Math 4340/7340: HW 2

Due: Feb. 17 2017

Do these problems from Fulton: 2.4, 2.7, 2.8, 2.12, 2.17, as well as the following problem.

If R be a local ring with maximal ideal m, note that $\mathbf{k} := R/m$ is a field, called the *residue field*.

1. Let k be a field, and $R = k[[X]]$ the ring of power series. Show that R is a local ring; what is its maximal ideal and residue field?

2. Let R be any local ring with residue field $\mathbf{k}$. Explain briefly why, for any integer t, the quotient m^t/m^{t+1} is a vector space over $\mathbf{k}$.

3. Let k be a field. Let $p = (0, \ldots, 0) \in \mathbb{A}^n_k$ be the origin in $\mathbb{A}^n_k$, and $R = \mathcal{O}_p(\mathbb{A}^n)$ the local ring of $\mathbb{A}^n$ at p. Let m be the maximal ideal of R. Show that the residue field of R is k itself. For each t, compute the dimension of m^t/m^{t+1} over the residue field.

This last piece is an example of a general construction, known as a *Hilbert-Samuel polynomial*, which plays an important role in both algebra and algebraic geometry.