UNIVERSITY OF MANITOBA

April 23, 2008, 6:00pm
\# 602

FINAL EXAMINATION
TITLE PAGE
TIME: 2 hours

DEPARTMENT & COURSE NO: MATH 1500
EXAMINATION: Introduction to Calculus
EXAMINER: Math 1500 Committee

NAME: (Print in ink) SOLUTIONS.

STUDENT NUMBER: ____________

SEAT NUMBER: ____________

SIGNATURE: (in ink) ____________
(I understand that cheating is a serious offense)

Please indicate your lecture section by checking the correct box below:

☐	A01	Slot 3, and 5T	MWF 10:30 and T 10:00	W. Korytowski
☐	A02	Slot 1	MWF 8:30	A. Gerhard
☐	A03	Slot 4	TTh 8:30	W. Korytowski
☐	A04	Slot 9	TTH 11:30	T. Kucera
☐	A05	Slot 15	TTh 4:00	D. Kalajdzievska
☐	A06	Slot 10	TTh 1:00	C. K. Gupta
☐	A92	Challenge for Credit	☐ SJR	
☐	Deferred			

INSTRUCTIONS TO STUDENTS:

This is a 2 hour exam.
Please show your work clearly.

No texts, notes, or other printed aids are permitted. You are not allowed to have calculators, cellphones, electronic translators, personal music devices or any other electronic devices or mechanical aids at the exam table with you.

This exam has a title page, 8 pages of questions and also 2 blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 100 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	20	
2	14	
3	10	
4	10	
5	12	
6	12	
7	22	
Total:	100	

This exam counts for 60% of your final grade.

1. Find $\frac{dy}{dx}$ in each case (DO NOT SIMPLIFY YOUR ANSWERS):

[3] (a) $y = x^\pi + \pi^\pi + \pi^x$

$$y' = \underbrace{\pi \cdot x^{\pi-1}}_{\text{power rule.}} + 0 + \underbrace{\pi^x \cdot \log(\pi)}_{\text{(using logarithm rules)}}$$

[6] (b) $y = \sqrt{1 + \ln(\sin(x))}$

$$y = (1 + \ln(\sin(x)))^{1/2} \Rightarrow y' = \frac{1}{2} \cdot (1 + \ln(\sin(x)))^{-1/2} \cdot \frac{d}{dx}(1 + \ln(\sin(x)))$$

$$= \frac{1}{2}(1 + \ln(\sin x))^{-1/2} \cdot \frac{1}{\sin(x)} \cdot \cos(x$$

[5] (c) $y = \int_1^{x^3} \ln(2 + \cos(\theta))\, d\theta$

Using FTC part I:

$$y' = \ln(2 + \cos(x^3)) \cdot \frac{d}{dx}(x^3)$$

$$= \ln(2 + \cos(x^3)) \cdot 3x^2.$$

[6] (d) $y = x^{(x^2+1)}$

use log. diff. : $\ln(y) = \ln\left(x^{(x^2+1)}\right)$

$$= (x^2+1) \cdot \ln(x)$$

$$\Rightarrow \frac{d}{dx}(\ln(y)) = \frac{y'}{y} = 2x \cdot \ln(x) + (x^2+1) \cdot \frac{1}{x}$$

$$\Rightarrow y' = y\left(2x\ln(x) + (x^2+1) \cdot \frac{1}{x}\right)$$

$$= x^{x^2+1}\left(2x\ln(x) + (x^2+1) \cdot \frac{1}{x}\right)$$

2. Evaluate the following indefinite and definite integrals:

[4] (a) $\int \left(3x^4 - \frac{4}{x}\right)\, dx$

$$= 3 \cdot \frac{x^5}{5} - 4 \cdot \ln(x) + C$$

using power rule:
$$\frac{d}{dx} \frac{x^{n+1}}{n+1} = x^n$$
and $\frac{d}{dx} \ln(x) = \frac{1}{x}$

[4] (b) $\int \left(e^{2t} + 4\sec^2(t)\right)\, dt$

$$= \frac{e^{2t}}{2} + 4\tan(t) + C$$

We know
$$\frac{d}{dt} \tan(t) = \sec^2(t)$$
$$\frac{d}{dt} e^{2t} = 2e^{2t}$$
so $\frac{d}{dt}\left(\frac{1}{2}e^{2t}\right) = e^{2t}$

[6] (c) $\int_{-\pi/4}^{\pi/3} \cos(2\theta)\, d\theta$

$$= \left[\frac{1}{2}\sin(2\theta)\right]_{-\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left(\underbrace{\sin\frac{2\pi}{3}}_{\sqrt{3}/2} - \underbrace{\sin\left(-\frac{\pi}{2}\right)}_{=-1}\right)$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{2}$$

have $\frac{d}{d\theta}\sin(2\theta) = \cos(2\theta)\cdot 2$

so $\frac{d}{d\theta}\left(\frac{1}{2}\sin(2\theta)\right) = \cos 2\theta$

[10] 3. Find the absolute maximum value and the absolute minimum value of $g(t) = t^3 - 3t^2 - 2$ on the interval $[-2, 1]$.

max/min can only occur at critical points (including endpoints)

Find critical points:

- endpoints $-2, 1$
- on interior $(-2,1)$, we have

$g'(t) = 3t^2 - 6t = 3t(t-2)$.

The only solution for $g'(t) = 0$ with t in $(-2,1)$

is $t = 0$.

$\Rightarrow$ critical points are $-2, 0, 1$.

Evaluate $g(t)$ at critical pts:

$g(-2) = (-2)^3 - 3\cdot(-2)^2 - 2 = -8 - 12 - 2 = -22 \Leftarrow$ min

$g(0) = 0 - 0 - 2 = -2 \Leftarrow$ max.

$g(1) = 1 - 3 - 2 = -4$.

$\therefore$ The max and min values are -2 (at $x = 0$)

and -22 (at $x = -2$)

respectively.

[10] 4. Prove that if $f'(x) < 0$ for all x in the interval I, then f is decreasing on the interval I.

We use the mean value theorem: for any x_1, x_2 in our interval, there is a value c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Choose any $x_2 > x_1$:

~~[illegible]~~, then $x_2 - x_1 > 0$

and $f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(c)$ for some c.

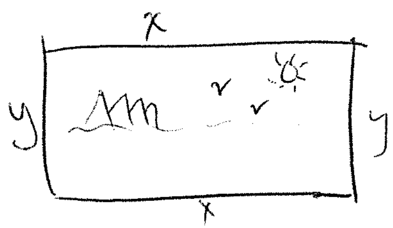
< 0 (since $f'(c) < 0$ by assumption, and $x_2 - x_1 > 0$.)

so $f(x_2) < f(x_1)$ for any $x_2 > x_1$,

Which is exactly what it means for f to be decreasing.

[12] 5. A rectangular picture frame is supposed to enclose an area of 600cm^2. The top edge costs twice as much per centimetre as either side or the bottom edge. What dimensions will minimize the cost of the frame?

Let x, y denote the dimensions of the frame:



area $= x \cdot y = 600$ cm^2.

~~cost~~ Suppose the cost of the each cm of sides + bottom edges is C (\$/cm)

Then the top edge costs \$ $2C$ / cm

$$\text{Total cost} = C\underbrace{(2y + x)}_{\text{side + bottom}} + \underbrace{2c(x)}_{\text{top}}$$

$$= C \cdot (2y + 3x)$$

$$T(x) = C\left(\frac{2 \cdot 600}{x} + 3x\right)$$

To minimize this, find derivative $T'(x) = C\left(-\frac{1200}{x^2} + 3\right)$.

critical points when $-\frac{1200}{x^2} + 3 = 0 \Rightarrow x^2 = 400$

$\Rightarrow x = +20$

($x = -20$ doesn't make physical sense)

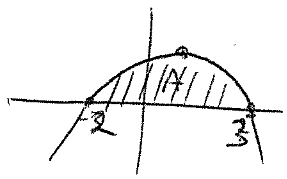
Note · $T'(x) < 0$ if $x < 20 \Rightarrow T$ is decreasing on $(0, 20)$

· $T'(x) > 0$ if $x > 20 \Rightarrow T$ increasing on $(20, \infty)$

$\Rightarrow$ minimum at $\boxed{x = 20}$ and so the required dimensions are $\boxed{x = 20\text{cm}, \; y = 30\text{cm}}$

[6] 6. (a) Find the area of the region bounded by the graph of $y = 6 + x - x^2$ and the x-axis.

Sketch:



$f(y) = 6 + x - x^2,$ $= (3-x)(2+x)$ x-intercepts at $x=-2$, $x=3$

$f'(y) = 1 - 2x$, critical pt at $x = 1/2$

$$A = \int_{-2}^{3} (6 + x - x^2)\,dx = \left[6x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^{3} = \left(6\cdot 3 + \frac{9}{2} - \frac{27}{3}\right) - \left(6\cdot(-2) + \frac{4}{2} - \frac{(-8)}{3}\right)$$

$$\cancel{= \left(6\cdot 3 + \frac{9}{2} - \frac{27}{3}\right) - \left(6(-2) + \frac{4}{2} + \frac{8}{3}\right)}$$

$$\cancel{= \left(14 - \frac{8}{3}\right) - \left(-9 + \frac{9}{2}\right) = 23 - \frac{8}{3} - \frac{9}{2}}$$

$$\cancel{= \frac{95}{6}}$$

$$= 9 + \frac{9}{2} - \left(-10 + \frac{8}{3}\right) = 19 + \frac{9}{2} - \frac{8}{3} = \frac{114 + 27 - 16}{6}$$

$$\boxed{= \frac{125}{6}}$$

[6] (b) If $f''(t) = 2 - 12t^2$ and $f'(0) = 3$ and $f(0) = 2$, find $f(t)$.

Take anti-deriv: $f'(t) = 2t - 4t^3 + c$ for some c

$3 = f'(0) = 0 - 0 + c \Rightarrow c = 3$

$\Rightarrow f'(t) = 2t - 4t^3 + 3$

so $f(t) = t^2 - t^4 + 3t + c$ for some c

$2 = f(0) = 0 - 0 + 0 + c \Rightarrow c = 2$

$$\boxed{\Rightarrow f(t) = t^2 - t^4 + 3t + 2}$$

7. Consider the function $f(x) = e^{-x}(x+1)$.

You are given the following information:

1. $f'(x) = -xe^{-x}$
2. $f''(x) = e^{-x}(x-1)$
3. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.

[2] (a) Find the intercepts of $f(x)$.

$e^{-x} \neq 0$ so

$f(x) = 0 \Rightarrow x = -1$ only x-intercept

[5] (b) Find the critical number(s) of $f(x)$ and find all the intervals on which $f(x)$ is increasing and all the intervals on which $f(x)$ is decreasing.

$f'(x) = -x \cdot e^{-x} = 0 \Rightarrow x = 0$ so $x=0$ is the only critical point.

	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	−

So f increasing on $(-\infty, 0)$
decreasing $(0, +\infty)$

[Note $e^{-x} > 0$ for all x]

[2] (c) Find the coordinates of the critical point(s) of $f(x)$, and classify each as a local maximum, a local minimum, or neither.

At $x=0$: $f(0) = e^0(0+1) = 1$.

By 1st deriv test, this is a local max.

[5] (d) Find all the intervals on which $f(x)$ is concave up and all the intervals on which $f(x)$ is concave down; and identify any inflection points

$f''(x) = e^{-x}(x-1) = 0 \Rightarrow x = 1$. $\left(f(1) = e^{-1}(1+1) = \frac{2}{e}\right)$

check signs:

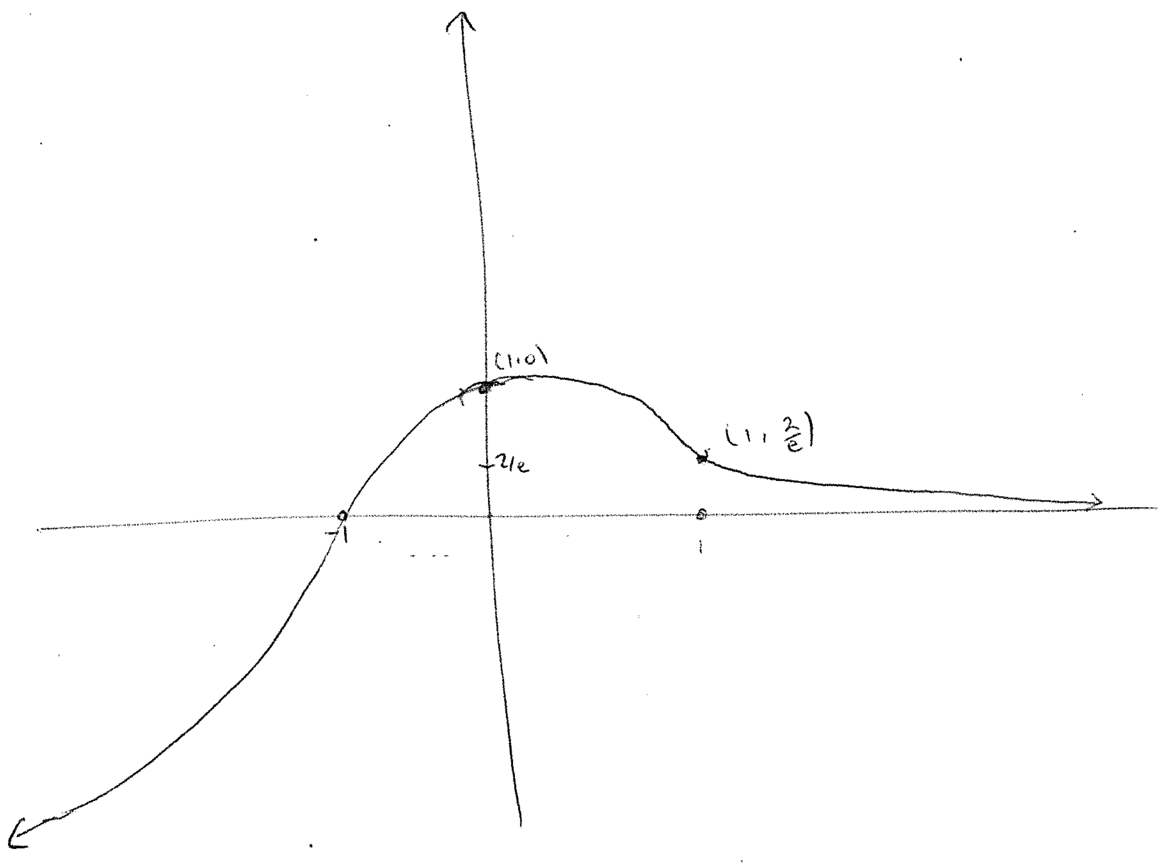
	$(-\infty, 1)$	$(1, \infty)$
f''	−	+
f	concave down	concave up.

Since we switch concavity at $x = 1$, it is (the only) inflection point

This question continues on the next page

Question 7, continued.

[8] (e) Give a neat sketch of the graph of $y = f(x)$.



April 21	MATH 1500	Siddarth Sankaran

Lab Quiz 5.1 20 minutes

Name: SOLUTIONS Student ID:

Always justify your answers!

Q1]. . . [3 points] Find

$$\frac{d}{dx}\int_{x^2}^{x^3} \cos\left(e^t\right) dt = \frac{d}{dx}\left(\int_{x^2}^{0} \cos(e^t)\,dt + \int_{0}^{x^3} \cos(e^t)\,dt\right)$$

$$= \frac{d}{dx}\left(-\int_{0}^{x^2} \cos(e^t)\,dt + \int_{0}^{x^3} \cos(e^t)\,dt\right)$$

(FTC + chain rule) $= -\cos(e^{x^2})\cdot 2x + \cos(e^{x^3})\cdot 3x^2$

Q2]. . . [3 points] Evaluate the definite integral

$$\int_1^3 \frac{x-2}{x^3}\,dx = \int_1^3 \left(\frac{x}{x^3} - \frac{2}{x^3}\right)dx = \int_1^3 \frac{1}{x^2} - \frac{2}{x^3}\,dx$$

$$= \int_1^3 x^{-2} - 2\cdot x^{-3}\,dx = \left[(-1)x^{-1} + x^{-2}\right]_1^3$$

$$= \left((-1)\cdot 3^{-1} + 3^{-2}\right) - (-1+1)$$

$$= -\frac{1}{3} + \frac{1}{9} = -\frac{2}{9}$$

(turn over)

Q3]... [4 points] Evaluate the definite integral

$$\int_0^{\sqrt{\pi}} \frac{e^x}{5} + x\cos(x^2) + 6x^{7/2}\,dx$$

Note $\frac{d}{dx}\sin(x^2) = 2x\cdot\cos(x^2)$

so $\frac{d}{dx}\left(\frac{1}{2}\sin(x^2)\right) = x\cdot\cos(x^2)$

$$= \left[\frac{e^x}{5} + \frac{1}{2}\sin(x^2) + 6\cdot\frac{x^{9/2}}{9/2}\right]_0^{\sqrt{\pi}}$$

$$= \left[\frac{e^x}{5} + \frac{1}{2}\sin(x^2) + \frac{4}{3}x^{9/2}\right]_0^{\sqrt{\pi}}$$

$$= \frac{e^{\sqrt{\pi}}}{5} + \frac{1}{2}\cancel{\sin(\pi)} + \frac{4}{3}\left(\sqrt{\pi}\right)^{9/2}$$

$$- \left(\underbrace{\frac{e^0}{5}}_{1/5} + \frac{1}{2}\cancel{\sin(0)} + 0\right)$$

$$= \frac{e^{\sqrt{\pi}}}{5} + \frac{4}{3}\pi^{9/4} - \frac{1}{5}$$