Name: ____________________

Student Number: ____________________

I understand that cheating is a serious offence: ____________________
(Signature – *In Ink*)

Please place a check mark beside your section number and instructor:

- ☐ A01 — A. Clay (M/W/F 10:30–11:20, St. John's 118)
- ☐ A02 — M. Sadeghi (M/W/F 9:30–10:20, Drake Centre 343)
- ☐ A03 — M. Virgilio (Tu/Th 8:30–9:45, Armes 208)
- ☐ A04 — Yong Zhang (Tu/Th 11:30–12:45, Armes 200)
- ☐ A05 — R. Borgersen (Tu/Th 1:00–2:15, St. Paul 100)
- ☐ A06 — S. Sankaran (M/W/F 3:30–4:20, Armes 208)
- ☐ D01 — N. Harland (Distance)

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 10 pages including this cover page and one blank page for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 60 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question.

V. Please do not call or e-mail your instructor to inquire about grades. They will be available shortly after they have been marked.

Question	Points	Score
1	12	
2	12	
3	7	
4	7	
5	5	
6	5	
7	5	
8	7	
Total:	60	

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1. Compute the following limits, if they exist. If a limit does not exist, indicate whether the limit tends to ∞ or $-\infty$, if applicable; otherwise write "does not exist". Be sure to justify your responses.

[3] (a) $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

Solution:

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} &= \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1} \frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} \\
&= \frac{1}{\sqrt{1+3}+2} = \frac{1}{4}.
\end{aligned}$$

[3] (b) $\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{6}{x^2-9}\right)$

Solution:

$$\begin{aligned}
\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{6}{x^2-9}\right) &= \lim_{x\to 3} \left(\frac{x+3}{x^2-9} - \frac{6}{x^2-9}\right) \\
&= \lim_{x\to 3} \frac{x-3}{x^2-9} \\
&= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)} \\
&= \lim_{x\to 3} \frac{1}{x+3} \\
&= \frac{1}{3+3} = \frac{1}{6}.
\end{aligned}$$

[3] (c) $\lim_{x\to 1^-} \frac{(x-\pi)^{10}}{(x-1)^{99}}$

Solution:

Plugging in 1 give a non-zero/zero form $(1-\pi)^{10}/0$ therefore the limit is either $\pm\infty$.

Checking the signs, the numerator is positive since the exponent is even.

Since $x < 1$, $x - 1$ is negative and therefore $(x-1)^{99}$ is negative. Hence the fraction is negative as $x \to 1^-$.

Therefore the limit is $-\infty$.

[3] (d) $\lim_{x\to 0} \frac{\tan 3x}{2x}$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan 3x}{2x} &= \lim_{x\to 0}\frac{\sin 3x}{\cos 3x\cdot 2x}\\ &= \lim_{x\to 0}\frac{\sin 3x\cdot 3x}{3x\cdot\cos 3x\cdot 2x}\\ &= \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{3}{2\cos 3x}\\ &= 1\cdot\frac{3}{2\cos 0}=\frac{3}{2}.\end{aligned}$$

2. Compute the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWER!

[3] (a) $f(x) = 4x^{2000} + \sqrt[3]{x^2} - \dfrac{4}{x^2} + e^{4\pi}$.

Solution:

$$f(x) = 4x^{2000} + x^{2/3} - 4x^{-2} + e^{4\pi}.$$

$$f'(x) = 8000x^{1999} + \frac{2}{3}x^{-1/3} + 8x^{-3} + 0.$$

[4] (b) $h(u) = \dfrac{u\cos u}{u^2+1}$.

Solution:
By the quotient rule and product rule

$$\begin{aligned}h'(u) &= \frac{(u\cos u)'(u^2+1)-(u\cos u)(u^2+1)'}{(u^2+1)^2}\\ &= \frac{((u)'\cos u + u(\cos u)')(u^2+1)-(u\cos u)(u^2+1)'}{(u^2+1)^2}\\ &= \frac{(\cos u + u(-\sin u))(u^2+1)-(u\cos u)(2u)}{(u^2+1)^2}\end{aligned}$$

[5] (c) $j(z) = e^{z^2}(z^3+\tan z)^4$.

Solution:
By the chain rule and product rule

$$\begin{aligned}j'(z) &= \left(e^{z^2}\right)'(z^3+\tan z)^4 + e^{z^2}\left((z^3+\tan z)^4\right)'\\ &= e^{z^2}\left(z^2\right)'(z^3+\tan z)^4 + e^{z^2}4(z^3+\tan z)^3\left(z^3+\tan z\right)'\\ &= e^{z^2}(2z)(z^3+\tan z)^4 + e^{z^2}4(z^3+\tan z)^3\left(3z^2+\sec^2 z\right)\end{aligned}$$

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[7] 3. Find the values for A and B that make the function

$$f(x) = \begin{cases} x^2 + A, & \text{if } x < 2 \\ B, & \text{if } x = 2 \\ \sin\left(\dfrac{3\pi}{x}\right) & \text{if } x > 2 \end{cases}$$

continuous at the point $x = 2$. Be sure to fully justify your answer.

Solution:

For the function to be continuous at $x = 2$, we need that

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sin\left(\frac{3\pi}{x}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 + A) = 2^2 + A = 4 + A$$

$$f(2) = B$$

Setting $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$ yields

$$4 + A = -1 \Rightarrow A = -5.$$

Setting $\lim_{x\to 2^+} f(x) = f(2)$ yields

$$B = -1.$$

Therefore $A = -5$ and $B = -1$.

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[7] 4. Compute $f'(x)$ using the definition of the derivative if $f(x) = \sqrt{3x+2}$. No credit will be given for any other method.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+2}-\sqrt{3x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+2}-\sqrt{3x+2})(\sqrt{3(x+h)+2}+\sqrt{3x+2})}{h(\sqrt{3(x+h)+2}+\sqrt{3x+2})} \\
&= \lim_{h\to 0} \frac{3x+3h+2-(3x+2)}{h(\sqrt{3(x+h)+2}+\sqrt{3x+2})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+2}+\sqrt{3x+2})} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+2}+\sqrt{3x+2}} \\
&= \frac{3}{\sqrt{3(x+0)+2}+\sqrt{3x+2}} \\
&= \frac{3}{2\sqrt{3x+2}}
\end{aligned}$$

[5] 5. For each of the following statements, indicate whether the statement is true or false. If the statement is true, simply write "true", if the statement is false then you must provide an example of a function and justification showing why it is false.

(a) If a function is differentiable at a, then it is continuous at a.

Solution: True. This is a theorem in section 2.8 in the text.

(b) If a function is continuous at a, then it is differentiable at a.

Solution:
This is false. An example of which is $f(x) = |x|$. This function is continuous at 0 since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = f(0).$$

However at 0

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Since the left and right limits are not equal, the limit, and therefore the derivative does not exist.

Hence f is continuous at 0, but not differentiable.

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[5] 6. Determine an equation of the line tangent to $f(x) = \sqrt{x^3 + 1}$ at $x = 2$.

Solution:

$$f'(x) = \frac{1}{2}(x^3 + 1)^{-1/2}(x^3 + 1)' = \frac{1}{2\sqrt{x^3 + 1}}(3x^2)$$

Hence the slope is

$$m = f'(2) = \frac{1}{2\sqrt{2^3 + 1}}(3(2)^2) = \frac{12}{2(3)} = 2.$$

The point has $x = 2$ and $y = f(2) = \sqrt{2^3 + 1} = \sqrt{9} = 3$. Therefore a point on the tangent line is (2, 3).

Hence an equation of the tangent line is $y - 3 = 2(x - 2)$ (or equivalently $y = 2x - 1$.)

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[5] 7. Suppose that y is a function of x that obeys the identity $y\tan(x) = \sin(y^2) + x$. Solve for $\dfrac{dy}{dx}$ in terms of x and y.

Solution:

Using implicit differentiation, we have

$$\begin{aligned}
&\frac{dy}{dx}\tan x + y\sec^2 x = 2y\frac{dy}{dx}\cos(y^2) + 1 \\
&\Rightarrow \frac{dy}{dx}\tan x - 2y\frac{dy}{dx}\cos(y^2) = 1 - y\sec^2 x \\
&\Rightarrow \frac{dy}{dx}\big(\tan x - 2y\cos(y^2)\big) = 1 - y\sec^2 x \\
&\Rightarrow \frac{dy}{dx} = \frac{1 - y\sec^2 x}{\tan x - 2y\cos(y^2)}
\end{aligned}$$

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[7] 8. A block of ice maintains the shape of a cube as it melts, resulting in its volume decreasing at a rate of 10cm^3 per minute. At what rate is the surface area changing when the block has dimensions $10\text{cm} \times 10\text{cm} \times 10\text{cm}$?

Solution:

Let x be the side length of the cube. We are attempting to find the surface area of the cube which is $A(x) = 6x^2$. We know that $\frac{dV}{dt} = -10cm^3/min$ and want to find $\frac{dA}{dt}$ when $x = 10$ cm.

Option 1: Solve for dx/dt first.

Since $V = x^3$ we differentiate to get $dV/dt = 3x^2 dx/dt$. At $x = 10$ we have

$$-10 = 3(100)dx/dt \Rightarrow dx/dt = -10/300 = -1/30.$$

Since $A = 6x^2$ we differentiate to get $dA/dt = 12x dx/dt$. At $x = 10$ we have

$$dA/dt = 12(10)(-1/30) = \frac{-120}{30} = -4.$$

Therefore the surface area is decreasing at a rate of $4cm^2/min$.

Option 2: Solve for A as a function of V

Since $V = x^3$ we have $x = V^{1/3}$. Therefore $A = 6x^2 = 6V^{2/3}$.

At $x = 10$ we have $V = 1000$. Taking the derivative we get $dA/dt = 4V^{-1/3}dV/dt$ and thus at $x = 10$ we have

$$dA/dt = 4(1000)^{-1/3}(-10) = \frac{-40}{10} = -4.$$

Therefore the surface area is decreasing at a rate of $4cm^2/min$.

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