap \overline{V} = \emptyset$ for all $\mathbf{v} \in S_2$. Then the exponential decay estimate (2.1.10) and standard arguments for the convergence of theta series imply that the sum

$$\sum_{\mathbf{v} \in S_2} \varphi_f(h_j^{-1}\mathbf{v})\ \mathfrak{g}^{\circ}\left(\sigma(\mathbf{v}) \cdot \sigma(\boldsymbol{\alpha}); \rho\right) = \sum_{\mathbf{v} \in S_2} \varphi_f(h_j^{-1}\mathbf{v}) \int_1^{\infty} \nu^{\circ}\left(\sqrt{t}\sigma(\mathbf{v}) \cdot \sigma(\boldsymbol{\alpha})\right) \frac{dt}{t^{\rho+1}} \tag{4.3.14}$$

converges for all $\rho \in \mathbb{C}$ to a family of smooth differential forms on V varying holomorphically in ρ, which in turn defines a current by integration.

□

We next show that by patching together the $\mathfrak{g}_j^{\circ}(T, \mathbf{y}, \varphi_f)_{\sigma_k}$ on $X_{\mathbb{V}[k]}$, we obtain a current satisfying an analogue of Green's equation (4.3.6).

Proposition 4.3.4. *Let* $\mathfrak{g}^{\circ}(T, \mathbf{y}, \varphi_f)$ *denote the current on* $\mathcal{X}(\mathbb{C}) = \coprod_k \mathcal{X}_{\sigma_k}(\mathbb{C})$ *whose restriction to a connected component* $X_j \subset \mathcal{X}_{\sigma_k}(\mathbb{C})$ *is* $\mathfrak{g}_j^{\circ}(T, \mathbf{y}, \varphi_f)_{\sigma_k}$. *Then there is an identity of currents*

$$\mathrm{dd^c}\mathfrak{g}^{\circ}(T, \mathbf{y}, \varphi_f) + \delta_{Z(T,\varphi_f)} \wedge \Omega_{\mathcal{E}^\vee}^{r-\mathrm{rk}(T)} = \omega(T, \mathbf{y}, \varphi_f) \tag{4.3.15}$$

where $\Omega_{\mathcal{E}^\vee} = c_1(\mathcal{E}^\vee, \nabla)^*$ *and* $\omega(T, \mathbf{y}, \varphi_f)$ *is the differential form on* $\mathcal{X}(\mathbb{C})$ *whose restriction to* $\mathcal{X}_{\sigma_k}(\mathbb{C})$ *is* $\omega(T, \mathbf{y}, \varphi_f)_{\sigma_k}$.

Proof. It suffices to prove the given identity, for each component X_j, at the level of Γ_j-invariant currents on $\mathbb{D}$. The estimates in the proof of Proposition 4.3.3 allow us to write

$$\begin{aligned} \mathrm{dd^c}\mathfrak{g}^\circ_j(T, \mathbf{y}, \varphi_f)_{\sigma_k}\Big|_{X_j} &= \mathrm{dd^c}\left[\mathrm{CT}_{\rho=0} \sum_{\mathbf{v}} \varphi_f(h_j^{-1}\mathbf{v})\, \mathfrak{g}^\circ\left(\sigma_k(\mathbf{v}) \cdot \sigma_k(\boldsymbol{\alpha}); \rho\right)\right] \\ &= \sum_{\mathbf{v}} \varphi_f(h_j^{-1}\mathbf{v})\ \mathrm{dd^c}\,\mathrm{CT}_{\rho=0}\, \mathfrak{g}^\circ\left(\sigma_k(\mathbf{v}) \cdot \sigma_k(\boldsymbol{\alpha}); \rho\right). \end{aligned} \tag{4.3.16}$$

The proposition follows immediately from Proposition 2.6.6.(vi).

□

Finally, in order to obtain agreement with the derivatives of Eisenstein series in our main theorem, we introduce a modified version of $\mathfrak{g}^\circ(T, \mathbf{y}, \varphi_f)$. We write $\det' A$ for the product of non-zero eigenvalues of a square matrix A, with the convention $\det'(\mathbf{0}) = 1$.

Definition 4.3.5. *Let* $\mathbf{y} = (y_v)_{v|\infty} \in \mathrm{Sym}_r(F_\mathbb{R})_{\gg 0}$ *and* $T \in \mathrm{Sym}_r(F)$, *and define a current* $\mathfrak{g}(T, \mathbf{y}, \varphi_f) \in D^*(\mathcal{X}(\mathbb{C}))$ *as follows: if* T *is not totally positive semidefinite, set*

$$\mathfrak{g}(T, \mathbf{y}, \varphi_f) := \mathfrak{g}^\circ(T, \mathbf{y}, \varphi_f)$$

and if T *is totally positive semidefinite, set*

$$\begin{aligned} \mathfrak{g}(T, \mathbf{y}, \varphi_f) &:= \mathfrak{g}^\circ(T, \mathbf{y}, \varphi_f) \\ &\qquad - \sum_{v|\infty} \log\left(\frac{\det' \sigma_v(T) \cdot \det y_v}{\det'\left(\sigma_v(T) y_v\right)}\right) \delta_{Z(T, \varphi_f)_{\sigma_v}} \wedge \Omega_{\mathcal{E}^\vee}^{r - \mathrm{rk}(T) - 1} \end{aligned}$$

where $\Omega_{\mathcal{E}^\vee} = c_1(\mathcal{E}^\vee, \nabla)^* = \frac{i}{2\pi} c_1(\mathcal{E}^\vee, \nabla)$. *Note that the additional term is closed, and vanishes if* T *is non-degenerate.*

Note also that when $\det T \neq 0$, the current defined by the Green form (4.3.7) agrees with the one in Definition 4.3.5 by Proposition 2.6.6.(*iv*).

4.4. **Unitary Shimura varieties.** The results in the previous section carry over, essentially verbatim, to the unitary case. To describe the setup, suppose that E is a CM extension of the totally real field F with $[F : \mathbb{Q}] = d$, and that $\mathbb{V}$ is a Hermitian space over E, with Hermitian form $Q(\cdot, \cdot)$.

Fix a CM type $\Phi = \{\sigma_1, \ldots, \sigma_d\} \subset \mathrm{Hom}(E, \mathbb{C})$. For each i, the space $\mathbb{V}_{\sigma_i} = \mathbb{V} \otimes_{\sigma_i, E} \mathbb{C}$ is a complex Hermitian space; we assume

$$\text{signature } \mathbb{V}_{\sigma_i} = \begin{cases} (p, q), & \text{if } i = 1 \\ (p+q, 0), & \text{if } i = 2, \ldots d. \end{cases} \tag{4.4.1}$$

for some integers $p, q > 0$.

Let

$$\mathbb{D}(\mathbb{V}_{\sigma_1}) := \{z \subset \mathbb{V}_{\sigma_1} \text{ negative-definite subspace, } \dim_\mathbb{C} z = q\},$$

(see (2.2.5)) be the symmetric space attached to the real points of the unitary group

$$\mathbf{H}_{\mathbb{V}} \ := \ \mathrm{Res}_{F/\mathbb{Q}}\ \mathrm{U}(\mathbb{V}). \tag{4.4.2}$$

Just as in Section 4.1, a fixed compact open subgroup $K \subset \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)$ determines a complex Shimura variety

$$X_{\mathbb{V}} = X_{\mathbb{V},K} := \mathbf{H}_{\mathbb{V}}(\mathbb{Q})\backslash \mathbb{D}(\mathbb{V}_{\sigma_1}) \times \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)/K \tag{4.4.3}$$

which is quasi-projective, and projective when $\mathbb{V}$ is anisotropic; choosing representatives $h_1, \dots, h_t$ for the double coset space $\mathbf{H}_{\mathbb{V}}(\mathbb{Q})\backslash \mathbf{H}_{\mathbb{V}}(\mathbb{A}_f)/K$ gives a decomposition

$$X_{\mathbb{V}} \ \simeq \ \coprod_j \Gamma_j\backslash \mathbb{D} \ =: \coprod_j X_j, \qquad \text{where } \Gamma_j := \mathbf{H}_{\mathbb{V}}(\mathbb{Q}) \cap \left(h_j K h_j^{-1} \right). \tag{4.4.4}$$

Let $\mathcal{X}$ denote the canonical model over E, so that $\mathcal{X}_{\sigma_1}(\mathbb{C}) \simeq X_{\mathbb{V}}$. For the other complex embeddings, the story is similar to Section 4.1. For each $k = 1, \dots, d$, let $\mathbb{V}[k]$ denote the (unique up to isometry) E-Hermitian space such that

- $\mathbb{V}[k]_{\sigma_k} \simeq \mathbb{V}_{\sigma_1}$;
- $\mathbb{V}[k]_{\sigma_j}$ is positive definite, for $j \neq k$; and
- $\mathbb{V}[k]_v \simeq \mathbb{V}_v$ at all finite places v.

Identifying $\mathbb{V}[k] \otimes_{\mathbb{Q}} \mathbb{A}_f \simeq \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{A}_f$, and in particular, viewing K as a subgroup of $\mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f)$, we may define the complex Shimura variety $X_{\mathbb{V}[k]} = X_{\mathbb{V}[k],K}$ in the same way as $X_{\mathbb{V},K}$.

Now suppose $\rho \in \mathrm{Hom}(E, \mathbb{C})$ and let σ_k be the element of the CM type such that $\rho|_F = \sigma_k|_F$. Then there is an identification

$$\mathcal{X}_\rho(\mathbb{C}) \simeq X_{\mathbb{V}[k]}; \tag{4.4.5}$$

this follows from the general considerations of [40] and [39, Section II.4], or [37, Section 3A] for the case at hand.

The special cycles are defined just as in Section 4.2: recall that a tuple $\mathbf{v} = (v_1, \dots, v_r) \in (\mathbb{V}[k])^r$ determines a section of $(\mathcal{E}^r)^\vee$, where $\mathcal{E}$ is the rank q tautological bundle on $\mathbb{D}(\mathbb{V}[k])$. Its vanishing locus $\mathbb{D}_{\mathbf{v}} \subset \mathbb{D}(\mathbb{V}[k])$ determines a cycle $c(\mathbf{v}, X_j)$ on each component $X_j = \Gamma_j\backslash \mathbb{D}(\mathbb{V}[k])$, which is either empty or of codimension $r'q$, where $r' = \dim \mathrm{span}\{v_1, \dots, v_r\}$. Given a K-invariant Schwartz function $\varphi_f \in \mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r)^K$ and a matrix $T \in \mathrm{Her}_r(E)$, define the (complex) special cycle

$$Z(T, \varphi_f) \ = \sum_{X_j \subset X_{\mathbb{V}[k]} \subset \mathcal{X}(\mathbb{C})} \ \sum_{\substack{\Omega_T(\mathbb{V}[k]) \\ \bmod \Gamma_j}} \varphi_f(h_j^{-1}\mathbf{v})\, c(\mathbf{v}, X_j) \tag{4.4.6}$$

exactly as in Definition 4.2.1, with the sum taken over all connected components of $\mathcal{X}(\mathbb{C})$; as before, these are the complex points of a rational cycle.

Given $\mathbf{y} \in \mathrm{Her}_r(E \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$, use the CM type Φ to identify

$$\mathbf{y} = (y_1, \dots, y_d) \in (\mathrm{Her}_r(\mathbb{C})_{>0})^d \simeq \mathrm{Her}_r(E \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}. \tag{4.4.7}$$

For $\rho\colon E \hookrightarrow \mathbb{C}$, let $\mathfrak{g}(T,\mathbf{y},\varphi_f)_\rho$ denote the current on $\mathcal{X}_\rho(\mathbb{C}) \simeq X_{\mathbb{V}[k]}$ whose restriction to a component $X_j \subset X_{\mathbb{V}[k]}$ is given by

$$\mathfrak{g}^\circ(T,\mathbf{y},\varphi_f)_\rho\big|_{X_j} = \mathrm{CT}_{s=0} \sum_{\mathbf{v}\in\Omega_T(\mathbb{V}[k])} \varphi_f(h_j^{-1}\mathbf{v})\ \mathfrak{g}^\circ\left(\rho(\mathbf{v})\cdot\sigma_k(\boldsymbol{\alpha});s\right). \tag{4.4.8}$$

Here $\boldsymbol{\alpha} \in \mathrm{GL}_r(E_\mathbb{R})$ is any matrix with totally positive determinant such that $\mathbf{y} = \boldsymbol{\alpha}\cdot{}^t\overline{\boldsymbol{\alpha}}$. The independence of the choice of $\boldsymbol{\alpha}$ follows again from Proposition 2.6.6.(ii).

The analogue of Proposition 4.3.4, which can be proved with straightforward modifications to the arguments in the previous section, is:

Proposition 4.4.1. *The following equation of currents on $\mathcal{X}_\rho(\mathbb{C})$ holds:*

$$\mathrm{dd^c}\mathfrak{g}^\circ(T,\mathbf{y},\varphi_f)_\rho \ + \ \delta_{Z(T,\varphi_f)_\rho}\wedge\Omega_{\mathcal{E}^\vee}^{r-\mathrm{rk}\,T} \ = \ \omega(T,\mathbf{y},\varphi_f)_\rho.$$

Here $\Omega_{\mathcal{E}^\vee} = c^{\mathrm{top}}(\mathcal{E}^\vee,\nabla)^*$ is the top Chern-Weil form of the Hermitian bundle $\mathcal{E}^\vee$.

As in the orthogonal case, we introduce a modified version of $\mathfrak{g}^\circ(T,\mathbf{y},\varphi_f)$ by adding a closed current.

Definition 4.4.2. *Let $\mathbf{y} = (y_v)_{v|\infty} \in \mathrm{Her}_r(E_\mathbb{R})_{\gg 0}$ and $T \in \mathrm{Her}_r(E)$. We set*

$$\mathfrak{g}(T,\mathbf{y},\varphi_f) := \mathfrak{g}^\circ(T,\mathbf{y},\varphi_f)$$

if T is not totally positive semidefinite and

$$\mathfrak{g}(T,\mathbf{y},\varphi_f) := \mathfrak{g}^\circ(T,\mathbf{y},\varphi_f) - \sum_{v|\infty}\log\left(\frac{\det'\sigma_v(T)\cdot\det y_v}{\det'\sigma_v(T)y_v}\right)\delta_{Z(T,\varphi_f)_{\sigma_v}}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-\mathrm{rk}(T)-1}$$

if T is totally positive semidefinite; when T is non-degenerate, the additional term vanishes.

Example 4.4.3. We treat the orthogonal and unitary cases simultaneously here. Assume that $\mathbb{V}$ is anisotropic and that

$$T = \begin{pmatrix} 0 & \\ & S\end{pmatrix} \tag{4.4.9}$$

with S non-degenerate, where $S \in \mathrm{Sym}_t(F)$ (resp. $\mathrm{Her}_t(E)$) in the orthogonal (resp. unitary) cases, and $t = \mathrm{rk}\,T$. Then any $\mathbf{x}$ with $T(\mathbf{x}) = T$ is of the form $\mathbf{x} = (0,\dots,0,\mathbf{x}')$ with $T(\mathbf{x}') = S$.

Suppose that $\varphi_f = \varphi_f' \otimes \varphi_f''$ with $\varphi_f' \in S(\mathbb{V}(\mathbb{A}_f)^{r-t})$ and $\varphi_f'' \in S(\mathbb{V}(\mathbb{A}_f)^t)$, so that $Z(T,\varphi_f) = \varphi_f'(0)\cdot Z(S,\varphi_f'')$. Suppose furthermore that $\mathbf{y}$ is of the form

$$\mathbf{y} = \begin{pmatrix}\mathbf{y}' & \\ & \mathbf{y}''\end{pmatrix}, \tag{4.4.10}$$

where $\mathbf{y}'$ and $\mathbf{y}''$ are totally positive definite of rank $r-t$ and t, respectively.

It follows from Example 2.6.8 that, after descending to the Shimura variety $\mathcal{X}_{\sigma_k}(\mathbb{C})$, we have the equation of currents

$$\begin{aligned}\mathfrak{g}(T,&\mathbf{y},\varphi_f)_{\sigma_k}\\ &= \varphi_f'(0)\cdot\Big[\mathfrak{g}(S,\mathbf{y}'',\varphi_f'')_{\sigma_k}\wedge\Omega_{\mathcal{E}^\vee}^{r-t}-\log(\det y')\,\delta_{Z(S,\varphi_f'')_{\sigma_k}}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-t-1}\\ &\qquad+(r-t)\int_1^\infty\Big[\omega(S,u\mathbf{y}'',\varphi_f'')_{\sigma_k\cdot}-\delta_{Z(S,\varphi_f'')_{\sigma_k}}\Big]\frac{du}{u}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-t-1}\Big].\end{aligned}\tag{4.4.11}$$

Since $\omega(S,u\mathbf{y}'',\varphi_f')$ and $\delta_{Z(\tau,\varphi_f'')}$ are cohomologous, the term appearing on the second line above is exact, i.e.

$$\begin{aligned}\mathfrak{g}(T,\mathbf{y},\varphi_f)_{\sigma_k}\equiv\varphi_f'(0)\cdot&\Big[\mathfrak{g}(S,\mathbf{y}'',\varphi_f'')_{\sigma_k}\wedge\Omega_{\mathcal{E}^\vee}^{r-t}-\log(\det y_k')\,\delta_{Z(S,\varphi_f'')_{\sigma_k}}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-t-1}\Big]\\ &\in D^{r-1,r-1}(\mathcal{X}_{\sigma_k}(\mathbb{C}))/\mathrm{im}\partial+\mathrm{im}\overline{\partial}.\end{aligned}\tag{4.4.12}$$

This expression generalizes a similar term appearing indirectly in the work of Kudla-Rapoport-Yang [34, p. 178], which dealt with the case of a Shimura curve over $\mathbb{Q}$ and $r=2, r'=1$.

Finally, we note that for $T=\mathbf{0}_r$ and $\mathbb{V}$ anisotropic, a computation along the same lines gives $\mathfrak{g}^\circ(\mathbf{0}_r,\mathbf{y},\varphi_f)=0$ and the pleasant expression

$$\mathfrak{g}(\mathbf{0}_r,\mathbf{y},\varphi_f)_{\sigma_k} \;=\; -\log(\det y_k)\cdot\varphi_f(0)\cdot c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-1}.\tag{4.4.13}$$

Remark 4.4.4. We continue to assume $\mathbb{V}$ is anisotropic, and note two useful invariance properties for $\mathfrak{g}(T,\mathbf{y},\varphi_f)$. Set $\mathbf{k}=F$ in the orthogonal case and $\mathbf{k}=E$ in the unitary one.

(1) Suppose $T=\begin{pmatrix}0&\\&S\end{pmatrix}$ for a non-degenerate matrix $S\in\mathrm{Sym}_t(F)$ (resp. $S\in\mathrm{Her}_t(E)$), and $\boldsymbol{\theta}\in\mathrm{SL}_r(\mathbf{k}_\mathbb{R})$ is of the form

$$\boldsymbol{\theta}=\begin{pmatrix}1_{r-t}&*\\&1_t\end{pmatrix}.\tag{4.4.14}$$

Then

$$\mathfrak{g}\left(T,\,\boldsymbol{\theta}\,\mathbf{y}\,{}^t\overline{\boldsymbol{\theta}},\varphi_f\right)\;=\;\mathfrak{g}(T,\mathbf{y},\varphi_f).\tag{4.4.15}$$

(2) Suppose $\gamma\in\mathrm{SL}_r(\mathbf{k})$, and let $T[\gamma]:={}^t\overline{\gamma}^{-1}\,T\,\gamma^{-1}$. Then

$$\mathfrak{g}\left(T[\gamma],\,\gamma\,\mathbf{y}\,{}^t\overline{\gamma},\varphi_f'\right)\;=\;\mathfrak{g}\left(T,\mathbf{y},\varphi_f\right)\tag{4.4.16}$$

where $\varphi_f'(\mathbf{x})=\varphi_f(\mathbf{x}\cdot\gamma)$.

Given T, one can always find an element $\gamma\in\mathrm{SL}_r(\mathbf{k})$ as above such that $T[\gamma]=\begin{pmatrix}0&\\&S\end{pmatrix}$ for some non-degenerate matrix S. Similarly, we may choose $\boldsymbol{\theta}$ as above, such that $\boldsymbol{\theta}\mathbf{y}{}^t\overline{\boldsymbol{\theta}}=\begin{pmatrix}\mathbf{y}'&\\&\mathbf{y}''\end{pmatrix}$ with $\mathbf{y}''$ of the same rank as S; thus, up to a factor

$$\log\left(\frac{\det'(T)}{\det'(T[\gamma])}\right)\delta_{Z(S,\varphi_f'')_{\sigma_k}}\wedge c_{\mathrm{rk}(\mathcal{E})-1}(\mathcal{E}^\vee,\nabla)^*\wedge\Omega_{\mathcal{E}^\vee}^{r-t-1},\tag{4.4.17}$$

we may always place ourselves in the setting of Example 4.4.3.

4.5. **Star products on X_K.** In this section, we continue to treat both the orthogonal and unitary cases. Let $\mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1)$ and $\mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2)$ denote two currents attached to special cycles $Z(T_1, \varphi_1)$ and $Z(T_2, \varphi_2)$. Assume that T_1 and T_2 are non-degenerate and that $Z(T_1, \varphi_1)$ and $Z(T_2, \varphi_2)$ intersect properly, and consider the star product

$$\begin{aligned} &\mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1) * \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2) \\ &\qquad := \ \mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1) \wedge \delta_{Z(T_2,\varphi_2)} \ + \ \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2) \wedge \omega(T_1, \mathbf{y}_1, \varphi_1) \end{aligned} \tag{4.5.1}$$

in $D^*(X_K)$.

Theorem 4.5.1. *Let $\varphi = \varphi_1 \otimes \varphi_2$. With assumptions as above,*

$$\mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1) * \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2) \equiv \sum_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}} \mathfrak{g}\left(T, \begin{pmatrix} \mathbf{y}_1 & \\ & \mathbf{y}_2 \end{pmatrix}, \varphi\right)$$

in $\tilde{D}^*(X_K) := D^*(X_K)/(\mathrm{im}\partial + \mathrm{im}\overline{\partial})$.

Proof. Fix an embedding σ_k, matrices $\alpha_i \in \mathrm{GL}_{r_i}(\mathbb{K})$ such that $\sigma_k(y_i) = \alpha_i \cdot {}^t\overline{\alpha_i}$ for $i = 1, 2$, and a component $X_j = \Gamma_j \backslash \mathbb{D}^+ \subset \mathcal{X}_{\sigma_k}(\mathbb{C})$; working with Γ_j-invariant currents on $\mathbb{D}^+$, the proof of Proposition 4.3.3(i) implies that

$$\mathfrak{g}(T_1, \mathbf{y}_1, \varphi_1) * \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_2)\big|_{X_j} = \sum_{\substack{\mathbf{x}_1 \in \Omega_{T_1}(\mathbb{V}[k]) \\ \mathbf{x}_2 \in \Omega_{T_2}(\mathbb{V}[k])}} \varphi_1(h_j^{-1}\mathbf{x}_1)\, \varphi_2(h_j^{-1}\mathbf{x}_2)\ \mathfrak{g}^\circ(\mathbf{v}_1) * \mathfrak{g}^\circ(\mathbf{v}_2) \tag{4.5.2}$$

where we write $\mathbf{v}_i := \sigma_k(\mathbf{x}_i) \cdot \alpha_i$; note that the proper intersection assumption implies that $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ is regular for any non-zero term above, i.e. $\mathbb{D}_{\mathbf{v}}$ is either empty or of codimension $(r_1 + r_2)\mathrm{rk}(\mathcal{E})$. By Theorem 2.7.2, the previous line becomes

$$\begin{aligned} &\sum_{\substack{\mathbf{x}_1 \in \Omega_{T_1}(\mathbb{V}[k]) \\ \mathbf{x}_2 \in \Omega_{T_2}(\mathbb{V}[k])}} \varphi_1(h_j^{-1}\mathbf{x}_1)\varphi_2(h_j^{-1}\mathbf{x}_2) \left\{\mathfrak{g}^\circ(\mathbf{v}) - \partial\alpha(\mathbf{v}_1, \mathbf{v}_2) - \overline{\partial}\beta(\mathbf{v}_1, \mathbf{v}_2)\right\} \\ &\quad = \sum_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}} \sum_{\mathbf{x} \in \Omega_T(\mathbb{V}[k])} \varphi(h_j^{-1}\mathbf{x})\mathfrak{g}^\circ(\mathbf{v}) - \sum_{\substack{\mathbf{x}_1 \in \Omega_{T_1}(\mathbb{V}[k]) \\ \mathbf{x}_2 \in \Omega_{T_2}(\mathbb{V}[k])}} \varphi_1(h_j^{-1}\mathbf{x}_1)\varphi_2(h_j^{-1}\mathbf{x}_2) \left\{\partial\alpha(\mathbf{v}_1, \mathbf{v}_2) + \overline{\partial}\beta(\mathbf{v}_1, \mathbf{v}_2)\right\} \\ &\quad = \sum_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}} \mathfrak{g}\left(T, \begin{pmatrix} \mathbf{y}_1 & \\ & \mathbf{y}_2 \end{pmatrix}, \varphi\right) - \sum_{\substack{\mathbf{x}_1 \in \Omega_{T_1}(\mathbb{V}[k]) \\ \mathbf{x}_2 \in \Omega_{T_2}(\mathbb{V}[k])}} \varphi_1(h_j^{-1}\mathbf{x}_1)\varphi_2(h_j^{-1}\mathbf{x}_2) \left\{\partial\alpha(\mathbf{v}_1, \mathbf{v}_2) + \overline{\partial}\beta(\mathbf{v}_1, \mathbf{v}_2)\right\} \end{aligned} \tag{4.5.3}$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ and $\alpha(\mathbf{v}_1, \mathbf{v}_2)$ and $\beta(\mathbf{v}_1, \mathbf{v}_2)$ are as in Theorem 2.7.2. Again, an argument along the lines of Proposition 4.3.3(i) shows that the sum

$$\sum_{\substack{\mathbf{x}_1 \in \Omega_{T_1}(\mathbb{V}[k]) \\ \mathbf{x}_2 \in \Omega_{T_2}(\mathbb{V}[k])}} \varphi_1(h_j^{-1}\mathbf{x}_1)\varphi_2(h_j^{-1}\mathbf{x}_2)\alpha(\mathbf{v}_1, \mathbf{v}_2), \tag{4.5.4}$$

and its analogue with α replaced by β, converge to currents on $\mathbb{D}^+$, that are moreover Γ_j-invariant by Theorem 2.7.2. The theorem follows upon descending to X_j. □

5. Local archimedean heights and derivatives of Siegel Eisenstein series

Here we prove Theorem 5.3.1, our main global result relating archimedean local heights and derivatives of Siegel Eisenstein series. We review the definition of these Eisenstein series and the Siegel-Weil formula in Section 5.1. For the proof we also need to explicitly determine the asymptotics of the Fourier coefficients $E'_T(\lambda \mathbf{y}, \Phi_f, s_0)$ as $\lambda \to \infty$; we do this in Section 5.2 and give the proof of Theorem 5.3.1 in Section 5.3.

In Section 5.5 we explain how, using our results, Kudla's conjectural arithmetic Siegel-Weil fomula can be rephrased in terms of Faltings heights of special cycles.

Fix a totally real number field F of degree d, and a CM extension E, and set

$$\mathbf{k} = \begin{cases} F, & \text{orthogonal case,} \\ E, & \text{unitary case} \end{cases} \qquad \text{and} \qquad \mathbb{K} = \begin{cases} \mathbb{R}, & \text{orthogonal case} \\ \mathbb{C}, & \text{unitary case.} \end{cases} \tag{5.0.1}$$

Let $\sigma_1, \ldots, \sigma_d$ the archimedean places of F in the orthogonal case, or the elements of a fixed CM type of E in the unitary case.

We fix an m-dimensional Hermitian $\mathbf{k}$-vector space $(\mathbb{V}, Q)$ such that $\mathbb{V}_{\sigma_i} := \mathbb{V} \otimes_{\mathbf{k}, \sigma_i} \mathbb{K}$ is positive definite when $i > 1$ and

$$\operatorname{sig} \mathbb{V}_{\sigma_1} = \begin{cases} (p, 2), & \text{orthogonal case} \\ (p, 1), & \text{unitary case} \end{cases} \tag{5.0.2}$$

with $p \geq 1$. *From now on we assume that* $\mathbb{V}$ *is anisotropic.*

Finally, let $\eta\colon F^\times \backslash \mathbb{A}_F^\times \to \{\pm 1\}$ the quadratic character corresponding to E and fix a unitary character $\chi\colon E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{A}_F^\times} = \eta^m$.

5.1. Siegel Eisenstein series and the Siegel-Weil formula.

5.1.1. Let $\mathrm{Mp}_{2r}(\mathbb{A}_F)$ be the metaplectic double cover of $\mathrm{Sp}_{2r}(\mathbb{A}_F)$ and for a positive integer r, set

$$\mathbf{G}'_r(\mathbb{A}) = \begin{cases} \mathrm{Mp}_{2r}(\mathbb{A}_F), & \text{case 1,} \\ \mathrm{U}(r,r)(\mathbb{A}_F), & \text{case 2,} \end{cases} \tag{5.1.1}$$

Denote by $P_r(\mathbb{A})$ the standard Siegel parabolic of $\mathbf{G}'_r(\mathbb{A})$; then $P_r(\mathbb{A}) = M_r(\mathbb{A}) \ltimes N_r(\mathbb{A})$, where

$$\begin{aligned} M_r(\mathbb{A}) &= \{(m(a), \epsilon) | a \in \mathrm{GL}_r(\mathbb{A}_F),\ \epsilon = \pm 1\}, \\ N_r(\mathbb{A}) &= \{(n(b), 1) | b \in \mathrm{Sym}_r(\mathbb{A}_F)\} \end{aligned} \tag{5.1.2}$$

in case 1 and

$$M_r(\mathbb{A}) = \{m(a) | a \in \mathrm{GL}_r(\mathbb{A}_E)\}, \quad N_r(\mathbb{A}) = \{n(b) | b \in \mathrm{Her}_r(\mathbb{A}_E)\} \tag{5.1.3}$$

in case 2. We also write

$$\underline{m}(a) = \begin{cases} (m(a), 1), & \text{for } a \in \mathrm{GL}_r(\mathbb{A}_F) \text{ in case 1,} \\ m(a), & \text{for } a \in \mathrm{GL}_r(\mathbb{A}_E) \text{ in case 2,} \end{cases}$$
$$\underline{n}(b) = \begin{cases} (n(b), 1), & \text{for } b \in \mathrm{Sym}_r(\mathbb{A}_F) \text{ in case 1,} \\ n(b), & \text{for } b \in \mathrm{Her}(\mathbb{A}_E) \text{ in case 2,} \end{cases} \tag{5.1.4}$$
$$\underline{w}_r = \begin{cases} (w_r, 1), & \text{case 1,} \\ w_r, & \text{case 2.} \end{cases}$$

where $w_r = \left(\begin{smallmatrix} & 1_r \\ -1_r & \end{smallmatrix} \right)$. The multiplication in $M_r(\mathbb{A})$ in case 1 is defined by

$$(m(a_1), \epsilon_1) \cdot (m(a_2), \epsilon_2) = (m(a_1 a_2), \epsilon_1 \epsilon_2 (\det a_1, \det a_2)_{\mathbb{A}}), \tag{5.1.5}$$

where $(\cdot, \cdot)_{\mathbb{A}}$ denotes the Hilbert symbol of F.

Define a character $\chi_{\mathbb{V}}$ of $M_r(\mathbb{A})$ as follows: in case 1, set

$$\chi_{\mathbb{V}}\left(m(a), \epsilon\right) = \Big(\det a, (-1)^{m(m-1)/2} \det V\Big)_{\mathbb{A}} \cdot \begin{cases} \epsilon \cdot \gamma_{\mathbb{A}}(\det a, \psi)^{-1}, & \text{if } m \text{ is odd,} \\ 1, & \text{if } m \text{ is even,} \end{cases} \tag{5.1.6}$$

where $\gamma_{\mathbb{A}}$ denotes the Weil index, see [23], and in case 2, set

$$\chi_{\mathbb{V}}(m(a)) = \chi(\det a). \tag{5.1.7}$$

We may extend $\chi_{\mathbb{V}}$ to a character of $P_r(\mathbb{A})$ by declaring it trivial on $N_r(\mathbb{A})$, and define

$$I_r(\mathbb{V}, s) = \mathrm{Ind}_{P_r(\mathbb{A})}^{\mathbf{G}'_r(\mathbb{A})}(\chi_{\mathbb{V}} |\cdot|^s) \tag{5.1.8}$$

(smooth induction), where the induction is normalized so that $s = 0$ belongs to the unitary axis. Concretely, elements of $I_r(\mathbb{V}, s)$ are smooth functions $\Phi(\cdot, s)\colon \mathbf{G}'_r(\mathbb{A}) \to \mathbb{C}$ satisfying

$$\Phi((m(a), \epsilon)(n(b), 1)g', s) = |\det a|_{\mathbb{A}_{\mathbf{k}}}^{s+\rho} \cdot \chi_{\mathbb{V}}(m(a), \epsilon) \cdot \Phi(g', s), \quad \rho = \frac{r+1}{2} \tag{5.1.9}$$

in case 1 and

$$\Phi(m(a)n(b)g', s) = |\det a|_{\mathbb{A}_{\mathbf{k}}}^{s+\rho} \cdot \chi_{\mathbb{V}}(m(a)) \cdot \Phi(g', s), \quad \rho = \frac{r}{2} \tag{5.1.10}$$

in case 2.

We say that a section $\Phi(s) \in I_r(\mathbb{V}, s)$ is standard if its restriction to the standard maximal compact $K'_{r,\mathbb{A}}$ of $G'_r(\mathbb{A})$ is $K'_{r,\mathbb{A}}$-finite and independent of s.

Let

$$\mathbf{G}'_r(F) = \begin{cases} \mathrm{Sp}_{2r}(F), & \text{orthogonal case,} \\ \mathrm{U}(r,r)(F), & \text{unitary case;} \end{cases} \tag{5.1.11}$$

then there is an embedding $\mathbf{G}'_r(F) \to \mathbf{G}'_r(\mathbb{A})$; given by simply by the diagonal embedding in case 2, and the canonical splitting of the metaplectic cover $\mathrm{Mp}_{2r}(\mathbb{A}_F) \to \mathrm{Sp}_{2r}(\mathbb{A}_F)$ over $\mathrm{Sp}_{2r}(F)$ in case 1. In the sequel we will tacitly identify $\mathbf{G}'_r(F)$ with its image under this embedding.

Given a standard section $\Phi(s) \in I_r(\mathbb{V}, s)$ and $g' \in G'_r(\mathbb{A})$, the Siegel Eisenstein series

$$E(g', \Phi, s) = \sum_{\gamma \in P_r(F) \backslash \mathbf{G}'_r(F)} \Phi(\gamma g', s) \tag{5.1.12}$$

converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$. It admits a Fourier expansion

$$E(g', \Phi, s) = \sum_T E_T(g', \Phi, s). \tag{5.1.13}$$

where T ranges over $\mathrm{Sym}_r(F)$ in case 1 (resp. $\mathrm{Her}_r(E)$ in case 2), and

$$E_T(g', \Phi, s) = \int_{N_r(F) \backslash N_r(\mathbb{A})} E(\underline{n}(b) g', \Phi, s)\, \psi(-\mathrm{tr}(Tb))\, d\underline{n}(b), \tag{5.1.14}$$

where $d\underline{n}(b)$ denotes the Haar measure on $N_r(\mathbb{A}_F)$ that is self-dual with respect to the pairing $(b, b') \mapsto \psi(\mathrm{tr}(bb'))$.

5.1.2. Let $\omega = \omega_{\psi,\chi}$ be the Weil representation of $\mathbf{G}'_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V}(\mathbb{A}))$ on $\mathcal{S}(\mathbb{V}(\mathbb{A}_F)^r)$. For $\phi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F)^r)$, $g' \in \mathbf{G}'_r(\mathbb{A}_F)$ and $h \in \mathrm{U}(\mathbb{V}(\mathbb{A}))$, define the theta series

$$\Theta(g', h; \phi) = \sum_{\mathbf{v} \in \mathbb{V}(\mathbf{k})^r} \omega(g', h)\, \phi(\mathbf{v}). \tag{5.1.15}$$

The Siegel-Weil formula relates the integral of this function over $\mathrm{U}(\mathbb{V})(F) \backslash \mathrm{U}(\mathbb{V})(\mathbb{A})$ to the value of an Eisenstein series. Rather than discussing the formula in full generality, it will be convenient to recast the theta integral in the context of the Shimura varieties discussed above. For $k = 1, \dots, d$, we have the "nearby" spaces $\mathbb{V}[k]$, obtained by switching invariants at σ_1 and σ_k. It follows immediately from definitions that $I_r(\mathbb{V}[k], s) = I_r(\mathbb{V}, s)$; in the sequel, we will implicitly identify these spaces without further mention.

Fix such a k and a compact open subgroup $K \subset \mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f)$, and let

$$X_{\mathbb{V}[k]} \;=\; X_{\mathbb{V}[k],K} \;=\; \mathbf{H}_{\mathbb{V}[k]}(\mathbb{Q}) \big\backslash \mathbb{D}(\mathbb{V}[k]) \times \mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_f) \big/ K. \tag{5.1.16}$$

Since $\mathbf{H}_{\mathbb{V}[k]}(\mathbb{R})$ acts transitively on $\mathbb{D}(\mathbb{V}[k])$, we may identify

$$X_{\mathbb{V}[k]} \simeq \mathbf{H}_{\mathbb{V}[k]}(\mathbb{Q}) \big\backslash \mathbf{H}_{\mathbb{V}[k]}(\mathbb{A}_{\mathbb{Q}}) \big/ K_\infty K \tag{5.1.17}$$

where $K_\infty \subset \mathbf{H}_{\mathbb{V}[k]}(\mathbb{R})$ is the stablilizer of a fixed point $z_0 \in \mathbb{D}(\mathbb{V}[k])$. Thus, if ϕ is $K_\infty K$-invariant, then the theta function $\Theta(g', h; \phi)$ descends to a well defined function $\Theta(g', h; \phi)$ on $\mathbf{G}'_r(\mathbb{A}) \times X_{\mathbb{V}[k]}$.

For any $\phi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F)^r)$, the function $\Phi(g') = \omega(g')\phi(0)$ belongs to $I_r(\mathbb{V}, s_0)$, where

$$s_0 = s_0(r) = \begin{cases} (m - r - 1)/2, & \text{orthogonal case,} \\ (m - r)/2, & \text{unitary case;} \end{cases} \tag{5.1.18}$$

this construction defines a $\mathbf{G}'_r(\mathbb{A})$-intertwining map that we denote $\lambda \colon \mathcal{S}(\mathbb{V}(\mathbb{A}_F)^r) \to I_r(\mathbb{V}, s_0)$.

Theorem 5.1.3 (Siegel-Weil formula)**.** *Suppose $\mathbb{V}$ is anisotropic and let $\phi \in \mathcal{S}(\mathbb{V}(\mathbb{A})^r)^{K_\infty K}$. Denote by $\Phi \in I_r(\mathbb{V}, s)$ the unique standard section such that $\Phi(\cdot, s_0(r)) = \lambda(\phi)$. Let Ω be a positive $\mathbf{G}(\mathbb{R})$-invariant differential form on $\mathbb{D}(\mathbb{V})$ of top degree. Then $E(g', \Phi, s)$ is regular (in the variable s) at $s = s_0(r)$, and*

$$\frac{\kappa_0}{\mathrm{vol}(X_{\mathbb{V}}, \Omega)} \int_{[X_{\mathbb{V}}]} \Theta(g', \cdot; \phi)\, \Omega = E(g', \Phi, s_0(r)),$$

where

$$\kappa_0 = \begin{cases} 1, & \text{if } s_0(r) > 0, \\ 2, & \text{if } s_0(r) = 0. \end{cases} \tag{5.1.19}$$

Proof. Recall the usual formulation of the Siegel-Weil formula: set $\mathbf{H}^1_{\mathbb{V}} := \mathrm{Res}_{F/\mathbb{Q}}\, \mathrm{O}(\mathbb{V})$ in the orthogonal case and $\mathbf{H}^1_{\mathbb{V}} = \mathbf{H}_{\mathbb{V}} = \mathrm{Res}_{F/\mathbb{Q}}\, \mathrm{U}(\mathbb{V})$ in the unitary case. The Siegel-Weil formula asserts that $E(g', \Phi, s)$ is regular at $s = s_0(r)$, and that the value is given by the formula

$$E(g', \Phi, s_0(r)) = \kappa_0 \int_{\mathbf{H}^1_{\mathbb{V}}(\mathbb{Q}) \backslash \mathbf{H}^1_{\mathbb{V}}(\mathbb{A})} \Theta(g', h; \phi)\, dh, \tag{5.1.20}$$

where the Haar measure dh is normalized so that $\mathbf{H}^1_{\mathbb{V}}(\mathbb{Q}) \backslash \mathbf{H}^1_{\mathbb{V}}(\mathbb{A})$ has volume one. This is proved in [18, 20] in the unitary case, and in [30] and [47] for the metaplectic cases with m even and odd, respectively; a convenient reference treating all cases simultaneously is [10].

We now claim that

$$\int_{X_{\mathbb{V}}} \Theta(g', h; \phi)\, \Omega = C \cdot E(g', \Phi, s_0(r)) \tag{5.1.21}$$

for some constant C independent of ϕ. To see this, note that

$$\begin{aligned} \int_{X_{\mathbb{V}}} \Theta(g', \cdot; \phi)\, \Omega &= \int_{\mathbf{H}_{\mathbb{V}}(\mathbb{Q}) \backslash \mathbf{H}_{\mathbb{V}}(\mathbb{A}) / K_\infty K} \Theta(g', h; \phi)\, d'h \\ &= \mathrm{vol}(K_\infty K) \int_{\mathbf{H}_{\mathbb{V}}(\mathbb{Q}) \backslash \mathbf{H}_{\mathbb{V}}(\mathbb{A})} \Theta(g', h; \phi)\, d'h \end{aligned} \tag{5.1.22}$$

for some Haar measure $d'h$. By (5.1.20), this establishes the claim in the unitary case. The orthogonal case follows from the fact that the action of $\mathbf{H}_{\mathbb{V}}(\mathbb{A}) = \mathrm{GSpin}(\mathbb{V}(\mathbb{A}))$ on $\mathcal{S}(\mathbb{V}(\mathbb{A})^r)$ factors through its quotient $\mathrm{SO}(\mathbb{V}(\mathbb{A}))$, together with [26, Thm. 4.1.(ii)], which shows that, up to multiplying by a non-zero constant, the integral over $\mathrm{O}(\mathbb{V}(\mathbb{Q})) \backslash \mathrm{O}(\mathbb{V}(\mathbb{A}))$ in (5.1.20) can be replaced by integration over $\mathrm{SO}(\mathbb{V}(\mathbb{Q})) \backslash \mathrm{SO}(\mathbb{V}(\mathbb{A}))$.

To evaluate the constant C, compare the constant terms in the Fourier expansion on both sides of (5.1.21); the left hand side is $\mathrm{vol}(X_{\mathbb{V}[k]}, \Omega) \cdot \omega(g')\phi(0)$, while, again using (5.1.20), the right hand side is $C \cdot \kappa_0 \cdot \omega(g')\phi(0)$. □

5.2. **Fourier coefficients of scalar weight Eisenstein series.** In this section we study the asymptotic behaviour of the Fourier coefficients $E_T(g',\Phi,s)$ as g' goes to infinity, under certain hypotheses on Φ. More precisely,let K'_r be the standard maximal compact subgroup of $G'_r = \mathbf{G}'_r(F_v)$ (where v is archimedean) described in Section 2.5.1, and let

$$l = \begin{cases} \dfrac{m}{2}, & \text{orthogonal case} \\ \left(\dfrac{m+k(\chi)}{2}, \dfrac{-m+k(\chi)}{2}\right), & \text{unitary case} \end{cases} \tag{5.2.1}$$

as in (3.3.11). Assume that $\Phi = \Phi_f \otimes \Phi_\infty$, with

$$\begin{aligned} &\Phi_\infty := \Phi^l \otimes \cdots \otimes \Phi^l \in I_r(\mathbb{V}(\mathbb{R}), s), \\ &\Phi_f \ = \lambda(\varphi_f) \text{ for some Schwartz form } \varphi_f \in \mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r). \end{aligned} \tag{5.2.2}$$

Here λ is as in Section 5.1.1.

Lemma 5.2.1. *With Φ as above, the Eisenstein series $E(g',\Phi,s)$ is regular at $s=s_0$.*

Proof. When $s_0 = 0$, this follows from [32, Thm. 1.1] and [48]. Suppose $s_0 > 0$ so that $r \le p$: let $z_0 \in \mathbb{D}$ denote the fixed based point as in Section 2.2.1, and consider the Schwartz form $\tilde{\varphi} \in \mathcal{S}(\mathbb{V}^r_{\sigma_1})$ defined by

$$\varphi(\mathbf{v}, z_0) \wedge \Omega^{p-r}(z_0) = \tilde{\varphi}(\mathbf{v})\,\Omega^p(z_0), \quad \mathbf{v} \in V^r. \tag{5.2.3}$$

where $\Omega = c_1(\mathcal{E},\nabla)^*$. As remarked in Section 2.5.3, it has weight l under K'_r. Since $\tilde{\varphi}(\mathbf{0}) = (-1)^r$, an argument as in the proof of Lemma 3.4.2 shows that $\lambda(\tilde{\varphi}) = (-1)^r\Phi^l(s_0)$. Thus the global element $\Phi = \otimes_v \Phi^l \otimes \Phi_f$ is in the image of λ, and the lemma follows from Theorem 5.1.3. □

For $\boldsymbol{\tau} = (x_j + iy_j)_{1\le j \le d} \in \mathbb{H}^d_r$, let $g'_{\boldsymbol{\tau}} = (g'_{\tau_j}) \in \mathbf{G}'_r(\mathbb{R})$ with $g'_{\tau_j} = \underline{n}(x_j)\underline{m}(\alpha_j)$ as in (3.3.10) and define

$$E(\boldsymbol{\tau},\Phi_f,s) := (\det y_1\cdots y_d)^{-\iota m/4}\, E(g'_{\boldsymbol{\tau}}, \Phi_f \otimes \Phi^l_\infty, s). \tag{5.2.4}$$

Then $E(\boldsymbol{\tau},\Phi_f,s)$ has a Fourier expansion

$$\begin{aligned} E(\boldsymbol{\tau},\Phi_f,s) &= \sum_T E_T(\boldsymbol{\tau},\Phi_f,s) \\ &= \sum_T C_T(\mathbf{y},\Phi_f,s)\, q^T, \qquad q^T := \prod_{v\mid\infty} e^{2\pi i \mathrm{tr}(\tau_v \sigma_v(T))}, \end{aligned} \tag{5.2.5}$$

where T runs over $\mathrm{Sym}_r(F)$ (resp. $\mathrm{Her}_r(E)$), and

$$C_T(\mathbf{y},\Phi_f,s) := (\det y_1\cdots y_d)^{-\iota m/4}\, E_T(g'_{\boldsymbol{\tau}},\Phi_f,s)\, q^{-T}. \tag{5.2.6}$$

In the rest of this section we will determine the asymptotic behaviour of $\frac{d}{ds} C_T(\lambda\mathbf{y},\Phi_f,s_0)$ as $\lambda \to +\infty$.

5.2.2. Consider first the case $\det T \neq 0$. Assuming that $\Phi = \prod_v \Phi_v$, we have

$$E_T(g', \Phi, s) \;=\; \prod_v W_{T,v}(g'_v, \Phi_v, s), \tag{5.2.7}$$

where

$$W_{T,v}(g'_v, \Phi_v, s) = \int_{N_r(F_v)} \Phi_v(\underline{w}_r^{-1}\underline{n}(b)g', s)\,\psi(-\mathrm{tr}(T\underline{n}(b)))\,d\underline{n}(b), \quad \mathrm{Re}(s) \gg 0. \tag{5.2.8}$$

Since T is nonsingular, each $W_{T,v}(g'_v, \Phi_v, s)$ admits analytic continuation to $s \in \mathbb{C}$ and we can write

$$\frac{d}{ds}E_T(g', \Phi, s)\bigg|_{s=s_0} = \sum_v E'_T(g', \Phi, s_0)_v, \tag{5.2.9}$$

where the sum runs over places of F and we set

$$E'_T(g', \Phi, s_0)_v = \prod_{w\neq v} W_{T,w}(g'_w, \Phi_w, s_0) \cdot \frac{d}{ds}W_{T,v}(g'_v, \Phi_v, s)\bigg|_{s=s_0}. \tag{5.2.10}$$

Lemma 5.2.3. *Suppose Φ satisfies (5.2.2) and $\det T \neq 0$. Then*

$$\beta(T, \Phi_f) := C_T(\mathbf{y}, \Phi_f, s_0(r))$$

and

$$\kappa(T, \Phi_f) := \lim_{\lambda\to\infty} \frac{d}{ds}C_T(\lambda\mathbf{y}, \Phi_f, s)|_{s=s_0(r)}$$

are independent of $\mathbf{y}$. Let $\iota = 1$ in the orthogonal case and $\iota = 2$ in the unitary one and set $\kappa = 1 + \frac{\iota}{2}(r-1)$. Then explicit values for these quantities are as follows:

(i) If T is not totally positive definite, then $\beta(T, \Phi_f) = \kappa(T, \Phi_f) = 0$.

(ii) Suppose T is totally positive definite and set $W_{T,f}(e, \Phi_f, s) = \prod_{v<\infty} W_{T,v}(e, \Phi_v, s)$ and

$$c(T) := \left(\frac{(-2\pi i)^{rl}}{2^{r(\kappa-1)/2}\,\Gamma_r(l)}\right)^d \cdot N_{F/\mathbb{Q}}(\det T)^{\iota\, s_0(r)}.$$

Then

$$\beta(T, \Phi_f) \;=\; c(T) \cdot W_{T,f}(e, \Phi_f, s_0(r))$$

and

$$\kappa(T, \Phi_f) = \left(\frac{\iota d}{2}\left(r\log\pi - \frac{\Gamma'_r(l)}{\Gamma_r(l)}\right) + \frac{\iota}{2}\log N_{F/\mathbb{Q}}\det T\right)\beta(T, \Phi_f) + c(T)\,W'_{T,f}(e, \Phi_f, s_0(r)).$$

Note that if $\beta(T,\Phi_f)\neq 0$, then the above may be rewritten more suggestively as

$$\kappa(T,\Phi_f)=\left[\frac{\iota d}{2}\left(r\log\pi-\frac{\Gamma_r'(l)}{\Gamma_r(l)}\right)+\frac{\iota}{2}\log N_{F/\mathbb{Q}}\det T\right.$$
$$\left.+\frac{W'_{T,f}(e,\Phi_f,s_0(r))}{W_{T,f}(e,\Phi_f,s_0(r))}\right]\beta(T,\Phi_f).$$

(iii) In both cases, $\frac{\partial}{\partial\lambda}\frac{\partial}{\partial s}C_T(\lambda\mathbf{y},\Phi_f,s)|_{s=s_0(r)}=O(\lambda^{-1-C})$ as $\lambda\to\infty$, for some $C>0$.

Proof. Writing (5.2.7) in classical coordinates, we have

$$C_T(\lambda\mathbf{y},\Phi_f,s)=\left(\prod_{v|\infty}\mathcal{W}_{\sigma_v(T)}(\lambda y_v,s)\right)\cdot W_{T,f}(e,\Phi_f,s) \tag{5.2.11}$$

where $\mathcal{W}_{\sigma_v(T)}(\lambda y_v,s)$ is the normalized archimedean Whittaker functional as in (3.3.12); all claims in the lemma follow easily from Propositions 3.3.3 and 3.3.4. □

5.2.4. We now consider the asymptotic behaviour of $\frac{d}{ds}C_T(\lambda\mathbf{y},\Phi_f,s_0)$ as $\lambda\to+\infty$ for a general matrix T. Our approach, which follows that of [30], involves relating these coefficients to certain Eisenstein series on GL_r.

Fix a matrix T and let $t=\mathrm{rk}(T)\leq r$. A change of variables in (5.1.14) shows that, for $\gamma\in\mathrm{SL}_r(\mathbf{k})$ and $g\in\mathbf{G}'_r(\mathbb{A})$ we have

$$E_{T[\gamma]}(g,\Phi,s)=E_T(\underline{m}(\gamma^{-1})g,\Phi,s),\quad T[\gamma]:={}^t\overline{\gamma}^{-1}T\gamma^{-1}. \tag{5.2.12}$$

Hence, by choosing an appropriate γ, it suffices to consider the case when T is of the form

$$T=\begin{pmatrix}0_{r-t} & \\ & S\end{pmatrix}, \tag{5.2.13}$$

where S is non-degenerate of rank t when $t\neq 0$. We assume T is of this form until Proposition 5.2.12.

For integers $1\leq k\leq k'$ and any ring R, consider the embedding $\mathrm{Mat}_{2k}(R)\to\mathrm{Mat}_{2k'}(R)$ given by

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\mapsto\left(\begin{array}{cc|cc}\mathbf{1}_{k'-k} & & \mathbf{0}_{k'-k} & \\ & A & & B\\ \hline \mathbf{0}_{k'-k} & & \mathbf{1}_{k'-k} & \\ & C & & D\end{array}\right),\qquad A,B,C,D\in\mathrm{Mat}_k(R) \tag{5.2.14}$$

It is easily checked that in the orthogonal case, this map induces an embedding

$$\eta_k^{k'}:\mathrm{Mp}_{2k}(\mathbb{A})\to\mathrm{Mp}_{2k'}(\mathbb{A}) \tag{5.2.15}$$

with $\eta_k^{k'}([1,\epsilon])=[1,\epsilon]$, and in the unitary case an embedding $\eta_k^{k'}:\mathrm{U}(k,k)(\mathbb{A})\to\mathrm{U}(k',k')(\mathbb{A})$; the same is true over F_v for any place v.

For integers $0\leq j\leq r$, define a parabolic subgroup of GL_r by

$$\mathcal{P}_{r,j}\;=\;\left\{\begin{pmatrix}* & *\\ \mathbf{0}_{j,r-j} & *\end{pmatrix}\right\}\cap\mathrm{GL}_r. \tag{5.2.16}$$

Lemma 5.2.5. *Suppose T is of the form (5.2.13) and $t = \mathrm{rk}(T)$. For a standard section $\Phi(s) \in I_r(\mathbb{V}, s)$, there is a decomposition*

$$E_T(g, \Phi, s) = \sum_{j=t}^{r} \sum_{\mathsf{a} \in \mathcal{P}_{r-t,j-t}(\mathbf{k}) \backslash \mathrm{GL}_{r-t}(\mathbf{k})} B_T^j\left(\underline{m}\begin{pmatrix} \mathsf{a} & \\ & \mathbf{1}_t \end{pmatrix} g, \Phi, s\right)$$

where

$$B_T^j(g, \Phi, s) := \int_{\underline{n}(b) \in \mathsf{N}_j(\mathbb{A})} \Phi\left(\eta_j^r(\underline{w}_j^{-1}\underline{n}(b)) \cdot g, s\right) \psi_T\begin{pmatrix} \mathbf{0}_{r-j} & \\ & -b \end{pmatrix} d\underline{n}(b). \tag{5.2.17}$$

and $w_j = \begin{pmatrix} & \mathbf{1}_j \\ -\mathbf{1}_j & \end{pmatrix}$.

Proof. This follows from the standard unfolding argument for the Fourier coefficients of Eisenstein series and the Bruhat decomposition. See e.g. [30, Lemma 2.4] for the symplectic case; the proofs for the cases required here are identical. ☐

If the section $\Phi = \otimes_v \Phi_v$ is factorizable, then there is a product expansion

$$B_T^j(g, \Phi, s) = \prod_v B_{T,v}^j(g_v, \Phi_v, s) \tag{5.2.18}$$

taken over the places of F, where

$$B_{T,v}^j(g_v, \Phi_v, s) := \int_{\mathsf{N}_j(F_v)} \Phi_v\left(\eta_j^r(\underline{w}_j^{-1}\underline{n}(b))g_v, s\right) \psi_{T,v}\begin{pmatrix} \mathbf{0}_{r-j} & \\ & -b \end{pmatrix} d\underline{n}(b). \tag{5.2.19}$$

Note that if $j = r$, then these factors are the usual local Whittaker functions

$$W_{T,v}(g_v, \Phi_v, s) = \int_{\mathsf{N}_r(F_v)} \Phi_v(\underline{w}_r^{-1}\underline{n}(b)g, s)\psi_{T,v}(-b)\, d\underline{n}(b). \tag{5.2.20}$$

We shall relate these, as well as the remaining terms, to Whittaker functions of lower rank; we require a bit more notation.

For positive integers $k \le k'$, pullback by $\eta_k^{k'}$ induces a map

$$(\eta_k^{k'})^* \colon I_{k',v}(\mathbb{V}, s) \to I_{k,v}\left(\mathbb{V}, s + \frac{k'-k}{2}\right) \tag{5.2.21}$$

preserving holomorphic standard sections. Given a place v of $\mathbf{k}$, define an operator $U_{k,k',v}(s)$ as follows: for $\Phi \in I_{k'}(\mathbb{V}, s)$ and $g \in \mathbf{G}'_k(F_v)$, and supposing $\mathrm{Re}(s)$ is sufficiently large, let

$$\left(U_{k,k',v}(s)\Phi\right)(g) = \int_{\substack{b_1 \in \mathsf{N}_{k'-k}(\mathbf{k}_v) \\ b_2 \in \mathrm{Mat}_{k'-k,k}(\mathbf{k}_v)}} \Phi\left(\underline{w}_{k'}^{-1}\underline{n}\begin{pmatrix} b_1 & b_2 \\ b_2^* & \mathbf{0}_k \end{pmatrix} \eta_k^{k'}(\underline{w}_k^{-1} \cdot g), s\right) db_1\, db_2, \tag{5.2.22}$$

generalizing the construction in [30, Section 7]. Also, let $\mathcal{M}_k(s) = \otimes_v \mathcal{M}_{k,v}(s)$ denote the standard intertwining operator, where

$$\mathcal{M}_{k,v}(s) \colon I_{k,v}(\mathbb{V}, s) \to I_{k,v}(\mathbb{V}, -s) \tag{5.2.23}$$

is defined for Re(s) sufficiently large by the integral

$$\mathcal{M}_{k,v}(s)\Phi(g) = \int\limits_{\mathsf{N}_k(F_v)} \Phi\left(\underline{w}_k^{-1}\,\underline{n}(b)\,g, s\right)\,d\underline{n}(b), \qquad \Phi \in I_{k,v}(\mathbb{V}, s). \tag{5.2.24}$$

Both $\mathcal{M}_{k,v}(s)$ and $\mathcal{M}_k(s)$ admit meromorphic continuation to $\mathbb{C}$.

Lemma 5.2.6. *Let v be a place of F and suppose $T = \left(\begin{smallmatrix} 0 & \\ & S\end{smallmatrix}\right)$ as in (5.2.13).*

(i) If $\Phi \in I_{k',v}(\mathbb{V}, s)$, then $U_{k,k',v}(s)\Phi \in I_{k,v}(\mathbb{V}, s - \frac{k'-k}{2})$.

(ii) Fix an integer j with $t \leq j \leq r$. If $t \neq 0$, define

$$U_v(s) := \left[U_{t,j,v}\left(s + \frac{r-j}{2}\right) \circ (\eta_j^r)^*\right]$$

which, by (i), defines a $\mathbf{G}'_t(F_v)$-intertwining map

$$U_v(s)\colon I_{r,v}(\mathbb{V}, s) \to I_{t,v}(\mathbb{V}, \sigma), \qquad \text{where } \sigma = s + \frac{r+t}{2} - j.$$

Then for any $\Phi_{r,v} \in I_{r,v}(\mathbb{V}, s)$ and $g \in \mathbf{G}'_r(F_v)$,

$$B^j_{T,v}(g, \Phi_{r,v}, s) = W_{S,v}\left(e, U_v(s)\left(r(g)\Phi_{r,v}\right), \sigma\right).$$

(iii) If $T \neq 0$, then

$$\mathcal{M}_{t,v}(\sigma) \circ U_v(s) \;=\; (\eta_t^j)^* \circ \mathcal{M}_{j,v}\left(s + \frac{r-j}{2}\right) \circ (\eta_j^r)^*.$$

(iv) If $T = 0$, then

$$B^j_{\mathbf{0},v}(g, s, \Phi_{r,v}) \;=\; \left[\mathcal{M}_{j,v}\left(s + \frac{r-j}{2}\right) \circ (\eta_j^r)^*\right]\left(r(g)\Phi_{r,v}\right)(e).$$

Proof. Consider the orthogonal case first. Suppose $b_1 \in \mathrm{Sym}_{k'-k}(F_v)$ and $b_2 \in \mathrm{Mat}_{k'-k,k}(F_v)$. For $\underline{n}(\beta) \in N_k(F_v)$, a direct computation yields

$$\underline{w}_{k'}^{-1}\,\underline{n}\begin{pmatrix} b_1 & b_2 \\ {}^tb_2 & 0\end{pmatrix}\eta_k^{k'}\left(\underline{w}_k^{-1}\cdot\underline{n}(\beta)\right) \;=\; m'\cdot\underline{n}\left(\begin{smallmatrix} 0 & \\ & \beta\end{smallmatrix}\right)\cdot\underline{w}_{k'}^{-1}\cdot\underline{n}\begin{pmatrix} b_1+b_2\beta\,{}^tb_2 & b_2 \\ {}^tb_2 & 0\end{pmatrix}\cdot\eta_k^{k'}\left(\underline{w}_k^{-1}\right) \tag{5.2.25}$$

where $m' = \underline{m}\left(\begin{smallmatrix} \mathbf{1} & -b_2\beta \\ & \mathbf{1}\end{smallmatrix}\right) \in M_{k'}(F_v)$. Similarly, if $\underline{m}(\alpha) \in M_k(F_v)$, then

$$\underline{w}_{k'}^{-1}\,\underline{n}\begin{pmatrix} b_1 & b_2 \\ {}^tb_2 & 0\end{pmatrix}\eta_k^{k'}\left(\underline{w}_k^{-1}\cdot\underline{m}(\alpha)\right) \;=\; \eta_k^{k'}(\underline{m}(\alpha))\cdot\underline{w}_{k'}^{-1}\cdot\underline{n}\begin{pmatrix} b_1 & b_2\alpha \\ {}^t\alpha\,{}^tb_2 & 0\end{pmatrix}\cdot\eta_k^{k'}\left(\underline{w}_k^{-1}\right). \tag{5.2.26}$$

Using these relations and applying an appropriate change of variables in (5.2.22) implies that for any $\Phi \in I_{k',v}(\mathbb{V}, s)$, we have the transformation formula

$$\left[U_{k,k',v}(s)\Phi\right]\left(\underline{m}(\alpha)\underline{n}(\beta)g\right) \;=\; \chi_{\mathbb{V}}(\underline{m}(\alpha))\cdot|\alpha|^{s-\frac{k'}{2}+k}\cdot\left[U_{k,k',v}(s)\Phi\right](g)\,; \tag{5.2.27}$$

thus $U_{k,k',v}(s)\Phi \in I_{k,v}(\mathbb{V}, s - \frac{k'-k}{2})$, proving (i).

To prove (ii), let $\Phi = \Phi_{r,v} \in I_{r,v}(\mathbb{V}, s)$ and set

$$\Phi'(s') := (\eta_j^r)^*\left(r(g)\Phi(s)\right) \;\in\; I_{j,v}(\mathbb{V}, s + (r-j)/2) \tag{5.2.28}$$

with $s' = s + (j-r)/2$. Then, identifying $N_k(F_v) \simeq \mathrm{Sym}_k(F_v)$, we may write

$$\begin{aligned} &W_{S,v}(e, U(s)\Phi, \sigma) \\ &\quad = \int_{\beta \in \mathrm{Sym}_t(F_v)} [U(s)\Phi] \left(\underline{w}_t^{-1} \cdot \underline{n}(\beta) \right) \; \psi_S(-\beta)\, d\beta \\ &\quad = \int_{\beta_t \in \mathrm{Sym}_t(F_v)} \int_{\substack{b_1 \in \mathrm{Sym}_{j-t}(F_v) \\ b_2 \in \mathrm{Mat}_{j-t,t}(F_v)}} \Phi' \left(\underline{w}_j^{-1}\, \underline{n} \left(\begin{smallmatrix} b_1 & b_2 \\ {}^t b_2 & 0 \end{smallmatrix} \right) \eta_t^j \left(\underline{w}_t^{-1} \cdot \underline{w}_t^{-1} \cdot \underline{n}(\beta) \right) \right), s' \Big)\, db_1\, db_2\, \psi_S(-\beta)\, d\beta \\ &\quad = \int_{\beta_t \in \mathrm{Sym}_t(F_v)} \int_{\substack{b_1 \in \mathrm{Sym}_{j-t}(F_v) \\ b_2 \in \mathrm{Mat}_{j-t,t}(F_v)}} \Phi' \left(\underline{w}_j^{-1}\, \underline{n} \left(\begin{smallmatrix} b_1 & b_2 \\ b_2^* & \beta \end{smallmatrix} \right),\; s' \right)\; db_1\; db_2\; \psi_S(-\beta)\; d\beta \\ &\quad = \int_{b \in \mathrm{Sym}_j(F_v)} \Phi'(\underline{w}_j^{-1} \cdot \underline{n}(b),\; s') \cdot \psi_S(-\beta)\; db \\ &\quad = \int_{b \in \mathrm{Sym}_j(F_v)} \Phi \left(\eta_j^r \left(\underline{w}_j^{-1} \cdot \underline{n}(b) \right) \cdot g,\; s \right) \psi_T \left(\begin{smallmatrix} \mathbf{0} & \\ & -b \end{smallmatrix} \right)\; db \;=\; B_{T,v}^j(g, \Phi, s); \end{aligned} \tag{5.2.29}$$

here we used the fact that $\mathrm{tr}(T(\begin{smallmatrix} 0 & \\ & b \end{smallmatrix})) = \mathrm{tr}(S\beta)$, and hence $\psi_T((\begin{smallmatrix} 0 & \\ & -b \end{smallmatrix})) = \psi_S(-\beta)$. The proofs of (iii) and (iv) are similar, as are the statements in the unitary case. ☐

Remark 5.2.7. (i) Since $\mathcal{M}_{t,v}(-\sigma) \circ \mathcal{M}_{t,v}(\sigma)$ is given by a meromorphic function in σ, part (iii) shows that $U_v(s)$ admits meromorphic continuation to $s \in \mathbb{C}$. Setting $U(s) = \otimes_v U_v(s)$ and using the meromorphic continuation of the global intertwining operator $\mathcal{M}_t(\sigma)$ we also conclude that the global operator $U(s)$ admits meromorphic continuation.

(ii) If $T \neq 0$ and $j = t$ or $j = r$, then $\mathcal{P}_{r-t,j-t} = \mathrm{GL}_{r-t}$ and so the sum over a for these terms in Lemma 5.2.5 is trivial. Noting that $U(s) = (\eta_t^r)^*$ when $j = t$, and $U(s) = U_{t,r}(s)$ when $j = r$, the corresponding summands in Lemma 5.2.5 are

$$B_T^t(g, \Phi, s) = W_S\left(e, [(\eta_t^r)^* \circ r(g)]\,\Phi,\; s + \frac{r-t}{2}\right)$$

and

$$B_T^r(g, \Phi, s) = W_S\left(e, [U_{t,r}(s) \circ r(g)]\,\Phi, s - \frac{r-t}{2}\right).$$

In particular, when T is non-degenerate we recover the expression $E_T(g, \Phi, s) = \prod_v W_{T,v}(g_v, \Phi_v, s)$.

In particular, the lemma implies the global identity

$$B_T^j(g, \Phi_r, s) = W_S\left(e, U(s)\left(r(g)\Phi_r\right), \sigma\right). \tag{5.2.30}$$

We will also need the following invariance property: suppose $x \in \mathrm{Mat}_{r-t,t}(\mathbb{A}_k)$ and let

$$\theta = \left(\begin{smallmatrix} 1_{r-t} & x \\ 0 & 1_t \end{smallmatrix} \right) \in \mathrm{GL}_r(\mathbb{A}_{\mathbf{k}}). \tag{5.2.31}$$

Then a direct computation using (5.2.17) yields the transformation formula

$$B_T^j\left(\underline{m}(\theta)g, \Phi, s\right) = B_T^j\left(g, \Phi, s\right). \tag{5.2.32}$$

5.2.8. Our next step is to relate the individual terms in Lemma 5.2.5 to Eisenstein series on GL_{r-t}, generalizing the discussion in [30]. Consider first the orthogonal case and let $\mathrm{GL}'_{r-t}(\mathbb{A})$ denote the metaplectic double cover of $\mathrm{GL}_{r-t}(\mathbb{A}_F)$: as a set, $\mathrm{GL}'_{r-t}(\mathbb{A}) = \mathrm{GL}_{r-t}(\mathbb{A}) \times \{\pm 1\}$, with multiplication

$$(\mathsf{a}_1, \epsilon_1) \cdot (\mathsf{a}_2, \epsilon_2) \;=\; (\mathsf{a}_1\mathsf{a}_2,\, (\det \mathsf{a}_1, \det \mathsf{a}_2)_{\mathbb{A}}\, \epsilon_1\, \epsilon_2). \tag{5.2.33}$$

It follows from the formulas in [43, §5] that there is an embedding

$$\iota\colon \mathrm{GL}'_{r-t}(\mathbb{A}) \hookrightarrow M_r(\mathbb{A}), \qquad \text{given by} \qquad (a, \epsilon) \mapsto \left(m\left(\begin{smallmatrix} a & \\ & \mathbf{1}_t \end{smallmatrix}\right), \epsilon\right) \tag{5.2.34}$$

Abusing notation, let $\chi_{\mathbb{V}}\colon \mathrm{GL}'_{r-t}(\mathbb{A}) \to \mathbb{C}$ denote the character $\chi_{\mathbb{V}}(\mathsf{a}) = \chi_{\mathbb{V}}(\iota(\mathsf{a}))$, where the latter $\chi_{\mathbb{V}}$ is defined in (5.1.6).

For the moment, fix an integer j with $t \le j \le r$; to lighten notation, we write

$$\mathcal{P} = \mathcal{P}_{r-t,j-t} \;=\; \left\{ \begin{pmatrix} \mathsf{p}_1 & * \\ \mathbf{0}_{j-t,r-j} & \mathsf{p}_2 \end{pmatrix} \;\middle|\; \mathsf{p}_1 \in \mathrm{GL}_{r-j},\ \mathsf{p}_2 \in \mathrm{GL}_{j-t} \right\} \;\subset\; \mathrm{GL}_{r-t}. \tag{5.2.35}$$

Let $\mathcal{P}'_{\mathbb{A}}$ denote the inverse image of $\mathcal{P}(\mathbb{A})$ with respect to the projection $\mathrm{GL}'_{r-t}(\mathbb{A}) \to \mathrm{GL}_{r-t}(\mathbb{A})$. Consider the (smooth normalized) induced representation

$$\widetilde{I}^j_{\mathcal{P}}(\mathbb{V}, s) \;:=\; \mathrm{Ind}_{\mathcal{P}'_{\mathbb{A}}}^{\mathrm{GL}'_{r-t}(\mathbb{A})} \left(\chi \,|\det(\mathsf{p}_1)|_{\mathbb{A}}^{s+\frac{r+1}{2}-\frac{j-t}{2}}\, |\det(\mathsf{p}_2)|_{\mathbb{A}}^{-s+\frac{j+1}{2}} \right); \tag{5.2.36}$$

concretely, $\widetilde{I}^j_{\mathcal{P},v}(\mathbb{V}, s)$ consists of smooth functions $\Psi(\cdot, s)\colon \mathrm{GL}'_{r-t}(\mathbb{A}) \to \mathbb{C}$ such that

$$\Psi(\mathsf{pa}, s) \;=\; \chi_{\mathbb{V}}(\mathsf{p})\,|\det(\mathsf{p}_1)|_{\mathbb{A}}^{s+\frac{r+1}{2}}\, |\det(\mathsf{p}_2)|_{\mathbb{A}}^{-s-\frac{r-1}{2}+j}\, \Psi(\mathsf{a}) \tag{5.2.37}$$

for all $\mathsf{a} \in \mathrm{GL}'_{r-t}(\mathbb{A})$ and $\mathsf{p} = [\left(\begin{smallmatrix} \mathsf{p}_1 & * \\ \mathbf{0} & \mathsf{p}_2 \end{smallmatrix}\right), \epsilon] \in \mathcal{P}'_{\mathbb{A}}$.

In the unitary case, we may be more direct: let $\widetilde{I}^j_{\mathcal{P}}(\mathbb{V}, s)$ denote the space of smooth functions $\Psi(\cdot, s)\colon \mathrm{GL}_{r-t}(\mathbb{A}_E) \to \mathbb{C}$ such that

$$\Psi(\mathsf{pa}, s) \;=\; \chi_{\mathbb{V}}(\mathsf{p})\,|\det(\mathsf{p}_1)|_{\mathbb{A}_E}^{s+\frac{r}{2}}\, |\det(\mathsf{p}_2)|_{\mathbb{A}_E}^{-s-\frac{r}{2}+j}\, \Psi(\mathsf{a}), \qquad \text{for all } \mathsf{p} = \left(\begin{smallmatrix} \mathsf{p}_1 & * \\ & \mathsf{p}_2 \end{smallmatrix}\right) \in \mathcal{P}(\mathbb{A}_E). \tag{5.2.38}$$

We also write $\iota\colon \mathrm{GL}_{r-t}(\mathbb{A}_E) \to \mathbf{G}'_r(\mathbb{A})$ for the embedding $a \mapsto m\left(\begin{smallmatrix} a & \\ & \mathbf{1}_t \end{smallmatrix}\right)$.

Finally, for a finite place v of F, let

$$\widetilde{\mathbf{K}}_v = \mathrm{GL}_{r-t}(\mathcal{O}_{\mathbf{k},v}) \subset \mathrm{GL}_{r-t}(\mathbf{k}_v); \tag{5.2.39}$$

in the orthogonal case we identify $\widetilde{\mathbf{K}}_v$ as a subset of $\mathrm{GL}'_{r-t}(F_v)$ via the map $k \mapsto [k, 1]$. For a real place v, set $\widetilde{\mathbf{K}}_v = \mathrm{O}(r)$ or $\widetilde{\mathbf{K}}_v = \mathrm{U}(r)$, and finally define $\widetilde{\mathbf{K}} = \prod \widetilde{\mathbf{K}}_v$.

5.2.9. To formulate the connection between $B^j_T(g,\Phi,s)$ and Eisenstein series on GL_{r-t}, assume for the moment that $t=\mathrm{rk}(T)>0$, and fix an integer j with $t\le j\le r$. Let $\Phi_r=\otimes_v\Phi_{r,v}\in I_r(\mathbb{V},s)$ be a standard section such that $\Phi_{r,v}=\Phi^l_r$ at each archimedean place v.

Suppose that $g\in\mathbf{G}'_r(\mathbb{A})$ is of the form

$$g=\iota(g')\,\eta^r_t(g''),\tag{5.2.40}$$

where $g''\in\mathbf{G}'_t(\mathbb{A})$ and $g'\in\mathrm{GL}'_{r-t}(\mathbb{A})$ or $g'\in\mathrm{GL}_{r-t}(\mathbb{A}_E)$ in the orthogonal or unitary cases, respectively. As a function of g'', the expression

$$U(s)\,(r(g)\Phi_r)\,(e)\;=\;U(s)\left(r(\iota(g'))\Phi_r\right)(g'')\tag{5.2.41}$$

defines an element of $I_t(\mathbb{V},\sigma)$ by Lemma 5.2.6, and a change of variables in the definition shows that it defines an element of $\widetilde{I}^j_{\mathcal{P}}(\mathbb{V},s)$ as a function of g'. In particular, the function

$$B^j_T\,(g,\,\Phi,\,s)=B^j_T\left(\iota(g'),\,r(g'')\Phi_v,\,s\right)\tag{5.2.42}$$

is in $\widetilde{I}_{\mathcal{P}}(\mathbb{V},s)$ when viewed as a function of g' with g'' fixed.

It follows from multiplicity one for K'_t-types in $I_t(\mathbb{V}_v,\sigma)$ for archimedean v that, as a function of g''_∞, the expression (5.2.41) is proportional to $\prod_{v|\infty}\Phi^l_t(g''_v,\sigma)$; evaluating at $g''_\infty=e$ to determine the constant of proportionality, we find that

$$U(s)\,(r(g)\Phi_r)\,(e)=\widetilde{\Phi}(g',g''_f,s)\prod_{v|\infty}\Phi^l_t(g''_v,\sigma),\tag{5.2.43}$$

where $\widetilde{\Phi}(g',g''_f,s)=U(s)\left(r(\iota(g')\,\eta^r_t(g''_f))\Phi_r\right)(e)$. Note that $\tilde{\Phi}(g',g''_f,s)$ is meromorphic in s and, as a function of g', is $\widetilde{\mathbf{K}}_f$-finite and $\widetilde{\mathbf{K}}_\infty$-invariant. Moreover $\tilde{\Phi}(\cdot,g''_f,s)\in\widetilde{I}^j_{\mathcal{P}}(\mathbb{V},s)$ for fixed g''_f and so we can write

$$\tilde{\Phi}(g',g''_f,s)=\alpha_1(s)\,\tilde{\Phi}_1(g',s)\Phi''_1(g''_f,\sigma)+\cdots+\alpha_N(s)\,\tilde{\Phi}_N(g',s)\Phi''_N(g''_f,\sigma),\tag{5.2.44}$$

where $\tilde{\Phi}_i(\cdot,s)$ are $\widetilde{\mathbf{K}}_\infty$-invariant standard sections of $\widetilde{I}^j_{\mathcal{P}}(\mathbb{V},s)$, $\Phi''_i(\cdot,\sigma)=\otimes_{v\nmid\infty}\Phi''_{i,v}(\cdot,\sigma)$ are standard sections of $I_t(\mathbb{V}(\mathbb{A}_f),\sigma)$, and the coefficients $\alpha_i(s)$ meromorphic in s.

Substituting (5.2.43) and (5.2.44) in (5.2.30) we conclude that, for g as above with $g''_f=1$,

$$\sum_{\mathsf{a}\in\mathcal{P}(\mathbf{k})\backslash\mathrm{GL}_{r-t}(\mathbf{k})}B^j_T\,(m(\begin{smallmatrix}\mathsf{a}&\\&1_t\end{smallmatrix})g,\Phi,s)=\left(\sum_{1\le i\le N}\gamma_i(s)\,\mathcal{G}^j(g',\tilde{\Phi}_i,s)\right)\prod_{v|\infty}W_{S,v}(g''_v,\Phi^l_t,\sigma),\tag{5.2.45}$$

where

$$\mathcal{G}^j(g',\tilde{\Phi}_i,s):=\sum_{\mathsf{a}\in\mathcal{P}(\mathbf{k})\backslash\mathrm{GL}_{r-t}(\mathbf{k})}\tilde{\Phi}_i(\mathsf{a}g',s)\tag{5.2.46}$$

is an Eisenstein series on $\mathrm{GL}'_{r-t}(\mathbb{A})$ (case 1) or $\mathrm{GL}_{r-t}(\mathbb{A}_E)$ (case 2), $\mathcal{P} = \mathcal{P}_{r-t,j-t}$ is as in (5.2.35), and

$$\gamma_i(s) \ := \alpha_i(s) \prod_{v \nmid \infty} W_{S,v}\left(e, \Phi''_{i,v}, \sigma\right), \qquad 1 \le i \le N, \tag{5.2.47}$$

is meromorphic in s.

We turn now to the case $T = 0$. If $1 \le j \le r-1$, then the same argument as above shows that, for $g' \in \mathrm{GL}'_r(\mathbb{A})$ (orthogonal case) or $g' \in \mathrm{GL}_r(\mathbb{A}_E)$ (unitary case), we can write

$$\sum_{\mathsf{a} \in \mathcal{P}_{r,j}(\mathbf{k})\backslash \mathrm{GL}_r(\mathbf{k})} B_0^j\left(\underline{m}(\mathsf{a})\iota(g'), \Phi, s\right) = \sum_{1 \le i \le N} \gamma_i(s)\, \mathcal{G}^j(g', \tilde{\Phi}_i, s) \tag{5.2.48}$$

for some standard $\tilde{\mathbf{K}}_\infty$-invariant sections $\tilde{\Phi}_i(s) \in \tilde{I}^j_{\mathcal{P}}(\mathbb{V}, s)$ and meromorphic functions $\gamma_i(s)$. In addition, Lemma 5.2.6 shows that

$$\begin{aligned} &B_0^0(\iota(g'), \Phi_r, s) = \Phi_r(\iota(g'), s),\\ &B_0^r(\iota(g'), \Phi_r, s) = \mathcal{M}(s)\Phi(\iota(g')) = W_0(\iota(g'), \Phi_r, s). \end{aligned} \tag{5.2.49}$$

5.2.10. We can now generalize Lemma 5.2.3 to arbitrary matrices T; we consider the case $T \neq 0$ first.

Definition 5.2.11. *Assume that $\Phi_f \in I_r(\mathbb{V}(\mathbb{A}_f), s)$ satisfies (5.2.2) and $T = \left(\begin{smallmatrix} 0 & \\ & S\end{smallmatrix}\right) \neq 0$ with S non-degenerate, cf. (5.2.13). Let $\Phi'_f, \Phi''_f \in I_t(\mathbb{V}(\mathbb{A}_f), \sigma)$ be given by*

$$\Phi'_f(\sigma) = (\eta^r_t)^* \Phi_f(s) \quad \text{and} \quad \Phi''_f(\sigma) = (\eta^r_t)^*(\mathcal{M}_r(-s)\Phi_f(-s)),$$

where $(\eta^r_t)^*$ is defined as in (5.2.21), $\mathcal{M}_r(s)$ is the standard intertwining operator (5.2.24), and $\sigma = s + \frac{r-t}{2}$.

Define constants $\beta(T, \Phi_f)$ and $\kappa(T, \Phi_f)$ as follows:

(i) If $s_0(r) > 0$ set $\beta(T, \Phi_f) = \beta(S, \Phi'_f)$ and $\kappa(T, \Phi_f) = \kappa(S, \Phi'_f)$, where these quantities are defined for the non-degenerate matrix S as in Lemma 5.2.3.

(ii) Suppose $s_0(r) = 0$, so that $r = m-1$ (resp. $r = m$) in the orthogonal (resp. unitary) case, and let

$$\mathsf{d}(s) \ = \ 2^{-\frac{r}{2}(\frac{\iota m}{2}-1)-\iota s}(2\pi i)^{\iota m r/2} \frac{\Gamma_r(\iota s)}{\Gamma_r\left(\frac{\iota}{2}(s+m)\right)\Gamma_r\left(\frac{\iota s}{2}\right)}. \tag{5.2.50}$$

Note that $\mathsf{d}(s)$ is holomorphic and non-vanishing at $s = 0$. Define

$$\beta(T, \Phi_f) \ = \ 2\beta(S, \Phi'_f)$$

and

$$\kappa(T, \Phi_f) \ := \ \kappa(S, \Phi'_f) \ - \ \left[d \cdot \frac{\mathsf{d}'(0)}{\mathsf{d}(0)} \beta(S, \Phi'_f) + \mathsf{d}(0)^d \kappa(S, \Phi''_f)\right].$$

Let us now define constants $\kappa(T, \Phi_f)$ and $\beta(T, \Phi_f)$ for an arbitrary (i.e. not necessarily block diagonal) matrix T of rank $t > 0$.

Let $\gamma \in \mathrm{SL}_r(\mathbf{k})$ such that

$$T[\gamma] := {}^t\overline{\gamma}^{-1} T \gamma^{-1} = \begin{pmatrix} 0_{r-t} & \\ & S \end{pmatrix}, \tag{5.2.51}$$

where S is non-degenerate.

Define[4]

$$\begin{aligned} \beta(T, \Phi_f) &:= \beta(T[\gamma], \underline{m}(\gamma)\Phi_f) \\ \kappa(T, \Phi_f) &:= \kappa(T[\gamma], \underline{m}(\gamma)\Phi_f) - \frac{\iota}{2} \log N_{F/\mathbb{Q}} \left(\frac{\det'(T)}{\det(S)} \right) \cdot \beta(T, \Phi_f); \end{aligned} \tag{5.2.52}$$

that $\beta(T, \Phi_f)$ and $\kappa(T, \Phi_f)$ are independent of γ (and hence are well-defined) follows from a direct computation, or alternatively from the invariance property

$$C_T(\mathbf{y}, \Phi_f, s) = C_{T[\gamma]}(\mathbf{y}[{}^t\overline{\gamma}^{-1}], \underline{m}(\gamma)\Phi_f, s), \tag{5.2.53}$$

(cf. (5.2.12)) and the next proposition, which shows that the constants $\beta(T, \Phi_f)$ and $\kappa(T, \Phi_f)$ determine the asymptotic behaviour of the derivative $\frac{d}{ds} C_T(\mathbf{y}, \Phi_f, s_0)$.

Proposition 5.2.12. *Assume that Φ_f satisfies (5.2.2) and let $\mathbf{y} = (y_v)_{v|\infty} \in \mathrm{Sym}_r(F_{\mathbb{R}})_{\gg 0}$ (resp. $\mathrm{Her}(E_{\mathbb{R}})_{\gg 0}$). Write $\det' A$ for the product of non-zero eigenvalues of a square matrix A, and let*

$$F(\mathbf{y}) := \frac{d}{ds} C_T(\mathbf{y}, \Phi_f, s) \Big|_{s = s_0(r)} - \kappa(T, \Phi_f) - \frac{\iota}{2} \beta(T, \Phi_f) \sum_{v|\infty} \log \left(\frac{\det' \sigma_v(T) \cdot \det y_v}{\det' (\sigma_v(T) y_v)} \right).$$

Then for every fixed $\mathbf{y}$, we have

$$\lim_{\lambda \to \infty} F(\lambda \mathbf{y}) = 0 \qquad \textit{and} \qquad \frac{\partial}{\partial \lambda} F(\lambda \mathbf{y}) = O(\lambda^{-1-C}) \tag{5.2.54}$$

as $\lambda \to \infty$, for some $C > 0$.

Proof. We begin with a few preliminary reductions. Choose $\gamma \in \mathrm{SL}_r(\mathbf{k})$ satisfying (5.2.51),and note that the quantity $\frac{\det y_v}{\det'(\sigma_v(T) y_v)}$ is unchanged upon simultaneously replacing T with $T[\gamma]$ and $\mathbf{y}$ with $\gamma\, \mathbf{y}\, {}^t\overline{\gamma}$. Moreover, $\det'(T[\gamma]) = \det(S)$. Combining these observations with (5.2.53), it suffices to prove the proposition for T of the form $T = \left(\begin{smallmatrix} 0 & \\ & S \end{smallmatrix} \right)$ with S non-degenerate of rank $t = \mathrm{rk}(T)$.

In this case, Lemma 5.2.5 gives a decomposition $C_T(\mathbf{y}, \Phi_f, s) = \sum_{j=t}^{r} C_T^j(\mathbf{y}, \Phi_f, s)$, where

$$C_T^j(\mathbf{y}, \Phi_f, s) = \prod_{v|\infty} \det(y_v)^{-\iota m/4} \sum_{\mathsf{a} \in \mathcal{P}_{r-t,j-t} \backslash \mathrm{GL}_{r-t}(\mathbf{k})} B_T^j \left(\underline{m}\left(\begin{smallmatrix} \mathsf{a} & \\ & 1_t \end{smallmatrix} \right) g'_{\mathbf{y}}, \Phi, s \right) \, q^{-T}. \tag{5.2.55}$$

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[4] In the published version of this article, there is an error in this definition: we had made the erroneous claim that given a degenerate matrix T, there exists $\gamma \in SL_r(\mathbf{k})$ such that $T[\gamma] = \left(\begin{smallmatrix} 0 & \\ & S \end{smallmatrix} \right)$ with $\det(S) = \det'(T)$. We have opted here for a correction that results in minimal deviation from the published version: with the definition of $\beta(T, \Phi_f)$ and $\kappa(T, \Phi_f)$ given here, all the results and proofs go through without change. Alternatively, the reader may elect to remove the superfluous terms involving $\log \det'(T)$ from (5.2.52), from Proposition 5.2.12, and from Definitions 4.3.5 and 4.4.2; in this case, the statements of all other results are unchanged and the proofs go through with trivial modifications.

Next, we may fix an element $\boldsymbol{\theta} = \begin{pmatrix} 1_{r-t} & * \\ & 1_t \end{pmatrix} \in \mathrm{SL}_r(\mathbf{k}_{\mathbb{R}}) \simeq \mathrm{SL}_r(\mathbb{K})^d$ such that

$$\boldsymbol{\theta} \cdot \mathbf{y} \cdot {}^t\overline{\boldsymbol{\theta}} \;=\; \begin{pmatrix} \mathbf{y}' & \\ & \mathbf{y}'' \end{pmatrix} \tag{5.2.56}$$

is block diagonal, where $\mathbf{y}'$ and $\mathbf{y}''$ are totally positive definite of rank $r-t$ and t, respectively. Let $\widetilde{\boldsymbol{\theta}} = (\boldsymbol{\theta}, e, \dots) \in \mathrm{SL}_r(\mathbb{A})$; then, using the invariance property (5.2.32) and the relation $g'_{\boldsymbol{\theta}\mathbf{y}^t\overline{\boldsymbol{\theta}}} = \underline{m}(\widetilde{\boldsymbol{\theta}}) g'_{\mathbf{y}}$, we have

$$B_T^j \left(\underline{m}\begin{pmatrix} \mathsf{a} & \\ & \mathbf{1}_t \end{pmatrix} g'_{\boldsymbol{\theta}\mathbf{y}^t\overline{\boldsymbol{\theta}}}, \Phi, s \right) = B_T^j \left(\underline{m}\begin{pmatrix} \mathsf{a} & \\ & \mathbf{1}_t \end{pmatrix} g'_{\mathbf{y}}, \Phi, s \right). \tag{5.2.57}$$

Moreover, for $T = \begin{pmatrix} 0 & \\ & S \end{pmatrix}$, the quantities $\det'(\sigma_v(T) y_v)$ are unchanged upon replacing $\mathbf{y}$ by $\boldsymbol{\theta}\mathbf{y}^t\overline{\boldsymbol{\theta}}$, as are the determinants $\det y_v$.

Thus, we may reduce to the case where $T = \begin{pmatrix} 0 & \\ & S \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} \mathbf{y}' & \\ & \mathbf{y}'' \end{pmatrix}$ are both block diagonal, so that

$$\frac{\det' \sigma_v(T) \cdot \det y_v}{\det' \sigma_v(T) y_v} = \frac{\det \sigma_v(S) \cdot \det y_v}{\det \sigma_v(S) \det y''_v} = \det y'_v. \tag{5.2.58}$$

Our approach will be to show that the value at $s_0(r)$ of the derivative of the expression (5.2.55), after possibly subtracting off a constant and a multiple of $\sum \log(\det y'_v)$, satisfies the condition (5.2.54).

Fix an element $\boldsymbol{\mu} \in \prod_{v|\infty} \mathrm{GL}_{r-t}(\mathbf{k}_v)^d$ of totally positive determinant with $\mathbf{y}' = \boldsymbol{\mu} \cdot {}^t\overline{\boldsymbol{\mu}}$, and let $\tilde{\boldsymbol{\mu}} = ((\boldsymbol{\mu}, \mathrm{Id}, \dots), 1) \in \mathrm{GL}'_{r-t,\mathbb{A}_F}$ in the orthogonal case, and $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}, \mathrm{Id}, \dots) \in \mathrm{GL}_{r-t}(\mathbb{A}_E)$ in the unitary case. Then

$$g'_{\mathbf{y}} \;=\; \iota(\tilde{\boldsymbol{\mu}}) \cdot \eta_t^r \left(g'_{\mathbf{y}''} \right). \tag{5.2.59}$$

Taking into account our normalizations, cf. (3.3.12), we may use (5.2.45) to write

$$C_T^j(\mathbf{y}, \Phi_f, s) = \left(\sum \gamma_i(s)\, \mathcal{G}^j(\tilde{\boldsymbol{\mu}}, s, \tilde{\Phi}_i) \right) \prod_{v|\infty} \det(y'_v)^{-\iota m/4} \mathcal{W}_{\sigma_v(S)}(y''_v, \sigma) \tag{5.2.60}$$

for some meromorphic functions $\gamma_i(s)$ and standard sections $\tilde{\Phi}_i \in \widetilde{I}^j_{\mathcal{P}}(\mathbb{V}, s)$, where $\sigma = \sigma(s) = s + \frac{r+t}{2} - j$,

Now for fixed $\mathbf{y}$ and a parameter $\lambda > 0$, replacing $\mathbf{y}$ by $\lambda \mathbf{y}$ corresponds to replacing $\mathbf{y}''$ by $\lambda\, \mathbf{y}''$, and replacing $\boldsymbol{\mu}$ by $\sqrt{\lambda}\, \boldsymbol{\mu}$. Using the transformation formulas (5.2.37) or (5.2.38) to determine the central character of the Eisenstein series $\mathcal{G}^j$ in the orthogonal or unitary case, respectively, a short computation yields

$$\mathcal{G}^j(\sqrt{\lambda}\tilde{\boldsymbol{\mu}}, \tilde{\Phi}, s) \;=\; \lambda^{\frac{\iota d}{2}(r+t-2j)(s-s_0(r)) + \frac{\iota d(r-t)m}{4} \;-\; \iota d(j-t)s_0(j)}\, \mathcal{G}^j(\tilde{\boldsymbol{\mu}}, \tilde{\Phi}, s) \tag{5.2.61}$$

for any section $\tilde{\Phi}$. Since each $\mathcal{G}^j(\tilde{\boldsymbol{\mu}}, \tilde{\Phi}_i, s)$ is itself meromorphic, it follows that for fixed $\mathbf{y}$, we have

$$C_T^j(\lambda\mathbf{y}, \Phi_f, s) = f(\mathbf{y}, s)\, \lambda^{\frac{\iota d}{2}(r+t-2j)(s-s_0(r)) - \iota d(j-t)s_0(j)} \prod_{v|\infty} \mathcal{W}_{\sigma_v(S)}(\lambda y''_v, \sigma) \tag{5.2.62}$$

for some meromorphic function $f(\mathbf{y},s)$ independent of λ. A priori, we do not know whether $f(\mathbf{y},s)$ has a pole at $s=s_0(r)$, though one could imagine that an analysis along the lines of [30, 47] could be used to determine its order. In any case, let

$$\mathcal{A}_1^j(\mathbf{y},\Phi_f) := \mathcal{A}_1(C_T^j(\mathbf{y},\Phi_f,s)) \tag{5.2.63}$$

denote the coefficient of $(s-s_0(r))$ in the Laurent expansion at $s=s_0(r)$, so that

$$\frac{\partial}{\partial s} C_T(\lambda\mathbf{y},\Phi_f,s)\Big|_{s=s_0(r)} \;=\; \sum_{j=t}^{r} \mathcal{A}_1^j(\lambda\mathbf{y},\Phi_f). \tag{5.2.64}$$

Then $\mathcal{A}_1^j(\lambda\mathbf{y},\Phi_f)$ can be written as a sum of terms of the form

$$a(\mathbf{y})\,(\log\lambda)^k\,\lambda^{-\iota d(j-t)s_0(j)}\;\frac{\partial^{k'}}{\partial s^{k'}}\left(\prod_{v|\infty}\mathcal{W}_{\sigma_v(S)}\left(\lambda y_v'',\sigma(s)\right)\right)\Bigg|_{s=s_0(r)} \tag{5.2.65}$$

for some Laurent coefficients $a(\mathbf{y})$ of $f(\mathbf{y},s)$ at $s=s_0(r)$, and integers $k,k'\geq 0$.

First, consider the case where S is not totally positive definite. If v is a place such that $\sigma_v(S)$ is not positive, then, by Proposition 3.3.4, the derivatives of $\mathcal{W}_{\sigma_v(S)}(\lambda y_v'',\sigma)$ in s and λ are all of exponential decay as $\lambda\to\infty$. It follows that $\mathcal{A}_1^j(\lambda\mathbf{y},\Phi_f)$ satisfies the properties (5.2.54) for each $t\leq j\leq r$. On the other hand, for non-positive S the constants $\kappa(S,\Phi_f)$ and $\beta(S,\Phi_f)$ are both zero by definition; this proves the proposition in this case.

Next, suppose S is totally positive definite. Assume further that $j>t$ and $s_0(j)\neq 0$, which implies that $-\iota d(j-t)s_0(j)\leq -1/2$; this is the exponent of λ in (5.2.65). For fixed $\mathbf{y}$, Proposition 3.3.3(i) implies that each $\mathcal{W}_{\sigma_v(S)}(\lambda y_v'',\sigma)$, along with its derivatives in s, are bounded as $\lambda\to\infty$. In light of (5.2.65), this implies

$$\lim_{\lambda\to\infty}\mathcal{A}_1^j(\lambda\mathbf{y},\Phi_f) \;=\; 0 \tag{5.2.66}$$

for such j. Similarly, differentiating (5.2.65) with respect to λ and applying Proposition 3.3.3(ii) yields the estimate

$$\frac{\partial}{\partial\lambda}\mathcal{A}_1^j(\lambda\mathbf{y},\Phi_f) \;=\; O(\lambda^{-1-C}) \tag{5.2.67}$$

for some $C>0$. In other words, for $j>t$ with $s_0(j)\neq 0$, the term $\mathcal{A}_1^j(\mathbf{y},\Phi_f)$ satisfies the condition (5.2.54).

Now suppose that S is totally positive definite and $j=t$; the corresponding term can be written more concretely, using Remark 5.2.7, as

$$C_T^t(\mathbf{y},\Phi_f,s) = \prod_{v|\infty}\det(y_v)^{-\iota m/4}\,W_S\left(e,\left[(\eta_t^r)^*\circ r(g_{\mathbf{y}}')\right]\Phi,\sigma\right)\,q^{-T} \tag{5.2.68}$$

where $\sigma = s+\frac{r-t}{2}$ and $(\eta_t^r)^*\colon I_r(\mathbb{V},s)\to I_t(\mathbb{V},\sigma)$ is the map defined in (5.2.21). For an archimedean place v, note that multiplicity one for $K_{t,v}'$-types immediately implies the relation $(\eta_t^r)^*\Phi_r^l = \Phi_t^l$. More generally, for block diagonal $\mathbf{y}$ as above and any $h_v \in \mathbf{G}'_{t,v}$ we have

$$\begin{aligned}(\eta_t^r)^*(r(g'_{y_v})\Phi_r^l)(h_v,\sigma) &= \Phi_r^l\left(\eta_t^r(h_v)\,\iota(\tilde{\mu}_v)\,\eta_t^r(g'_{y''_v}),\ s\right)\\ &= \Phi_r^l\left(\iota(\tilde{\mu}_v)\,\eta_t^r(h_v\,g'_{y''_v}),\ s\right)\\ &= \det(y'_v)^{\frac{\iota}{2}(s+\rho_r)}\,\Phi_r^l\left(\eta_t^r(h_v\,g'_{y''_v}),\ s\right)\\ &= \det(y'_v)^{\frac{\iota}{2}(s+\rho_r)}\,\Phi_t^l\left(h_v\,g'_{y''_v},\ \sigma\right).\end{aligned} \tag{5.2.69}$$

Hence

$$C_T^t(\mathbf{y},\Phi_f,s) = \prod_{v|\infty}\left(\det y'_v\right)^{\frac{\iota}{2}(s-s_0(r))} C_S(\mathbf{y}'',\Phi'_f,\sigma), \tag{5.2.70}$$

where

$$C_S(\mathbf{y}'',\Phi'_f,\sigma) := W_{S,f}(e,\Phi'_f,\sigma)\prod_{v|\infty}\mathcal{W}_{\sigma_v(S)}(y''_v,\sigma) \tag{5.2.71}$$

and $\Phi'_f = (\eta_t^r)^*\Phi_f \in I_t(\mathbb{V}(\mathbb{A}_f),\sigma)$.

Now consider the special point $s = s_0(r)$, and note that

$$\sigma|_{s=s_0(r)} \;=\; s_0(r) + \frac{r-t}{2} \;=\; s_0(t). \tag{5.2.72}$$

Since S is non-degenerate, the term $C_S(\mathbf{y}'',\Phi'_f,\sigma)$ is holomorphic at $s = s_0(r)$, and therefore the same is true for $C_T^t(\mathbf{y},\Phi_f,s)$. Therefore,

$$\frac{\partial}{\partial s}C_T^t(\mathbf{y},\Phi_f,s)\Big|_{s=s_0(r)} = \frac{\iota}{2}\sum_{v|\infty}\log\det(y'_v)\ \beta(S,\Phi'_f)\ +\ \frac{\partial}{\partial\sigma}C_S(\mathbf{y}'',\Phi'_f,\sigma)\Big|_{\sigma=s_0(t)}. \tag{5.2.73}$$

Applying Lemma 5.2.3 to $C_S(...)$, it follows immediately that the difference

$$\frac{\partial}{\partial s}C_T^t(\mathbf{y},s,\Phi)\Big|_{s=s_0(r)} - \frac{\iota}{2}\beta(S,\Phi'_f)\sum_{v|\infty}\log\det y'_v - \kappa(S,\Phi'_f) \tag{5.2.74}$$

satisfies the conditions in (5.2.54).

Finally, it remains to consider the case $j > t$ and $s_0(j) = 0$. Since we are assuming $s_0(r) \geq 0$, we must have $s_0(j) = s_0(r) = 0$ and hence $j = r$ as well. On the other hand, if $s_0(r) > 0$, then such a term does not arise and we have completed the proof of the proposition at this stage.

Otherwise, by Remark 5.2.7, the corresponding term is

$$C_T^r(\mathbf{y},\Phi_f,s) = \prod_{v|\infty}\det(y_v)^{-\iota m/4}\ W_S\left(e,\left[U_{r,t}(s)\circ r(g'_{\mathbf{y}})\right]\Phi,\sigma\right)\ q^{-T} \tag{5.2.75}$$

with $\sigma = s - \frac{r-t}{2}$. Since $s_0(r) = 0$, we have $m = r+1$ in the orthogonal case, and $m = r$ in the unitary case; it follows in both cases that

$$\sigma|_{s=s_0(r)=0} \;=\; -\frac{r-t}{2} \;=\; -s_0(t). \tag{5.2.76}$$

To compute this term, note that the functional equation of the genus t Eisenstein series implies

$$W_S(e,\Phi_t,\sigma) \;=\; W_S(e,\mathcal{M}_t(\sigma)\Phi_t,-\sigma) \tag{5.2.77}$$

for any $\Phi_t \in I_t(\mathbb{V},\sigma)$. To apply this to the present case, recall that

$$\mathcal{M}_t(\sigma)\circ U(s) \;=\; (\eta_t^j)^*\circ\mathcal{M}_j\left(s+\frac{r-j}{2}\right)\circ(\eta_j^r)^* \tag{5.2.78}$$

as in Lemma 5.2.6(iii). Consider an archimedean place $v|\infty$, and define a meromorphic function $\mathtt{d}(s)$ by the relation

$$\mathcal{M}_r(s)\Phi^l_{r,v}(s) \;=\; \mathtt{d}(s)\Phi^l_{r,v}(-s). \tag{5.2.79}$$

To determine the function $\mathtt{d}(s)$, we may evaluate both sides at the identity, and apply [45, (1.31)]; a little algebra, using the fact $s_0(r)=0$, shows that $\mathtt{d}(s)$ is given by the formula (5.2.50).

On the other hand, by Lemma 5.2.6(iii) with $j=r$, we find

$$\begin{aligned}[\mathcal{M}_{t,v}(\sigma)\circ U_{r,t,v}(s)]\left(r(g'_{y_v})\,\Phi^l_{r,v}(s)\right) \;&=\; [(\eta_t^r)^*\circ\mathcal{M}_{r,v}(s)]\left(r(g'_{y_v})\,\Phi^l_{r,v}(s)\right)\\ &=\; \mathtt{d}(s)\cdot(\eta_t^r)^*\left[r(g'_{y_v})\,\Phi^l_{r,v}(-s)\right]\\ &=\; \mathtt{d}(s)\left(\det y'_v\right)^{\frac{\iota}{2}(-s+\rho_r)}\left(r(g'_{y''_v})\Phi^l_{t,v}\right)(-\sigma).\end{aligned} \tag{5.2.80}$$

Thus, applying the functional equation for $W_S(\ldots)$, taking into account our normalizations, and noting that $s_0(r)=0$, we have

$$C^r_T(\mathbf{y},\Phi_f,s) \;=\; \mathtt{d}(s)^d\cdot\prod_{v|\infty}\det(y'_v)^{-\frac{\iota s}{2}}\cdot C_S(\mathbf{y}'',\Phi''_f,-\sigma), \tag{5.2.81}$$

where $\Phi''_f(-\sigma)=(\eta_t^r)^*(\mathcal{M}_r(s)\Phi_f(s))$. Therefore

$$\begin{aligned}\frac{\partial}{\partial s}C^r_T(\mathbf{y},\Phi_f,s)\Big|_{s=s_0(r)=0} \;&=\; \mathtt{d}(0)^d\left[d\,\frac{\mathtt{d}'(0)}{\mathtt{d}(0)}-\frac{\iota}{2}\sum_{v|\infty}\log\det y'_v\right]\cdot C_S(\mathbf{y}'',\Phi''_f,s_0(t))\\ &\qquad -\,\mathtt{d}(0)^d\,\frac{\partial}{\partial\sigma}C_S(\mathbf{y}'',\Phi''_f,\sigma)\Big|_{\sigma=s_0(t)}.\end{aligned} \tag{5.2.82}$$

Setting

$$\beta' = -\mathtt{d}(0)^d\cdot C_S(\mathbf{y}'',\Phi''_f,s_0(t)) = -\mathtt{d}(0)^d\cdot\beta(S,\Phi''_f) \tag{5.2.83}$$

and

$$\kappa' = \mathtt{d}(0)^d\left[d\,\frac{\mathtt{d}'(0)}{\mathtt{d}(0)}\beta(S,\Phi''_f)\;-\;\kappa(S,\Phi''_f)\right] \tag{5.2.84}$$

it follows, in the same manner as the previous case, that the difference

$$\frac{\partial}{\partial s}C^r_T(\lambda\mathbf{y},\Phi_f,s)\Big|_{s=s_0(r)=0}\;-\;\frac{\iota}{2}\cdot\beta'\sum_{v|\infty}\log\det y'_v-\kappa' \tag{5.2.85}$$

satisfies the conditions of (5.2.54). Combining this fact with the previous computations for the terms with $t \leq j < r$, we conclude that

$$\frac{\partial}{\partial s} C_T(\lambda \mathbf{y}, \Phi_f, s)\Big|_{s=0} - \frac{\iota}{2} \cdot \left[\beta(S, \Phi') + \beta'\right] \sum_{v|\infty} \log\det y'_v - \left[\kappa(S, \Phi'_f) + \kappa'\right] \tag{5.2.86}$$

It remains to identify the two terms in square brackets in the preceding display as $\beta(T, \Phi_f)$ and $\kappa(T, \Phi_f)$, respectively. First, consider the value of $C_T(\lambda\mathbf{y}, \Phi, s)$ at $s = s_0(r) = 0$; by eqs. (5.2.70) and (5.2.81) and an argument analogous to the one leading to (5.2.65), we may write

$$\begin{aligned} C_T(\lambda \mathbf{y}, \Phi_f, 0) \ &= \ \sum_{j=1}^{t} \ \mathrm{CT}_{s=0}\, C_T^j(\lambda\mathbf{y}, \Phi_f, s) \\ &= \ \beta(S, \Phi'_f) - \beta' + O(\lambda^{-C}) \end{aligned} \tag{5.2.87}$$

for some $C > 0$. On the other hand, the Eisenstein series $E(g, \Phi, s)$ is incoherent, in the sense of [32], and hence vanishes identically at $s = 0$ (this result is [32, Theorem 4.10] in the orthogonal case with m even; see [10, Proposition 6.2] and the references therein for more details in the remaining cases). In particular, $C_T(\mathbf{y}, \Phi_f, 0) = 0$ and $\beta(S, \Phi'_f) = \beta'$, and, comparing with Definition 5.2.11, we find

$$\beta(S, \Phi'_f) + \beta' = 2\beta(S, \Phi'_f) = \beta(T, \Phi'_f) \tag{5.2.88}$$

and

$$\kappa(S, \Phi'_f) + \kappa' = \kappa(T, \Phi_f) \tag{5.2.89}$$

as required.

□

It remains to consider the case $T = \mathbf{0}$.

Proposition 5.2.13. *Let*

$$\beta(\mathbf{0}, \Phi_f) \ := \ \begin{cases} \Phi_f(e), & \text{if } s_0(r) > 0 \\ 2\,\Phi_f(e), & \text{if } s_0(r) = 0. \end{cases}$$

and

$$\kappa(\mathbf{0}, \Phi_f) \ := \ \begin{cases} 0, & \text{if } s_0(r) > 0 \\ -d \cdot \frac{\mathtt{d}'(0)}{\mathtt{d}(0)} \Phi_f(e) \ - \ \mathtt{d}(0)^d\, W'_{0,f}(e, \Phi_f, 0), & \text{if } s_0(r) = 0. \end{cases}$$

Then the difference

$$F(\mathbf{y}) = \frac{d}{ds} C_{\mathbf{0}}(\mathbf{y}, \Phi_f, s)\Big|_{s=s_0(r)} \ - \ \frac{\iota}{2} \sum_{v|\infty} \log\det y_v \cdot \beta(\mathbf{0}, \Phi_f) - \kappa(\mathbf{0}, \Phi_f)$$

satisfies the conditions (5.2.54).

Proof. Using Lemma 5.2.5 and (5.2.48), we may write $C_{\mathbf{0}}(\mathbf{y},\Phi,s) = \sum_{j=0}^{r} C_{\mathbf{0}}^{j}(\mathbf{y},\Phi,s)$, where

$$C_{\mathbf{0}}^{j}(\mathbf{y},\Phi_f,s) = \prod_{v|\infty} \det(y_v)^{-\iota m/4} \cdot \sum_{1\leq i\leq N} \gamma_i(s)\mathcal{G}^j(\boldsymbol{\alpha},\tilde{\Phi}_i,s), \tag{5.2.90}$$

for some meromorphic functions $\gamma_i(s)$ and standard sections $\tilde{\Phi}_i \in \widetilde{I}^j_{\mathcal{P}}(\mathbb{V},s)$. Applying the transformation formula (5.2.61), with $t=0$, we have

$$C_{\mathbf{0}}^{j}(\lambda\mathbf{y},\Phi_f,s) = \lambda^{\frac{d}{2}(r-2j)(s-s_0(r))-djs_0(j)} \cdot C_{\mathbf{0}}^{j}(\mathbf{y},\Phi_f,s). \tag{5.2.91}$$

Thus $\frac{\partial}{\partial s}C_{\mathbf{0}}^{j}(\lambda\mathbf{y},\Phi_f,s)|_{s=s_0(r)}$ is a finite sum of terms of the form

$$\log(\lambda)^k \cdot \lambda^{-d\,j\,s_0(j)} \cdot a(\mathbf{y}) \tag{5.2.92}$$

for some integer $k\geq 0$ and Laurent coefficient $a(\mathbf{y})$ of $C_{\mathbf{0}}^{j}(\mathbf{y},\Phi_f,s)$.

In particular, if $j\geq 1$ and $s_0(j)\neq 0$, then

$$\lim_{\lambda\to\infty} \frac{\partial}{\partial s}C_{\mathbf{0}}^{j}(\lambda\mathbf{y},\Phi_f,s)|_{s=s_0(r)} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial\lambda}\frac{\partial}{\partial s}C_{\mathbf{0}}^{j}(\lambda\mathbf{y},\Phi_f,s)|_{s=s_0(r)} = O(\lambda^{-1-C}). \tag{5.2.93}$$

Next, consider the case $j=0$, so that, by (5.2.49), we have

$$\begin{aligned} C_{\mathbf{0}}^{0}(\mathbf{y},\Phi_f,s) &= \prod_{v|\infty} \det(y_v)^{-\iota m/4} B_{\mathbf{0}}^{0}\left(g'_{\mathbf{y}},\Phi_f,s\right) \\ &= \prod_{v|\infty} \det(y_v)^{-\iota m/4} \cdot \Phi(g'_{\mathbf{y}},s) \\ &= \prod_{v|\infty} \det(y_v)^{\frac{\iota}{2}(s-s_0(r))}\Phi_f(e). \end{aligned} \tag{5.2.94}$$

Note here that $\Phi_f(e,s)=\Phi_f(e)$ is independent of s. Therefore

$$\frac{\partial}{\partial s}C_{\mathbf{0}}^{0}(\mathbf{y},\Phi_f,s)|_{s=s_0(r)} = \frac{\iota}{2}\sum_{v|\infty}\log\det y_v \cdot \Phi_f(e). \tag{5.2.95}$$

Finally, consider the case $s_0(j)=0$, so that $j=r$ with $s_0(r)=0$ and

$$C_{\mathbf{0}}^{r}(\mathbf{y},\Phi_f,s) = \prod_{v|\infty} \det(y_v)^{-\iota m/4} \cdot [\mathcal{M}(s)\Phi(s)](g'_{\mathbf{y}}). \tag{5.2.96}$$

We have

$$\begin{aligned} [\mathcal{M}(s)\Phi(s)](g'_{\mathbf{y}}) &= \prod_{v|\infty} [\mathcal{M}_v(s)\Phi^l_{r,v}(s)](g'_{y_v}) \cdot [\mathcal{M}_f\Phi_f(s)](e) \\ &= \mathtt{d}(s)^d \prod_{v|\infty} \Phi^l_{r,v}(g'_{y_v},-s) \cdot W_{0,f}(e,\Phi_f,-s) \\ &= \mathtt{d}(s)^d \prod_{v|\infty} \det(y_v)^{\frac{\iota}{2}(-s+\rho_r)} W_{0,f}(e,\Phi_f,-s) \end{aligned} \tag{5.2.97}$$

Moreover, applying [10, Lemma 6.3] to the incoherent section Φ gives

$$\mathcal{M}(0)\Phi(0)(g'_{\mathbf{y}}) \;=\; -\prod_{v|\infty}\det(y_v)^{\iota m/4}\cdot\Phi(e) \;=\; -\prod_{v|\infty}\det(y_v)^{\iota m/4}\cdot\Phi_f(e) \tag{5.2.98}$$

and hence, after a little algebra we obtain

$$\begin{aligned}\frac{d}{ds}C^r_{\mathbf{0}}(\mathbf{y},\Phi_f,s)\Big|_{s=s_0(r)=0} \;=\; &\frac{\iota}{2}\sum_{v|\infty}\det\log y_v\,\Phi_f(e)\\ &-d\cdot\frac{\mathtt{d}'(0)}{\mathtt{d}(0)}\Phi_f(e)-\mathtt{d}(0)^dW'_{0,f}(e,\Phi_f,0).\end{aligned} \tag{5.2.99}$$

The proposition follows immediately from these observations. □

5.3. **Archimedean height pairings.** In this section, we prove our main theorem relating the integrals of the Green forms $\mathfrak{g}(T,\mathbf{y},\varphi)$ constructed in Section 4 to Eisenstein series.

We continue to assume $\mathbb{V}$ is anisotropic and $\mathrm{rk}(\mathcal{E})=1$. Let $\mathcal{X}$ denote the canonical model of the Shimura variety $X_{\mathbb{V}}=X_{\mathbb{V},K}$, for a fixed open compact subgroup $K\subset\mathbf{H}(\mathbb{A}_f)$, and recall that there is a decomposition $\mathcal{X}(\mathbb{C})=\coprod X_{\mathbb{V}[k]}$ in terms of the nearby spaces $\mathbb{V}[k]$ for $k=1,\dots,d$, see Sections 4.1 and 4.4. For each k, let

$$\mathrm{Vol}(X_{\mathbb{V}[k]},\Omega)=\int_{[X_{\mathbb{V}[k]}]}\Omega^p \tag{5.3.1}$$

where $\Omega=\Omega_{\mathcal{E}}=c_1(\mathcal{E},\nabla)^*$ is the first Chern form of the positive line bundle $\mathcal{E}$ on $X_{\mathbb{V}[k]}$. Then $\mathrm{Vol}(X_{\mathbb{V}[k]},\Omega)=\deg_{\mathcal{E}}(X_{\mathbb{V}[k]})$ is a positive rational number (a positive integer if K is neat). As remarked in 2.2.2, the line bundle $\mathcal{E}$ is a rational multiple of the canonical bundle; in particular we have $\mathrm{Vol}(X_{\mathbb{V}[k]},\Omega)=\mathrm{Vol}(X_{\mathbb{V}},\Omega)$ for every k.

Theorem 5.3.1. *Assume that $\mathbb{V}$ is anisotropic and $\mathrm{rk}(\mathcal{E})=1$. Let $r\geq 1$ and*

$$s_0(r)=\begin{cases}(m-r-1)/2, & \textit{orthogonal case,}\\ (m-r)/2, & \textit{unitary case,}\end{cases}$$

and assume that $s_0(r)\geq 0$. Given $\varphi_f\in\mathcal{S}(\mathbb{V}(\mathbb{A}_f)^r)^K$, let $\Phi_f(\cdot,s)$ be the unique standard section of $I_r(\mathbb{V}(\mathbb{A}_f),s)$ such that $\Phi_f(\cdot,s_0)=\lambda(\varphi_f)$ and set

$$\Phi=\otimes_{v|\infty}\Phi^l_r\otimes\Phi_f\in I_r(\mathbb{V}(\mathbb{A}),s), \tag{5.3.2}$$

where Φ^l_r is the standard weight l section given by (3.3.11), (3.1.10) and (3.2.11). For $\boldsymbol{\tau}=\mathbf{x}+i\mathbf{y}\in\mathbb{H}^d_r$, let

$$E(\boldsymbol{\tau},\Phi_f,s)=(\det y_1\cdots y_d)^{-\iota m/4}\,E(g'_{\boldsymbol{\tau}},\Phi_f,s) \;=\; \sum_T E_T(\boldsymbol{\tau},\Phi_f,s)$$

be the corresponding Eisenstein series of scalar weight l defined in (5.2.4), and write $E'_T(\boldsymbol{\tau},\Phi_f,s)=\frac{d}{ds}E_T(\boldsymbol{\tau},\Phi_f,s)$. Then for any $T\in\mathrm{Sym}_r(F)$ (resp. $T\in\mathrm{Her}_r(E)$) we *have*

$$\frac{(-1)^r\kappa_0}{2\mathrm{Vol}(X_{\mathbb{V}},\Omega)}\int_{[\mathcal{X}(\mathbb{C})]}\mathfrak{g}(T,\mathbf{y},\varphi_f)\wedge\Omega^{p+1-r}\,q^T \;=\; E'_T(\boldsymbol{\tau},\Phi_f,s_0(r))-\kappa(T,\Phi_f)\,q^T,$$

where $\kappa(T,\Phi_f)$ *is explicit and defined in Definition 5.2.11, and* $\kappa_0=2$ *if* $s_0(r)=0$ *and* $\kappa_0=1$ *otherwise.*

Proof. Fix an archimedean embedding σ_k, and a component $X_{\mathbb{V}[k]}\subset\mathcal{X}(\mathbb{C})$. Recall that the restriction of the Green form $\mathfrak{g}(T,\mathbf{y},\varphi_f)$ to $X_{\mathbb{V}[k]}$ is given by

$$\mathfrak{g}(T,\mathbf{y},\varphi_f)|_{X_{\mathbb{V}[k]}} \;=\; \mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f)_{\sigma_k}-\log\left(\frac{\det'\sigma_k(T)\cdot\det y_k}{\det'(\sigma_k(T)y_k)}\right)\delta_{Z(T,\varphi_f)_{\sigma_k}}\wedge\Omega_{\mathcal{E}^\vee}^{r-\mathrm{rk}(T)-1} \tag{5.3.3}$$

when T is positive semi-definite, and $\mathfrak{g}(T,\mathbf{y},\varphi_f)|_{X_{\mathbb{V}[k]}}=\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f)_{\sigma_k}$ otherwise (see Definitions 4.3.5 and 4.4.2); here

$$\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f)\;=\;\underset{\rho=0}{\mathrm{CT}}\,\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f;\rho) \tag{5.3.4}$$

as in Propositions 4.3.4 and 4.4.1.

Consider the contribution of $\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f)_{\sigma_k}$ to the integral over $\mathcal{X}(\mathbb{C})$; by definition, it equals $\mathrm{CT}_{\rho=0}\,I(\rho,\sigma_k)$, where

$$I(\rho,\sigma_k)=\frac{(-1)^r\kappa_0}{2\mathrm{Vol}(X_{\mathbb{V}[k]},\Omega)}\int_{[X_{\mathbb{V}[k]}]}\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f;\rho)_{\sigma_k}\wedge\Omega^{p+1-r}. \tag{5.3.5}$$

Let us compute $I(\rho,\sigma_1)$. Define the archimedean Schwartz function

$$\phi=\tilde{\nu}\otimes\varphi_+\otimes\cdots\otimes\varphi_+\;\in\;\mathcal{S}(\mathbb{V}^r\otimes_F\mathbb{R})=\bigotimes_{1\le i\le d}\mathcal{S}(\mathbb{V}^r_{\sigma_i}), \tag{5.3.6}$$

with $\tilde{\nu}\in\mathcal{S}(\mathbb{V}^r_{\sigma_1})$ as in (3.4.2) and where φ_+ denotes the Gaussian for the positive definite spaces $\mathbb{V}^r_{\sigma_k}$ $(k>1)$, given by

$$\varphi_+(\mathbf{v})=e^{-\pi\sum_i Q(v_i,v_i)}, \tag{5.3.7}$$

for $\mathbf{v}=(v_1,\dots,v_r)\in\mathbb{V}^r_{\sigma_k}$.

Consider the theta series

$$\Theta(g'_{\boldsymbol{\tau}},h;\phi\otimes\varphi_f)\;=\;\sum_T\Theta_T(g'_{\boldsymbol{\tau}},h;\phi\otimes\varphi_f) \tag{5.3.8}$$

defined in (5.1.15), and write

$$C_{\Theta,T}(\mathbf{y},h;\phi\otimes\varphi_f)\;:=\;(\det y_1\cdots y_d)^{-\iota m/4}\cdot\Theta_T(g'_{\boldsymbol{\tau}},h;\phi\otimes\varphi_f)\,q^{-T}. \tag{5.3.9}$$

For $z=hz_0\in\mathbb{D}^+$ $(h\in\mathrm{U}(\mathbb{V}_{\sigma_1}))$ and $\mathrm{Re}(\rho)\gg0$, we have

$$\mathfrak{g}^{\circ}(T,\mathbf{y},\varphi_f;\rho)_{\sigma_1}(z)\wedge\Omega^{p+1-r}(z)=\int_1^\infty C_{\Theta,T}(t\mathbf{y},h;\phi\otimes\varphi_f)\frac{dt}{t^{\rho+1}}\wedge\Omega^p(z), \tag{5.3.10}$$

where the estimates in the proof of Proposition 4.3.3 justify the interchange of sum and integral. Using the Siegel-Weil formula (Theorem 5.1.3) to compute the integral over $X_{\mathbb{V}}$, we find

$$I(\rho,\sigma_1) = \frac{(-1)^r}{2}\int_1^\infty C_{E,T}\left(t\mathbf{y}, \lambda(\phi\otimes\varphi_f), s_0(r)\right)\,\frac{dt}{t^{\rho+1}}; \tag{5.3.11}$$

here the relevant Eisenstein series is

$$E(g',\lambda(\phi\otimes\varphi_f),s) = \sum_T E_T(g',\lambda(\phi\otimes\varphi_f),s) \tag{5.3.12}$$

where $\lambda\colon S(\mathbb{V}(\mathbb{A})^r)\to I_r(\mathbb{V},s_0(r))$ is as in Section 5.1.2, and we have written

$$C_{E,T}(\mathbf{y},\lambda(\phi\otimes\varphi_f)),s) \;:=\; (\det y_1\cdots y_d)^{-\iota m/4}\cdot E_T(g'_{\boldsymbol{\tau}},\lambda(\phi\otimes\varphi_f),s)\; q^{-T}. \tag{5.3.13}$$

Our next step is to relate this expression to the coefficient

$$C_T(\mathbf{y},\Phi_f,s) \;:= E_T(\boldsymbol{\tau},\Phi_f,s)\; q^{-T} \tag{5.3.14}$$

of the scalar weight l Eisenstein series in the statement. Comparing archimedean components, for $i\geq 2$ we have $\lambda(\phi_{\sigma_i}) = \lambda(\varphi_+) = \Phi^l(s_0)$, while $\lambda(\phi_{\sigma_1}) = \lambda(\tilde{\nu}) = (-1)^{r-1}\tilde{\Phi}(s_0)$ as in (3.4.6). Thus, writing $C'_T(\mathbf{y},\Phi_f,s_0) = \frac{d}{ds}C_T(\mathbf{y},\Phi_f,s)|_{s=s_0}$ and $\mathbf{y}=(y_1,\dots,y_d)$, the argument in the proof of Lemma 3.4.3 shows that

$$C_{E,T}(t\mathbf{y},\lambda(\phi\otimes\varphi_f),s_0) \;=\; (-1)^{r-1}\frac{2}{\iota}\cdot t\frac{d}{dt}C'_T((ty_1,t'y_2,\dots,t'y_r)),\Phi_f,s_0)\Big|_{t=t'}. \tag{5.3.15}$$

Adding the contributions from all archimedean places, we conclude that

$$\begin{aligned}\frac{(-1)^r\kappa_0}{2\mathrm{Vol}(X_K,\Omega)}&\int_{[\mathcal{X}(\mathbb{C})]}\mathfrak{g}^\circ(T,\mathbf{y},\varphi_f)\wedge\Omega^{p+1-r}\\ &= -\mathrm{CT}_{\rho=0}\int_1^\infty\frac{d}{dt}\left(C'_T(t\mathbf{y},\Phi_f,s_0)\right)\frac{dt}{t^\rho}.\end{aligned} \tag{5.3.16}$$

This integral can be evaluated using the following lemma, whose proof is straightforward.

Lemma 5.3.2. *Let $f\colon\mathbb{R}_{>0}\to\mathbb{C}$ be a smooth function such that for some constants $a,b\in\mathbb{C}$, the function $F(t)=f(t)-a-b\log t$ satisfies $\lim_{t\to\infty}F(t)=0$ and $F'(t)=O(t^{-1-C})$ as $t\to\infty$, for some positive constant C. Then*

$$-\mathrm{CT}_{\rho=0}\int_1^\infty f'(t)\,\frac{dt}{t^\rho} = f(1)-a.$$

By Proposition 5.2.12, the function $C'_T(t\mathbf{y},\Phi_f,s_0)$, regarded as a function of t, satisfies the hypotheses of the lemma with

$$a = \kappa(T,\Phi_f)+\frac{\iota}{2}\beta(T,\Phi_f)\sum_{v|\infty}\log\left(\frac{\det'\sigma_v(T)\cdot\det y_v}{\det'\sigma_v(T)y_v}\right). \tag{5.3.17}$$

To finish the proof, it suffices to show that the contributions from the second terms in (5.3.3) for the various components $X_{\mathbb{V}[k]} \subset \mathcal{X}(\mathbb{C})$ sum to

$$\iota \sum_{k=1}^{d} \log\left(\frac{\det' \sigma_k(T) \cdot \det y_k}{\det' (\sigma_k(T) y_k)}\right) \frac{(-1)^{\mathrm{rk}(T)} \kappa_0}{2\mathrm{Vol}(X_{\mathbb{V}}, \Omega)} \int_{[X_{\mathbb{V}[k]}]} \delta_{Z(T,\varphi_f)_{\sigma_k}} \wedge \Omega^{p-\mathrm{rk}(T)} \stackrel{?}{=} \frac{\iota}{2} \sum_k \log\left(\frac{\det' \sigma_k(T) \cdot \det y_k}{\det' (\sigma_k(T) y_k)}\right) \beta(T, \Phi_f), \tag{5.3.18}$$

where we used the identity $\Omega_{\mathcal{E}^\vee} = -\Omega_{\mathcal{E}} = -\Omega$. Note that on account of the logarithms, both sides vanish if T is non-degenerate. When T is degenerate, the claim is essentially contained in [29]; we outline the argument in Lemma 5.3.4 below. □

When the matrix T is non-degenerate, we recover the identities (1.1.19) and (1.1.20) by using the explicit expression for $\kappa(T, \Phi_f)$ given by Lemma 5.2.3.

Combining Theorems 4.5.1 and 5.3.1, we obtain the following corollary generalizing the main result of [25] (for $\mathrm{U}(p,1)$ this corollary is due to Liu [37]; recently Bruinier and Yang [15] have treated the $\mathrm{O}(p,2)$ case).

Corollary 5.3.3. *Assume that T_1 and T_2 are non-degenerate and that $Z(T_1, \varphi_{f,1})$ and $Z(T_2, \varphi_{f,2})$ intersect transversely. Assume also that $r_1 + r_2 = p + 1$. Then*

$$\frac{(-1)^{p+1}}{\mathrm{Vol}(X_{\mathbb{V}}, \Omega)} \int_{[\mathcal{X}(\mathbb{C})]} \mathfrak{g}(T_1, \mathbf{y}_1, \varphi_{f,1}) * \mathfrak{g}(T_2, \mathbf{y}_2, \varphi_{f,2})\, q^T = \sum_{T = \left(\begin{smallmatrix} T_1 & * \\ * & T_2 \end{smallmatrix}\right)} E'_T\left(\left(\begin{smallmatrix} \boldsymbol{\tau}_1 & \\ & \boldsymbol{\tau}_2 \end{smallmatrix}\right), \lambda(\varphi_{f,1} \otimes \varphi_{f,2}), 0\right)_\infty,$$

It remains to establish the following lemma, which is an application of the results of [29].

Lemma 5.3.4. *Suppose $T = \left(\begin{smallmatrix} 0 & \\ & S \end{smallmatrix}\right)$ is degenerate. Then for any $k = 1, \ldots, d$,*

$$\frac{(-1)^{\mathrm{rk}(T)} \kappa_0}{\mathrm{Vol}(X_{\mathbb{V}}, \Omega)} \int_{[X_{\mathbb{V}[k]}]} \delta_{Z(T,\varphi_f)_{\sigma_k}} \wedge \Omega^{p-\mathrm{rk}(T)} = \beta(T, \Phi_f),$$

with notation as in Theorem 5.3.1.

Proof. Suppose first that $t = \mathrm{rk}(T) > 0$, so that S is non-degenerate of rank t. As both sides of the desired identity are linear in φ_f, we may also assume that φ_f is of the form

$$\varphi_f = \varphi'_f \otimes \varphi''_f \in S(\mathbb{V}(\mathbb{A}_f)^{r-t}) \otimes S(\mathbb{V}(\mathbb{A}_f)^t). \tag{5.3.19}$$

By construction, we have $Z(T, \varphi_f) = \varphi'_f(0) Z(S, \varphi''_f)$. On the other hand, let $\Phi(\varphi'_f)$ and $\Phi(\varphi''_f)$ denote the standard sections corresponding to φ'_f and φ''_f, respectively. Then Definition 5.2.11 and a direct computation using explicit formulas for the Weil representation (see e.g. [47, Proposition 2.2.5]) give

$$\begin{aligned}\beta(T,\Phi_f) = \kappa_0\cdot\beta\left(S,\eta^*\Phi_f\right) &= \kappa_0\cdot C_S(e,\eta^*\Phi_f,s_0(t)))\\ &= \kappa_0\cdot\varphi'_f(0)\cdot C_S\left(e,\Phi(\varphi''_f),s_0(t)\right)\\ &= \kappa_0\cdot\varphi'_f(0)\cdot\beta(S,\Phi(\varphi''_f)).\end{aligned} \tag{5.3.20}$$

where $\eta^* = (\eta^r_t)^*$ as in (5.2.21).

To prove the lemma, say for $k=1$, let

$$\Theta_{\mathrm{KM}}(\boldsymbol{\tau}'',\varphi''_f)_{\sigma_1} \;=\; \sum_S \omega(S,\mathbf{y}'',\varphi''_f)_{\sigma_1}\,q^S \tag{5.3.21}$$

be the Kudla-Millson theta series of genus t, as defined in (4.3.5); here $\boldsymbol{\tau}'' = \mathbf{x}''+i\mathbf{y}''\in\mathbb{H}^d_t$. It is shown in [29] that $\omega(S,\mathbf{y}'',\varphi''_f)_{\sigma_1}$ is a closed form on $X_{\mathbb{V}}$ whose cohomology class is $[Z(S,\varphi''_f)_{\sigma_1}]$. In particular, we have

$$\int_{[X_{\mathbb{V}}]}\delta_{Z(T,\varphi_f)_{\sigma_1}}\wedge\Omega^{p-t} = \varphi'_f(0)\int_{[X_{\mathbb{V}}]}\delta_{Z(S,\varphi''_f)_{\sigma_1}}\wedge\Omega^{p-t} = \varphi'_f(0)\int_{[X_{\mathbb{V}}]}\omega(S,\mathbf{y}'',\varphi_f)_{\sigma_1}\wedge\Omega^{p-t}; \tag{5.3.22}$$

this also follows from Propositions 4.3.4 and 4.4.1. To compute the latter integral, define an archimedean Schwartz function

$$\varphi_\infty \;:=\; \widetilde{\varphi}\otimes\varphi_+\otimes\cdots\otimes\varphi_+ \;\in\; \bigotimes S(\mathbb{V}^t_{\sigma_k}) \tag{5.3.23}$$

where $\widetilde{\varphi}$ is the Schwartz function on $\mathbb{V}^t_{\sigma_1}$ defined by (5.2.3), and φ_+ is the standard Gaussian. Note that

$$\lambda(\varphi_\infty) = (-1)^t\otimes_{v|\infty}\Phi^l_t(s_0), \tag{5.3.24}$$

as in the proof of Lemma 5.2.1, and

$$\omega(S,\mathbf{y}'',\Phi''_f)_{\sigma_1}(z)\wedge\Omega^{p-t}(z) \;=\; C_{\Theta,S}(\mathbf{y}'',h,\varphi_\infty\otimes\varphi''_f)\,\Omega^p(z),\quad z=hz_0, \tag{5.3.25}$$

where $C_{\Theta,S}(\mathbf{y}'',h,\varphi_\infty\otimes\varphi''_f)$ is the coefficient of q^S of the theta series attached to $\varphi_\infty\otimes\varphi''_f$. Applying the Siegel-Weil formula (Theorem 5.1.3) again, noting that $s_0(t)>0$ here, we conclude

$$\frac{(-1)^t}{\mathrm{Vol}(X_{\mathbb{V}},\Omega)}\int_{[X_{\mathbb{V}}]}\omega(S,\mathbf{y}'',\varphi''_f)_{\sigma_1}\wedge\Omega^{p-t} \;=\; C_S\left(\mathbf{y}'',\Phi(\varphi_f),s_0(t)\right) \;=\; \beta(S,\Phi''_f). \tag{5.3.26}$$

Comparing this with (5.3.20) and (5.3.22) proves the lemma for $k=1$, and the proof for all other values of k follows in exactly the same way.

Finally, when $T=0_r$, the left hand side is $\kappa_0\varphi_f(0)=\kappa_0\Phi_f(e)$, which is by definition equal to $\beta(0,\Phi_f)$, cf. Proposition 5.2.13. □

Remark 5.3.5. Theorems 3.4.10 and 5.3.1 can be used to give a different expression, in certain cases, for the constant $C_{T,\Gamma_{\mathbf{v}}}$ appearing in Lemma 3.4.5. For simplicity, suppose that T is totally positive definite, the Schwartz form $\varphi_f=\varphi_L\in S(\mathbb{V}(\mathbb{A}_f)^r)$ is the characteristic function of $(L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})^r$ for a lattice $L\subset\mathbb{V}$, and that the level structure K is neat and is chosen so that $X_{\mathbb{V}} = X_{\mathbb{V},K}$ is connected; thus $X_{\mathbb{V}} = \Gamma\backslash\mathbb{D}^+$ with $\Gamma = K \cap \mathbf{H}(\mathbb{Q})^+$. Assume further that Γ stabilizes the lattice L.

A slight adaptation of the proof of Theorem 5.3.1, together with a little algebra, yields the formula

$$\begin{aligned} &\frac{(-1)^r}{2\mathrm{Vol}(X_{\mathbb{V}},\Omega)}\int_{X_{\mathbb{V}}} \mathfrak{g}(T,\mathrm{Id},\varphi_L)\wedge\Omega^{p+1-r} \\ &\qquad = \left(\frac{\mathcal{W}'_{\sigma_1(T)}(Id,s_0)}{\mathcal{W}_{\sigma_1(T)}(Id,s_0)} - \frac{\iota}{2}\left[\log\det\pi\sigma_1(T) - \frac{\Gamma'_r(\iota m/2)}{\Gamma_r(\iota m/2)}\right]\right)\cdot\beta(T,\Phi_L) \end{aligned} \tag{5.3.27}$$

for the integral over the single component $X_{\mathbb{V}}$; here Φ_L is the standard section corresponding to φ_L, and $\mathcal{W}_{\sigma_1(T)}(g,s)$ is the normalized archimedean Whittaker functional, as in (3.3.12). On the other hand, setting $\Omega(T) = \{\mathbf{v}\in\mathbb{V}^r \mid T(\mathbf{v}) = T\}$, we have

$$\int_{X_{\mathbb{V}}} \mathfrak{g}(T,\mathrm{Id},\varphi_L)\wedge\Omega^{p+1-r} = \int_{\Gamma\backslash\mathbb{D}^+}\sum_{\mathbf{v}\in\Omega(T)\cap L^r}\mathfrak{g}^{\circ}(\sigma_1(\mathbf{v}))\wedge\Omega^{p+1-r} \tag{5.3.28}$$

$$= \sum_{\substack{\mathbf{v}\in\Omega(T)\cap L^r \\ \mathrm{mod}\ \Gamma}}\int_{\Gamma_{\mathbf{v}}\backslash\mathbb{D}^+}\mathfrak{g}^{\circ}(\sigma_1(\mathbf{v}))\wedge\Omega^{p+1-r}; \tag{5.3.29}$$

note here that the sum over Γ-orbits is finite, see [24, §5]. Applying Theorem 3.4.10 and comparing with (5.3.27), it follows that

$$\sum_{\substack{\mathbf{v}\in\Omega(T)\cap L^r \\ \mathrm{mod}\ \Gamma}} C_{\sigma_1(T),\Gamma_{\mathbf{v}}} = 2\,(-1)^r\,\mathrm{Vol}(X_{\mathbb{V}},\Omega)\,\frac{\beta(T,\Phi_L)}{\mathcal{W}_{\sigma_1(T)}(Id,s_0)} \tag{5.3.30}$$

Finally, note that $\mathbf{H}_{\mathbb{V}}(\mathbb{Q})$ acts transitively on $\Omega(T)$; it follows from the expression in Lemma 3.4.5 that $C_{T,\Gamma_{\mathbf{v}}}$ is the same for all $\mathbf{v}$ appearing above. Thus, setting $r(T,\Gamma) := \#\Omega(T)\cap L^r/\Gamma$, and using Proposition 3.3.3(iii) for the value $\mathcal{W}_{\sigma(T)}(Id,s_0)$, we have

$$C_{\sigma_1(T),\Gamma_{\mathbf{v}}} = r(T,\Gamma)^{-1}\cdot 2\,(-1)^r\,\mathrm{Vol}(X_{\mathbb{V}},\Omega)\,\left(\frac{2^{r(\kappa-1)/2}\,\Gamma_r(\iota m/2)}{(-2\pi i)^{\iota mr/2}\,\det\sigma_1(T)^{\iota s_0(r)}}\right)\beta(T,\Phi_L). \tag{5.3.31}$$

5.4. **Classes in arithmetic Chow groups.** In this section, we describe how the currents $\mathfrak{g}(T,\mathbf{y},\varphi_f)$ arise as the archimedean parts of classes in arithmetic Chow groups lifting the cycles $Z(T,\varphi_f)$. As we will ultimately have little to say about arithmetic aspects of the theory, we shall gloss over many serious difficulties regarding suitable integral models, bad reduction, etc. A key point is a natural geometric context for the analogue (4.3.15) of Green's equation in the degenerate case.

We continue to assume that $\mathbb{V}$ is anisotropic, but drop the assumption $\mathrm{rk}(\mathcal{E}) = 1$. We also assume the level structure $K \subset \mathbf{H}(\mathbb{A}_f)$ is neat.

Let $\mathbf{k} = F$ (resp. $\mathbf{k} = E$) in the orthogonal (resp. unitary) case, so that $\mathcal{X}$ is proper over $\mathrm{Spec}\,\mathbf{k}$. Suppose $\mathfrak{X}$ is a regular integral model, proper and flat over $\mathrm{Spec}(\mathcal{O}_{\mathbf{k}})$, with an extension of the tautological bundle that we continue to denote $\mathcal{E}$. Finally, for each T

and φ_f, let $\mathcal{Z}(T,\varphi_f)$ denote a cycle on $\mathfrak{X}$ extending $Z(T,\varphi_f)$ on the generic fibre whose codimension in $\mathfrak{X}$ is equal to the codimension of $Z(T,\varphi_f)$ in $\mathcal{X}=\mathfrak{X}_{\mathbf{k}}$.

Let $\widehat{\mathrm{CH}}^{\bullet}_{\mathbb{C}}(\mathfrak{X}) = \bigoplus_r \widehat{\mathrm{CH}}^r_{\mathbb{C}}(\mathfrak{X})$ denote the Gillet-Soulé arithmetic Chow ring (with $\mathbb{C}$ coefficients), as in [46]. Classes in $\widehat{\mathrm{CH}}^r_{\mathbb{C}}(\mathfrak{X})$ are represented by pairs (Z,g_Z), where Z is codimension r cycle on $\mathscr{X}$ (with $\mathbb{C}$-coefficients) and g_Z is an $(r-1,r-1)$ current on $\mathcal{X}(\mathbb{C}) = \coprod_{\sigma\colon \mathbf{k}\to\mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$ that is invariant under complex conjugation, and satisfies Green's equation

$$\mathrm{dd^c} g_Z \;+\; \delta_{Z(\mathbb{C})} \;= [\omega_Z]$$

for some smooth differential form ω_Z on $\mathcal{X}(\mathbb{C})$; we may also view $g_Z = \{g_{Z,\sigma}\}_\sigma$ as a collection consisting of a current $g_{Z,\sigma}$ on $\mathcal{X}_\sigma(\mathbb{C})$ for each complex embedding σ.

When T is non-degenerate, Proposition 4.3.4 (and the discussion around (4.3.6)) or Proposition 4.4.1 gives rise to a class

$$(\mathcal{Z}(T,\varphi_f),\, \mathfrak{g}(T,\mathbf{y},\varphi_f)) \;\in\; \widehat{\mathrm{CH}}^{rq'}_{\mathbb{C}}(\mathfrak{X}), \tag{5.4.1}$$

where $q' = \mathrm{rk}(\mathcal{E})$.

Now consider a pair (T,φ_f) with $T\in \mathrm{Sym}_r(F)$ (resp. $T\in\mathrm{Her}_r(E)$) a degenerate matrix, and set $t=\mathrm{rank}(T)$. Let

$$\widehat{c}_{q'}(\mathcal{E}^\vee) \;\in\; \widehat{\mathrm{CH}}^{q'}_{\mathbb{C}}(\mathfrak{X})$$

denote the arithmetic Chern class attached to $\mathcal{E}^\vee$, as in [46, Section IV]. The class of $\widehat{c}_{q'}(\mathcal{E}^\vee)^{r-t}$ may be represented by a pair $(\mathcal{Z}_0, g_0)$ such that the generic fibre of $\mathcal{Z}_0$ intersects properly with $Z(T,\varphi_f)$, and where the current g_0 satisfies the equation

$$\mathrm{dd^c} g_0 \;+\; \delta_{\mathcal{Z}_0(\mathbb{C})} \;=\; \Omega^{r-t}_{\mathcal{E}^\vee}$$

and is of logarithmic type, see [46, §II.2]. On the other hand, consider the set of currents $\mathfrak{g}$ of degree $(r-1,r-1)$ satisfying the analogue

$$\mathrm{dd^c}\mathfrak{g} \;+\; \delta_{Z(T,\varphi_f)(\mathbb{C})}\wedge\Omega^{r-t}_{\mathcal{E}^\vee} \;=\; [\omega] \tag{5.4.2}$$

of Green's equation, for some smooth form ω; a short computation reveals that the map

$$\mathfrak{g} \;\mapsto\; \mathfrak{g} + g_0\wedge\delta_{Z(T,\varphi_f)(\mathbb{C})}$$

defines a bijection between the solutions of (5.4.2) and Green currents for the intersection $\mathcal{Z}(T,\varphi_f)\cdot\mathcal{Z}_0$. Therefore, applying Propositions 4.3.4 and 4.4.1, we obtain a class

$$\widehat{\mathfrak{Z}}(T,\mathbf{y},\varphi_f) \;:=\; \Big(\mathcal{Z}(T,\varphi_f)\cdot\mathcal{Z}_0,\; \mathfrak{g}(T,\mathbf{y},\varphi_f) + g_0\wedge\delta_{Z(T,\varphi_f)(\mathbb{C})}\Big) \;\in\; \widehat{\mathrm{CH}}^{rq'}_{\mathbb{C}}(\mathfrak{X}).$$

To see that this construction is independent of the choice of $(\mathcal{Z}_0,g_0)$ representing $\widehat{c}_{q'}(\mathcal{E}^\vee)^{r-t}$, choose any Green current $\mathfrak{g}'$ for $\mathcal{Z}(T,\varphi_f)$, and note that

$$\begin{aligned}\widehat{\mathfrak{Z}}(T,\mathbf{y},\varphi_f) \;&=\; (\mathcal{Z}(T,\varphi_f),\mathfrak{g}')\cdot(\mathcal{Z}_0,g_0) \;+\; (0,\, \mathfrak{g}(T,\mathbf{y},\varphi_f) - \mathfrak{g}'\wedge\Omega^{r-t}_{\mathcal{E}^\vee})\\ &=\; (\mathcal{Z}(T,\varphi_f),\mathfrak{g}')\cdot\widehat{c}_{q'}(\mathcal{E}^\vee)^{r-t} \;+\; (0,\, \mathfrak{g}(T,\mathbf{y},\varphi_f) - \mathfrak{g}'\wedge\Omega^{r-t}_{\mathcal{E}^\vee})\end{aligned} \tag{5.4.3}$$

To preserve uniformity of notation, set $\widehat{\mathfrak{Z}}(T,\mathbf{y},\varphi_f) = (\mathcal{Z}(T,\varphi_f),\mathfrak{g}(T,\mathbf{y},\varphi_f))$ when T is non-degenerate. For any (possibly degenerate) T, restricting the cycle $\widehat{\mathfrak{Z}}(T,\mathbf{y},\varphi_f)$ to the generic fibre (and forgetting the current) yields a cycle that coincides with the construction in [24].

As an example of our construction, take $T = \mathbf{0}_r$; applying the computation (4.4.13) for $\mathfrak{g}(0, \mathbf{y}, \varphi_f)$ gives the concrete expression

$$\widehat{\mathfrak{Z}}(\mathbf{0}_r, \mathbf{y}, \varphi_f) \;=\; \varphi_f(0) \cdot \Big(\widehat{c}_{q'}\left(\mathcal{E}^\vee\right)^r \;-\; \Big(0, \left\{\log\det y_k \cdot c_{q'-1}(\mathcal{E}^\vee, \nabla)^* \wedge \Omega_{\mathcal{E}^\vee}^{r-1}\right\}_{k=1,\dots,d}\Big)\Big). \tag{5.4.4}$$

5.5. **Kudla's arithmetic height conjecture.** We recast our results in the setting of Kudla's conjectures on the arithmetic heights of the cycles $\widehat{\mathfrak{Z}}(T, \mathbf{y}, \varphi_f)$ considered in the previous section.

Assume that $\mathbb{V}$ is anisotropic and $\mathrm{rk}(\mathcal{E}) = 1$. Let

$$\widehat{\omega} \;:=\; \widehat{c}_1(\mathcal{E}^\vee) \;\in\; \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathscr{X}) \tag{5.5.1}$$

denote the arithmetic class attached to $\mathcal{E}^\vee$, or more precisely, to its integral extension as in the preceding section, and consider the generating series

$$\phi_{\widehat{\omega}}(\tau) \;:=\; \sum_T \widehat{\deg}\left(\widehat{\mathfrak{Z}}(T, \mathbf{y}, \varphi_f) \cdot \widehat{\omega}^{p+1-r}\right) q^T \tag{5.5.2}$$

where $\widehat{\deg}\colon \widehat{\mathrm{CH}}^{p+1}_{\mathbb{C}}(\mathscr{X}) \to \mathbb{C}$ is the arithmetic degree map [46, Section III]. A rough form of Kudla's conjectural programme, as outlined in e.g. [27], suggests that $\phi_{\widehat{\omega}}(\tau)$ is, up to a normalization, equal to the special derivative of an Eisenstein series.

More precisely, let $\Phi_f \in I_r(\mathbb{V}(\mathbb{A}_f), s)$ denote the standard section of parallel scalar weight l determined by φ as in (5.3.2), and consider the parallel weight l Eisenstein series

$$\mathcal{E}(\boldsymbol{\tau}, \Phi_f, s) = A_r(s) E(\boldsymbol{\tau}, \Phi_f, s) =: \sum_T \mathscr{C}_T(\mathbf{y}, \Phi_f, s) q^T, \tag{5.5.3}$$

for an appropriate normalizing factor $A_r(s)$; then one should have an identity

$$\widehat{\deg}\left(\widehat{\mathfrak{Z}}(T, \mathbf{y}, \varphi_f) \cdot \widehat{\omega}^{p+1-r}\right) \;\overset{?}{\sim}\; \mathscr{C}'_T(\mathbf{y}, \Phi_f, s_0(r)) \tag{5.5.4}$$

up to correction terms involving rational multiples of $\log p$ with p in a fixed, finite set of primes that might depend on φ, the level structure K, and the choices of integral models. As these correction terms are expected to arise as contributions from (components of) cycles at primes of bad reduction, it is reasonable to assume that they are independent of the parameter $\mathbf{y}$.

Let

$$h(\mathcal{Z}(T, \varphi_f)) := \widehat{\deg}\left(\widehat{\omega}^{p+1-\mathrm{rk}(T)} \,\middle|\, \mathcal{Z}(T, \varphi_f)\right) \tag{5.5.5}$$

denote the Bost-Gillet-Soulé height of $\mathcal{Z}(T,\varphi_f)$ along $\widehat{\omega}$, as defined in [6, Proposition 2.3.1]. Using [6, (2.3.3)] and (5.4.3) above, a brief computation gives

$$\widehat{\deg}\left(\widehat{\mathfrak{Z}}(T,\mathbf{y},\varphi_f)\cdot\widehat{\omega}^{p+1-r}\right) = h(\mathcal{Z}(T,\varphi_f)) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})}\mathfrak{g}(T,\mathbf{y},\varphi_f)\wedge\Omega_{\mathcal{E}^\vee}^{p+1-r}$$
$$= h(\mathcal{Z}(T,\varphi_f)) + \frac{(-1)^{p+1-r}}{2}\int_{\mathcal{X}(\mathbb{C})}\mathfrak{g}(T,\mathbf{y},\varphi)\wedge\Omega_{\mathcal{E}}^{p+1-r}.$$

By Theorem 5.3.1, the conjecture (5.5.4) is equivalent to the statement

$$h(\mathcal{Z}(T,\varphi_f)) \overset{?}{\sim} \left(A_r(s_0(r)) - \frac{(-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})}{\kappa_0}\right)C'_T(\mathbf{y},\Phi_f,s_0(r))$$
$$+ A'_r(s_0(r))\cdot C_T(\mathbf{y},\Phi_f,s_0(r)) + \frac{(-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})}{\kappa_0}\cdot\kappa(T,\Phi_f) \tag{5.5.6}$$

for all T and φ_f, where $\operatorname{vol}(X_{\mathbb{V}}) = \operatorname{vol}(X_{\mathbb{V}},\Omega_{\mathcal{E}}^p)$.

Note that only the values $A_r(s_0(r))$ and $A'_r(s_0(r))$ appear in this expression. To pin these values down further, suppose for the moment that $s_0(r) > 0$, and take $\mathbf{y} = \lambda\cdot\mathrm{Id}$, say; then Proposition 5.2.12 implies that

$$C'_T(\lambda\cdot\mathrm{Id},\Phi_f,s_0(r)) = \kappa(T,\Phi_f) + \frac{\iota d\cdot(r-\mathrm{rk}T)}{2}\cdot\log\lambda\cdot\beta(T,\Phi_f) + F(\lambda) \tag{5.5.7}$$

for some function F satisfying $\lim_{\lambda\to\infty}F(\lambda)=0$. Similarly, $C_T(\lambda\cdot Id,\Phi_f,s_0(r)) = \beta(T,\Phi_f)$.

Thus, choosing T and φ_f such that $\mathrm{rk}(T) < r$ and $\beta(T,\Phi_f)\neq 0$, and noting that $h(\mathcal{Z}(T,\varphi_f))$ is evidently independent of $\mathbf{y}$, a necessary condition for (5.5.6) to hold is

$$A_r(s_0(r)) = \frac{(-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})}{\kappa_0}. \tag{5.5.8}$$

Now for the derivative $A'_r(s_0(r))$, assume that $\varphi(0)=1$, and consider the matrix $T=0$. Then

$$h(\mathcal{Z}(0,\varphi_f)) \sim \widehat{\deg}_{\mathscr{X}}\,\widehat{\omega}^{p+1} =: \widehat{\operatorname{vol}}_{\widehat{\omega}}(\mathscr{X}) \tag{5.5.9}$$

On the other hand, by Proposition 5.2.13,

$$\kappa(\mathbf{0},\Phi_f) = 0, \tag{5.5.10}$$

and $C_{\mathbf{0}}(\mathbf{y},s_0,\Phi) = \varphi(0) = 1$. Therefore (5.5.6) for $T=\mathbf{0}$ and $s_0(r)>0$ gives the further necessary condition

$$A'_r(s_0(r)) \sim \widehat{\operatorname{vol}}_{\widehat{\omega}}(\mathscr{X}). \tag{5.5.11}$$

In other words, taking $A_r(s)$ such that $A_r(s_0(r)) = (-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})\kappa_0^{-1}$ and $A'_r(s_0(r)) = \widehat{\operatorname{vol}}_{\widehat{\omega}}(\mathscr{X})$, the conjecture (5.5.4), for $s_0(r)>0$, takes the form

$$h(\mathcal{Z}(T,\varphi_f)) \overset{?}{\sim} \widehat{\operatorname{vol}}_{\widehat{\omega}}(\mathscr{X})\cdot\beta(T,\Phi_f) + \frac{(-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})}{\kappa_0}\kappa(T,\Phi_f). \tag{5.5.12}$$

When $s_0(r)=0$, note that $C_T(\mathbf{y},\Phi_f,0)=0$ and the conjecture (5.5.6) becomes

$$h(\mathcal{Z}(T,\varphi_f)) \overset{?}{\sim} \frac{(-1)^{p+1}\operatorname{vol}(X_{\mathbb{V}})}{\kappa_0}\cdot\kappa(T,\Phi_f) \tag{5.5.13}$$

The point here is that the right hand sides of these relations involve explict constants depending only on T and φ_f. This conjecture has been verified in the case of full level Shimura curves for $r = 1$ [33], for $r = 2$ [34], and for Hilbert modular surfaces with $r = 1$ in [7] (these results also include contributions from places of bad reduction). A slightly weaker version of this conjecture (i.e. away from an explicit set of primes determined by T and φ) was proved for general orthogonal Shimura varieties over $\mathbb{Q}$ by Hörmann [17]. Several cases involving cycles of top arithmetic codimension supported at finite primes were also established by Kudla and Rapoport, see e.g. [22], or the discussion in [27, §II] for more details.

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