Zero-dimensional Shimura varieties and central derivatives of Eisenstein series

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Abstract

We formulate and prove a version of the arithmetic Siegel-Weil formula for (zero dimensional) Shimura varieties attached to tori, equipped with some additional data. More precisely, we define a family of "special" divisors in terms of Green functions at archimedean and non-archimedean places, and prove that their degrees coincide with the Fourier coefficients of the central derivative of an Eisenstein series. The proof relies on the usual Siegel-Weil formula to provide a direct link between both sides of the identity, and in some sense, offers a more conceptual point of view on prior results in the literature.

1 Introduction

In this paper, we formulate and prove a version of the "arithmetic Siegel-Weil formula" for zero-dimensional Shimura varieties. Broadly speaking, the arithmetic Siegel-Weil formula, which is a conjectural formula due to Kudla, predicts that the arithmetic heights of certain "special" cycles on Shimura varieties are related to the Fourier coefficients of derivatives of Eisenstein series. For zero-dimensional varieties, a number of special cases have been worked out in the literature; see for example [KRY99, How12, HY12] as well as [AGHMP18], where an identity of this form was a crucial ingredient in their proof of the average Colmez conjecture.

Here we take a more general, abstract point of view. Our starting point is an arbitrary Shimura datum (T, h), where T is a rational torus, and $h \colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to T_{/\mathbb{R}}$ is a morphism. Upon fixing a compact open subgroup $K \subset T(\mathbb{A}_f)$, this data determines a zero-dimensional Shimura variety that admits a canonical model over the reflex field $E(T, h)$. Choose a CM field E containing $E(T, h)$ and let $\mathcal{M}$ denote the base change of the canonical model to $\mathrm{Spec}(E)$; then the theory of complex multiplication provides for an action of $E^{\times} \backslash \mathbb{A}_E^{\times}$ on the geometric points $\mathcal{M}(E_v^{\mathrm{alg}})$ for any place v of E.

We also fix the following collection of data, which we describe in loose terms here, and refer to Section 3 for more precise details:

- an "incoherent family" $\mathcal{V} = (\mathcal{V}_v)_v$ whose members are one-dimensional local E_v-Hermitian spaces, one for each place v of F (here $F \subset E$ is the maximal totally real subfield);

- a family of lattices $\mathcal{L}_v \subset \mathcal{V}_v$ for non-archimedean places v; and

- a family of pairs $\left\{ (\mathbb{V}^{(v)}, \beta^{(v)}) \right\}$ indexed by the places of F that are non-split in E; here $\mathbb{V}^{(v)}$ is a local system of one-dimensional E-Hermitian spaces on the base change $\mathcal{M}_{/E_v^{\mathrm{alg}}}$, and $\beta^{(v)}$ is a map relating this local system to the incoherent collection $\mathcal{V}$. In particular, at each geometric point $z \in \mathcal{M}(E_v^{\mathrm{alg}})$, the map $\beta^{(v)}$ identifies the fibre of $\mathbb{V}^{(v)} \otimes \mathbb{A}_E$ at z with the coherent collection that differs from $\mathcal{V}$ at precisely the place v.

In addition, the action of $E^\times \backslash \mathbb{A}_E^\times$ on the geometric points of $\mathcal{M}$ restricts to an action of $H(F) \backslash H(\mathbb{A}_F)$, where, for an F-algebra R, we write $H(R) = \{ x \in R \otimes_F E \mid N_{E/F}(x) = 1 \}$; we further require that the data above satisfies a certain equivariance property with respect to the action of $H(F) \backslash H(\mathbb{A}_F)$.

With this data in place, and for a fixed $\alpha \in F^\times$, we construct a family of "Green functions"

$$g_v(\alpha, \tau) \colon \mathcal{M}(E_v^{\mathrm{alg}}) \to \mathbb{R} \tag{1.1}$$

indexed by the places v of F, and depending on an additional parameter $\tau \in \mathbb{H}_E$, the Hilbert upper-half space attached to E. When v is an archimedean place, this coincides with the Green function defined by Kudla. The novelty of the present article is an analogous construction at non-archimedean places, which can be roughly described as a non-archimedean factor of an incomplete Mellin transform. Packaging these Green functions together as v varies over all the places of F yields a "special cycle"

$$\widehat{Z}(\alpha, \tau) \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(\mathcal{M}) \tag{1.2}$$

where $\widehat{\mathrm{Pic}}_{\mathbb{A}}(\mathcal{M})$ is the group of isomorphism classes of adelically metrized line bundles.

On the other hand, we use the same data to define a Hilbert modular Eisenstein series $E(\tau, s, \Phi^*)$ of parallel weight one that vanishes at the centre $s = 0$ of its functional equation. Our main theorem relates the two constructions; the proof hinges on an application of the classical Siegel-Weil formula, which provides the concrete link between the two sides of the identity.

Theorem 1.1. *For every $\alpha \in F^\times$, we have*

$$\widehat{\deg} \, \widehat{Z}(\alpha, \tau) = -\#\mathcal{M}(\mathbb{C}) \cdot E_\alpha'(\tau, 0, \Phi^*) \tag{1.3}$$

where $E_\alpha'(\tau, 0, \Phi)$ is the α'th Fourier coefficient of the central derivative

$$E'(\tau, 0, \Phi^*) := \frac{d}{ds} E(\tau, s, \Phi^*)\big|_{s=0}. \tag{1.4}$$

The aforementioned special cases can be placed in the context of our main theorem, though the translation is not at all trivial. In these cases, the p-adic Green functions we construct encode arithmetic degrees in a natural way (see Proposition 4.8). In particular, our formulation might be viewed as offering a conceptual explanation for the appearance of Eisenstein series in these results; we present a few examples illustrating this point in the appendix.

Acknowledgments

2 Preliminaries

2.1 Adelic line bundles on zero-dimensional spaces and arithmetic degrees

In this section, we give a very naive description of the theory of adelic line bundles, as introduced by Zhang, see e.g. [Zha95]: as our interest lies solely in the zero-dimensional case, we will give an ad-hoc presentation here relevant to this setting.

Suppose X is a zero dimensional variety over a number field L, and ω is a line bundle on X. For a place v of L, choose an algebraic closure L_v^{alg} of L_v. A *v-adic metric*

$$\| \cdot \|_v = \{\| \cdot \|_x\}_{x \in X(L_v^{\mathrm{alg}})} \tag{2.1}$$

on L is a collection of (real-valued) metrics

$$\| \cdot \|_x \colon \omega_x \to \mathbb{R}_{\geq 0}, \qquad x \in X(L_v^{\mathrm{alg}}) \tag{2.2}$$

on the fibres ω_x of ω at geometric points. These metrics are required to satisfy

$$\|\lambda s\|_x = |\lambda|_v \cdot \|s\|_x, \tag{2.3}$$

for all $\lambda \in L_v^{\mathrm{alg}}, s \in \omega_x$, where $| \cdot |_v$ is the unique extension of the norm on L_v to L_v^{alg}.

At a non-archimedean place v, a key role is played by *model metrics*, defined as follows. Suppose $X_v = X \times_L \mathrm{Spec}(L_v)$ admits a projective model $\mathcal{X}$ over $\mathrm{Spec}(\mathcal{O}_{L,v})$ and that ω extends to a line bundle $\tilde{\omega}$ over $\mathcal{X}$. Then an L_v^{alg} valued point $z \in X(L_v^{\mathrm{alg}})$ lifts uniquely to a point $\tilde{z} \in \mathcal{X}(\mathcal{O}_{L_v^{\mathrm{alg}}})$, and hence we obtain an $\mathcal{O}_{L_v^{\mathrm{alg}}}$-lattice

$$\widetilde{\omega}_{\tilde{z}} \subset \omega_z. \tag{2.4}$$

This inclusion determines a metric, called the *model metric relative to* $(\mathcal{X}, \tilde{\omega})$, by the formula

$$\|s\|_{z,(\mathcal{X},\tilde{\omega})} := \inf\{|a|_v \mid a \in (L_v^{\mathrm{alg}})^\times, \ a^{-1}s \in \widetilde{\omega}_{\tilde{z}}\} \tag{2.5}$$

Definition 2.1. An adelic line bundle $\widehat{\omega} = (\omega, (\| \cdot \|_v)_v)$ consists of a line bundle ω together with v-adic metrics $\| \cdot \|_v$ for every place v, subject to the following condition: there exists an integer N and a model $(\mathcal{X}, \tilde{\omega})$ over $\mathrm{Spec}\ \mathcal{O}_L[1/N]$ such that the metric $\| \cdot \|_v$ is induced by this model at every non-archimedean place induced by a prime ideal relatively prime to N.

Two adelic line bundles $\widehat{\omega}$ and $\widehat{\omega}'$ are *isomorphic* if there is an isomorphism $\omega \simeq \omega'$ that induces an isometry $(\omega_x, \| \cdot \|_x) \simeq (\omega'_x, \| \cdot \|'_x)$ at every geometric point x.

Of course, for zero dimensional schemes these definitions are overkill, as all line bundles are isomorphic to the trivial one. When $\omega = \mathcal{O}_X$ is the trivial bundle, specifying an adelic metric $(\| \cdot \|_v)_v$ is equivalent to choosing a family (g_v) of "Green functions" $g_v \colon X(L_v^{\mathrm{alg}}) \to \mathbb{R}$ whose members are identically zero for almost all v; this equivalence is encoded by the relation

$$\|1\|_z^2 = e^{-g_v(z)}, \qquad z \in X(L_v^{alg}). \tag{2.6}$$

Let $\widehat{\mathrm{Pic}}_{\mathbb{A}}(X)$ denote the group of isomorphism classes of adelic line bundles; as every bundle on X is isomorphic to the trivial one, every element of $\widehat{\mathrm{Pic}}_{\mathbb{A}}(X)$ is determined uniquely by a set of Green functions $g_v \colon X(L_v^{\mathrm{alg}}) \to \mathbb{R}$ as above.

Given an adelic bundle $\widehat{\omega}$, we define its arithmetic degree

$$\widehat{\deg}\,\widehat{\omega} = \sum_{v \le \infty} \sum_{z \in X(L_v^{\mathrm{alg}})} -\log \|\eta\|_z^2 \tag{2.7}$$

where η is a non-vanishing section of ω. The product formula implies that this map is invariant on isomorphism classes, and therefore defines a function on $\widehat{\mathrm{Pic}}_{\mathbb{A}}(X)$. In particular, if $\widehat{\omega} = (\mathcal{O}_X, (g_v)_v) \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(X)$ is given by a collection of Green functions as above, then

$$\widehat{\deg}\,\widehat{\omega} = \sum_{v \le \infty} \sum_{z \in X(L_v^{\mathrm{alg}})} g_v(z). \tag{2.8}$$

2.2 Hermitian spaces

Let E be a CM field, with maximal totally real subfield F. Suppose that v is a place of F, let $E_v := E \otimes_F F_v$ and suppose that V_v is a Hermitian space of dimension m over E_v. We define the local invariant

$$\mathrm{inv}_v(V_v) := ((-1)^{m(m-1)/2} \det V_v, \Delta_{E/F})_v \in F_v^\times / N_{E_v/F_v}(E_v^\times) \tag{2.9}$$

where $\Delta_{E/F}$ is the discriminant, and $(\cdot, \cdot)_v$ is the Hilbert symbol. At a non-archimedean place, two E_v-Hermitian spaces of the same dimension are isomorphic if and only if they have the same invariants. In particular, at split places, there is only one isometry class of E_v-Hermitian space of a given dimension, while at non-split places, there are two. At archimedean places, the isometry class of a Hermitian space over $E_v \simeq \mathbb{C}$ is determined by its signature.

Now suppose V is a global Hermitian space over E. Almost all of its local invariants are equal to one, and together they satisfy the product formula

$$\prod_{v \le \infty} \mathrm{inv}_v(V) = 1. \tag{2.10}$$

Conversely, suppose we have a collection of signs $\epsilon_v \in \{\pm 1\}$ for $v \le \infty$, almost all of which are 1, and a collection of pairs of non-negative integers (r_v, s_v) for $v | \infty$, with $r_v + s_v = n$ independent of v. Then there exists a Hermitian space V of signature (r_v, s_v) at each $v | \infty$ and $\mathrm{inv}_v(V) = \epsilon_v$ if and only if $\prod_{v \le \infty} \epsilon_v = 1$ and $\epsilon_v = (-1)^{s_v+1}$ for each $v | \infty$.

Following the terminology of Kudla [Kud97], we say that a collection

$$\mathcal{V} = (\mathcal{V}_v)_{v \le \infty} \tag{2.11}$$

of local Hermitian spaces is *incoherent* if there is no global Hermitian space that is locally isometric to $\mathcal{V}_v$ for all v.

2.3 Whittaker functions and Eisenstein series

We briefly recall the theory of (Hermitian) Hilbert Eisenstein series, mainly to fix notation. We fix, once and for all, a character

$$\chi \colon E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times. \tag{2.12}$$

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Consider the group $G = U(1,1)$, viewed as an algebraic group over F. Concretely, for an F-algebra R, the points of G are

$$G(R) \ := \ \left\{ g \in \mathrm{GL}_2(R \otimes_F E) \mid g \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \right) = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \right) \overline{g}^t \right\} \tag{2.13}$$

Let $B \subset G$ denote the subgroup of upper-triangular matrices, with Levi decomposition $B = MN$, where, for an F-algebra R, we have

$$M(R) = \left\{ m(a) = \left(\begin{smallmatrix} a & \\ & \overline{a}^{-1} \end{smallmatrix} \right) \mid a \in (R \otimes_F E)^\times \right\} \tag{2.14}$$

and

$$N(R) = \left\{ n(b) = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix} \right) \mid b \in R \right\}. \tag{2.15}$$

Now fix a place v of F, and let $|\cdot|_v$ denote the corresponding absolute value. For every $s \in \mathbb{C}$, we obtain a character

$$\chi |\cdot|^s \colon B(F_v) \to \mathbb{C}^\times, \qquad \left(\begin{smallmatrix} a & b \\ & \overline{a}^{-1} \end{smallmatrix} \right) \mapsto \chi_v(a)|N(a)|_v^s, \tag{2.16}$$

where $a \in E_v^\times$, $b \in F_v$ and $N(a) = N_{E_v/F_v}(a)$. We define the degenerate principal series representation to be the (smooth) normalized induction

$$I_v(s,\chi) := \mathrm{Ind}_{B_v}^{G_v} \left(\chi_v |\cdot|_v^s \right) \tag{2.17}$$

where we abbreviate $G_v = G(F_v)$ and $B_v = B(F_v)$. Concretely, elements of $I_v(s,\chi)$ are smooth function $\Phi(\cdot, s) \colon G_v \to \mathbb{C}$ such that

$$\Phi \left(\left(\begin{smallmatrix} a & b \\ & \overline{a}^{-1} \end{smallmatrix} \right) g, s \right) \ = \ \chi_v(a)|N(a)|_v^{s+\frac{1}{2}} \Phi(g,s), \tag{2.18}$$

for all $\left(\begin{smallmatrix} a & b \\ & \overline{a}^{-1} \end{smallmatrix} \right) \in B_v$.

We say that a section $\Phi(g,s)$ is holomorphic (resp. meromorphic) if for every fixed g, the map $s \mapsto \Phi(g,s)$ is holomorphic (resp. meromorphic). Additionally, consider the maximal compact subgroup[1]

$$K_v = \begin{cases} G(\mathcal{O}_{F,v}), & v \text{ non-archimedean,} \\ U(1) \times U(1), & v \text{ archimedean;} \end{cases} \tag{2.19}$$

we say $\Phi(g,s)$ is *standard* if $\Phi(k,s)$ is independent of s for all $k \in K_v$.

An important source of standard sections is the Weil representation: suppose that V_v is a Hermitian space over E_v of dimension m, and assume that the restriction of the character χ to $\mathbb{A}_F$ coincides with $(\chi_{E/F})^m$, where $\chi_{E/F} \colon F^\times \backslash \mathbb{A}_F^\times \to \{\pm 1\}$ is the quadratic character corresponding to the extension E/F.

Let $S(V_v)$ denote the space of Schwartz-Bruhat functions on V_v, and let

$$\omega_{\mathcal{V}_v} \colon G(F_v) \to \mathrm{Aut}\, S(V_v) \tag{2.20}$$

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[1]In the archimedean case, we view $U(1) \times U(1)$ as a subgroup of $G(\mathbb{R})$ via the map

$$(e^{i\theta}, e^{i\varphi}) \mapsto e^{i\varphi} \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$$

.

denote the Weil representation, cf. [Kud94]. This representation depends on the additional choice of an additive character ψ on $\mathbb{A}_F$; we take $\psi = \psi_{\mathbb{Q}} \circ \operatorname{tr}_{F/\mathbb{Q}}$, where $\psi_{\mathbb{Q}}$ is the standard additive character on $\mathbb{A}_{\mathbb{Q}}$, and suppress this choice from the notation.

Explicit formulas for the action of elements of M_v and N_v are given as follows:

$$\omega_{\mathcal{V}_v}\left(m(a)\right)\varphi_v(x) \;=\; |N(a)|_v^{m/2}\chi_v(a)\varphi_v(ax), \qquad\qquad a \in E_v^{\times} \qquad (2.21)$$

$$\omega_{\mathcal{V}_v}\left(n(b)\right)\varphi_v(x) \;=\; \psi_v(bQ(x))\varphi_v(x), \qquad\qquad b \in F_v. \qquad (2.22)$$

Comparing these formulas with (2.18), we find that there is an G_v-equivariant map

$$\lambda_{V_v} \colon S(V_v) \to I_v(s_0, \chi), \qquad s_0 := m - 1 \qquad (2.23)$$

given by the formula

$$\lambda_{V_v}(\varphi)(g) = \left(\omega_{V_v}(g)\varphi\right)(0). \qquad (2.24)$$

Given $\varphi \in S(V_v)$, we denote by

$$\Phi(g, s, \lambda_{V_v}(\varphi)) \in I_v(s, \chi) \qquad (2.25)$$

the unique standard section whose value at s_0 coincides with $\lambda_{V_v}(\varphi)$.

Next, for $m \in F_v$ and $\Phi(g, s) \in I_v(s, \chi)$, consider the Whittaker functional

$$W_{m,v}(g, s, \Phi) := \int_{F_v} \Phi(wn(b)g, s)\,\psi_v(-mb)\,db \qquad (2.26)$$

where $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ and the measure db is normalized so that it is self-dual for the pairing $(b_1, b_2) \mapsto \psi_v(b_1 b_2)$.

Proposition 2.2. *[KS97]* *Suppose v is a non-split place of E, and let V_v be a one-dimensional Hermitian space over E_v. Let $R(V_v)$ denote the image of the Rallis map λ_{V_v}.*

(i) Let $m \in F_v$. If V_v does not represent m, then

$$W_{m,v}(g, 0, \Phi) = 0 \qquad (2.27)$$

for all $\Phi \in R(V_v)$.

(ii) Let V_v^+ and V_v^- denote two one-dimensional Hermitian spaces with $\operatorname{inv}_v(V_v^{\pm}) = \pm 1$. Then

$$I_v(0, \chi) = R(V_v^+) \oplus R(V_v^-). \qquad (2.28)$$

$\square$

We now turn to the global setting. Let

$$I(s, \chi) = \operatorname{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left(\chi \,|\cdot|_{\mathbb{A}}^s\right) \qquad (2.29)$$

denote the global degenerate principal series representation consisting of smooth functions $\Phi(g, s)\colon G(\mathbb{A}_F) \to \mathbb{C}$ satisfying

$$\Phi\left(\left(\begin{smallmatrix} a & b \\ & a^{-1} \end{smallmatrix}\right)g, s\right) = \chi(a)|N(a)|_{\mathbb{A}_F}^{s+\frac{1}{2}}\Phi(g, s) \qquad (2.30)$$

for all $\left(\begin{smallmatrix} a & b \\ & \bar{a}^{-1} \end{smallmatrix}\right) \in B(\mathbb{A}_F)$.

As before, a section $\Phi(g,s) \in I(s,\chi)$ is said to be holomorphic if for each fixed g, it defines a holomorphic function in S, and is standard if the restriction to $K = \prod_v K_v$ is independent of s. There is a natural factorization

$$I(s,\chi) = \bigotimes_v I_v(s,\chi) \tag{2.31}$$

and we say a section $\Phi(g,s)$ is *factorizable* if it can be written as a product $\Phi(g,v) = \otimes_v \Phi_v(g,s)$.

If $\Phi(g,s) \in I(s,\chi)$ is a holomorphic standard section, we define the Eisenstein series

$$E(g,s,\Phi) := \sum_{\gamma \in B(F)\backslash G(F)} \Phi(\gamma g, s). \tag{2.32}$$

Standard results in the theory assert that this series is absolutely convergent for $Re(s) \gg 0$, has meromorphic continuation to $s \in \mathbb{C}$, and a functional equation $s \leftrightarrow -s$; in particular, our Eisenstein series have been normalized so their centre of symmetry is at $s = 0$.

Writing the Fourier expansion of $E(g,s,\Phi)$ as

$$E(g,s,\Phi) = \sum_{m \in F} E_m(g,s,\Phi), \tag{2.33}$$

and assuming that $\Phi(g,s)$ is factorizable, we have that

$$E_m(g,s,\Phi) = W_m(g,s,\Phi) = \prod_v W_{m,v}(g_v,s,\Phi_v) \tag{2.34}$$

for all $m \neq 0$.

We conclude this section with a discussion of the values of Eisenstein series at the central point $s = 0$, in the cases of relevance to this paper.

Suppose that $\mathcal{V} = (\mathcal{V}_v)_v$ is a collection of one-dimensional local Hermitian spaces indexed by the places of F, and we are given a Schwartz function

$$\varphi = \otimes \varphi_v \in S(\mathcal{V}), \qquad \varphi_v \in S(\mathcal{V}_v) \tag{2.35}$$

with corresponding standard section $\Phi(g,s,\lambda_{\mathcal{V}}(\varphi)) = \otimes_v \Phi_v(g,s,\lambda_{\mathcal{V}_v}(\varphi_v))$.

If $\mathcal{V}$ is an *incoherent* collection, then for any $m \in F^\times$, there is at least one place v such that $\mathcal{V}_v$ does not represent m; in light of (2.34) and Proposition 2.2(i), we find

$$E(g,0,\Phi) = 0 \tag{2.36}$$

When $\mathcal{V}$ is a coherent collection, the central value is computed by the Siegel-Weil formula, as follows. Suppose V is a global one-dimensional Hermitian space over E. Let H denote the algebraic group over F whose points, for an F-algebra A, are given by

$$H(A) = \{x \in A \otimes_F E \mid x\bar{x} = 1\}; \tag{2.37}$$

in particular, we may identify $H(A)$ as the unitary group of $V \otimes_F A$.

For an Schwartz function $\varphi \in S(V \otimes_F \mathbb{A}_F)$ and an element $h \in H(\mathbb{A}_F)$, we define the theta series

$$\Theta(g,h,\varphi) = \sum_{x \in V} \omega_V(g)\varphi(h^{-1}x) \tag{2.38}$$

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which defines an automorphic form on $G \times H$.

The following proposition is a special case of the Siegel-Weil formula, which in the unitary setting is due to Ichino.

Proposition 2.3 ([Ich04, Theorem 4.2])**.** *Let V be a one-dimensional Hermitian space over E, let $\varphi \in S(V \otimes \mathbb{A}_E)$, and let $\Phi(g, s) = \Phi(g, s, \lambda_V(\varphi))$ denote the corresponding section. Then*

$$E(g, 0, \Phi) = 2 \int_{[H]} \Theta(g, h, \varphi) \, dh \tag{2.39}$$

where dh is the Haar measure on $[H] = H(F) \backslash H(\mathbb{A}_F)$ normalized so that the total volume is one. $\qquad\square$

3 Description of the data

In this section, we give a detailed description of the data and assumptions that go into our main theorem.

3.1 Zero dimensional Shimura varieties

Let T be a rational torus, and $h \colon \mathbb{S} \to (T)_{\mathbb{R}}$ a homomorphism, where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is Deligne's torus.

For a neat compact open subgroup $K \subset T(\mathbb{A}_{\mathbb{Q}, f})$, we have a zero dimensional Shimura variety

$$Sh_K(T, h) := T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_{\mathbb{Q}, f})/K. \tag{3.1}$$

Suppose that $E \subset \mathbb{C}$ is a CM field that contains the reflex field $E(T, h)$, so that $Sh_K(T, h)$ admits a canonical model $\mathcal{M}$ over $\mathrm{Spec}(E)$. As per the theory of canonical models, there is an action of $E^{\times} \backslash \mathbb{A}_E^{\times}$ on $\mathcal{M}(\mathbb{C}) = \mathcal{M}(E^{\mathrm{alg}}) = Sh_K(T, h)$ that factors through the Galois group $\mathrm{Gal}(E^{\mathrm{alg}}/E)^{\mathrm{ab}}$, where E^{alg} is a fixed algebraic closure of E, see e.g. [Mil05] for details. For any place v of E, we fix an embedding $E^{\mathrm{alg}} \subset E_v^{\mathrm{alg}}$, so we may view $E^{\times} \backslash \mathbb{A}_E^{\times}$ acting on $\mathcal{M}(E_v^{\mathrm{alg}}) = \mathcal{M}(E^{\mathrm{alg}})$ as well.

Let $F \subset E$ denote the maximal totally real subfield, and consider the algebraic group H over F defined by (2.37); viewing $H(\mathbb{A}_F)$ and $H(F)$ as a subgroups of $\mathbb{A}_E^{\times}$ and $E^{\times}$, respectively, we have a natural inclusion $H(F) \backslash H(\mathbb{A}_F) \hookrightarrow E^{\times} \backslash \mathbb{A}_E^{\times}$, and so in this way, we obtain an action of $H(F) \backslash H(\mathbb{A}_F)$ on $\mathcal{M}(E_v^{\mathrm{alg}})$.

3.2 Linear algebraic data

Let v be a place of F, and set $E_v := E \otimes_F F_v$. We say that the place v is *non-split* if E_v is a field; in particular, the archimedean places of F are all non-split.

Our next piece of data is an *incoherent collection*

$$\mathcal{V} = (\mathcal{V}_v)_{v \leq \infty} \tag{3.2}$$

indexed by the places of F; each member $(\mathcal{V}_v, \langle \cdot, \cdot \rangle_v)$ is a Hermitian space over E_v, and we assume that

$$\mathcal{V}_v \text{ is positive definite for all } v | \infty. \tag{3.3}$$

For each non-archimedean place $v < \infty$, we also fix an $\mathcal{O}_E$-stable lattice

$$\mathcal{L}_v \subset \mathcal{V}_v \tag{3.4}$$

and assume that $\mathcal{L}_v$ is self-dual for almost all v. Note that we are not assuming that $\mathcal{L}_v$ is integral for all v.

For each non-archimedean place v, we define the Schwartz function

$$\varphi_{\mathcal{L},v} \in S(\mathcal{V}_v) \tag{3.5}$$

to be the characteristic function of $\mathcal{L}_v$. At an archimedean place v, we set

$$\varphi_{\mathcal{V},v}(x) = e^{-\pi \langle x,x \rangle_v}. \tag{3.6}$$

For convenience, we set

$$\varphi_{\mathcal{L},f} = \bigotimes_{v<\infty} \varphi_{\mathcal{L},v}, \qquad \varphi_{\mathcal{V},\infty} = \bigotimes_{v|\infty} \varphi_{\mathcal{V},v}, \qquad \varphi_{\mathcal{L},\mathbb{A}} = \varphi_{\mathcal{V},\infty} \otimes \varphi_{\mathcal{L},f}. \tag{3.7}$$

Next, let v be a non-split place of F. Define an element $\varsigma_v \in F_v^\times$ as follows: if v is non-archimedean and inert, let $\varsigma_v = \varpi$, where ϖ is a uniformizer of F_v. If v is ramified, choose $\varsigma_v \in \mathcal{O}_{F,v}^\times$ such that $\varsigma_v \notin N(E_v^\times)$. If v is archimedean, we take $\varsigma_v = -1$.

In any of the above cases, let $\mathcal{V}_v'$ denote the E_v-Hermitian space whose underlying space is $\mathcal{V}_v$, with Hermitian form given by

$$\langle x,y \rangle_v' := \varsigma_v \cdot \langle x,y \rangle_v \tag{3.8}$$

By construction, $\mathcal{V}_v$ and $\mathcal{V}_v'$ are representatives for the two distinct isomorphism classes of E_v Hermitian spaces of dimension 1.

Finally, we note that as $\mathcal{V}_v$ and $\mathcal{V}_v'$ share the same underlying vector space, we view $\mathcal{L}_v$ as a lattice in either space; in particular, we may view $\varphi_{\mathcal{L},v}$ as a Schwartz function for $\mathcal{V}_v'$ as well.

3.3 Local systems

For each non-split place v of F, fix an algebraically closed extension E_v^{alg} of E_v, and set

$$\mathcal{M}_{/E_v^{\mathrm{alg}}} := \mathcal{M} \times_E \mathrm{Spec}(E_v^{\mathrm{alg}}). \tag{3.9}$$

Our final piece of data is a family of pairs

$$\left\{ (\mathbb{V}^{(v)}, \beta^{(v)}) \mid v \text{ non-split} \right\} \tag{3.10}$$

whose entries are as follows:

(a) $\mathbb{V}^{(v)}$ is a local system of one-dimensional E-Hermitian spaces over $\mathcal{M}_{/E_v^{\mathrm{alg}}}$ equipped with an $H(\mathbb{A}_F)$ action lifting the action on $\mathcal{M}_{/E_v^{\mathrm{alg}}}$.

(b) Let

$$\mathcal{V}^v \times \mathcal{V}_v' = \left(\prod_{\substack{w \leq \infty \\ w \neq v}} \mathcal{V}_w \right) \times \mathcal{V}_v' \tag{3.11}$$

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which we view as a Hermitian $\mathbb{A}_E$-module, and denote by $\underline{\mathcal{V}^v \times \mathcal{V}'_v}$ the constant local system on $\mathcal{M}_{/E_v^{\mathrm{alg}}}$, whose fibre at each point is a copy of $\mathcal{V}^v \times V'_v$

The datum $\beta^{(v)}$ is an isometry

$$\beta^{(v)} \colon \mathbb{V}^{(v)} \otimes_E \mathbb{A}_E \to \underline{\mathcal{V}^v \times \mathcal{V}'_v} \tag{3.12}$$

of Hermitian $\mathbb{A}_E$-local systems, for which we require the following equivariance property to hold: given $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ and $t \in H(\mathbb{A}_F)$, we have an isometry

$$r(t) \colon \mathbb{V}^{(v)}|_z \to \mathbb{V}^{(v)}|_{t \cdot z} \tag{3.13}$$

arising from part (a) of the definition, and we require that the diagram

$$\begin{array}{ccc}
\mathbb{V}^{(v)}|_z \otimes_E \mathbb{A}_E & \xrightarrow{\beta_z^{(v)}} & \mathcal{V}^v \times \mathcal{V}'_v \\
{\scriptstyle r(t) \otimes 1} \downarrow & & \downarrow {\scriptstyle \times t} \\
\mathbb{V}^{(v)}|_{t \cdot z} \otimes_E \mathbb{A}_E & \xrightarrow{\beta_{t \cdot z}^{(v)}} & \mathcal{V}^v \times \mathcal{V}'_v
\end{array} \tag{3.14}$$

commutes up to multiplication by an element of $\widehat{\mathcal{O}}_E^\times$. In other words, for any $t \in H(\mathbb{A}_F)$, we require that the composition

$$\beta_z^{(v)} \circ \left(r(t^{-1}) \otimes t \right) \circ \left(\beta_{t \cdot z}^{(v)} \right)^{-1} \in \mathrm{End}\left(\mathcal{V}^v \times \mathcal{V}'_v \right) \tag{3.15}$$

is given by multiplication by an element of $\widehat{\mathcal{O}}_E^\times$.

4 Eisenstein series and Green functions

We fix a collection of data

$$\left(\mathcal{M} \to \mathrm{Spec}(E), \ \mathcal{V}, \ (\mathcal{L}_v), \ \left(\mathbb{V}^{(v)}, \beta^{(v)} \right)_v \right) \tag{4.1}$$

as described in Section 3. In the following three subsections, we show how this data defines a section $\Phi^*(g, s) \in I(s, \chi)$, a collection of "special cycles" $\widehat{Z}(\alpha, \tau) \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(\mathcal{M}_K)$, and then state and prove our main theorem.

4.1 The section $\Phi^*(s)$

We begin with the non-archimedean setting. Fix a non-archimedean place v of F that is non-split in E, and let $E_v = E \otimes_F F_v$, which is a field. Then $\mathcal{V}_v$ and $\mathcal{V}'_v$ are the two non-isomorphic one dimensional Hermitian spaces over E_v.

Suppose for the moment that $\mathcal{V}_v$ represents 1, i.e. there exists a vector $x \in \mathcal{V}_v$ such that $\langle x, x \rangle_{\mathcal{V}_v} = 1$. Then there are isometries

$$\mathcal{V}_v \xrightarrow{\sim} \mathcal{V}_v^+ := (E_v, x\overline{y}) \qquad \text{and} \qquad \mathcal{V}'_v \xrightarrow{\sim} \mathcal{V}_v^- := (E_v, \varsigma_v x\overline{y}) \tag{4.2}$$

where for $a \in F_v^{\times}$, the space $(E_v, ax\overline{y})$ denotes the Hermitian space with underlying space E_v and Hermitian form $\langle x, y \rangle = ax\overline{y}$.

If $\mathcal{V}_v$ does not represent 1, then $\mathcal{V}_v'$ does, and so there are isometries

$$\mathcal{V}_v \xrightarrow{\sim} \mathcal{V}_v^-, \qquad \mathcal{V}_v' \xrightarrow{\sim} \mathcal{V}_v^+. \tag{4.3}$$

In either case, these isomorphisms carry the lattice $\mathcal{L}_v \subset \mathcal{V}_v = \mathcal{V}_v'$ to some lattice $\Lambda \subset E_v$. It follows easily from definitions that

$$\Phi_v(g, s, \lambda_{\mathcal{V}_v}(\varphi_{\mathcal{L}_v})) = \Phi_v(g, s, \lambda_{\mathcal{V}_v^\epsilon}(\varphi_\Lambda)). \tag{4.4}$$

where

$$\epsilon = \chi_v(\det \mathcal{V}_v) = \pm 1. \tag{4.5}$$

Similarly, we have

$$\Phi_v\left(g, s, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L}_v})\right) = \Phi_v\left(g, s, \lambda_{\mathcal{V}_v^{-\epsilon}}(\varphi_\Lambda)\right) \tag{4.6}$$

Now let $\varphi_v^0(x)$ denote the characteristic function of $\mathcal{O}_{E,v}$. Write $\Lambda = t \cdot \mathcal{O}_{E,v}$ for some $t \in E_v^{\times}$, so that

$$\varphi_\Lambda(x) = \varphi_v^0(t^{-1}x) = |N(t)|_v^{\frac{1}{2}} \chi(t) \left(\omega_{\mathcal{V}_v^\pm}\left(m(t^{-1})\right) \varphi_v^0 \right)(x). \tag{4.7}$$

This in turn implies that the difference

$$\delta_v(g, s) := \Phi_v(g, s, \lambda_{\mathcal{V}_v}(\varphi_{\mathcal{L}})) - |N(t)|_v^{\frac{1}{2}-s} \chi(t) \Phi_v\left(gm(t)^{-1}, s, \lambda_{\mathcal{V}_v^\epsilon}(\varphi_v^0)\right) \tag{4.8}$$

vanishes identically at $s = 0$.

Remark 4.1. If v corresponds to a prime of F that is unramified in E and $\mathcal{L}_v = \mathcal{O}_{E,v}$ is self-dual, then we may take $t = 1$ above, and $\delta_v(g, s) \equiv 0$. In particular, $\delta_v(g, s) = 0$ for almost all v. In this case, $\Phi_v(g, s, \lambda_{V_v}(\varphi_{\mathcal{L}}))$ equals $\Phi_v(g, s, \lambda_{V_v^+}(\varphi_v^0))$, which is the standard spherical section.

Next, recall that

$$I_v(0, \chi) = R(\mathcal{V}_v) \oplus R(\mathcal{V}_v'), \tag{4.9}$$

so there are Schwartz functions $\phi_v, \phi_v' \in S(E_v) = S(\mathcal{V}_v) = S(\mathcal{V}_v')$ such that

$$\delta_v(g, s) = s \cdot \left(\Phi\left(g, s, \lambda_{\mathcal{V}_v}(\phi_v)\right) + \Phi\left(g, s, \lambda_{\mathcal{V}_v'}(\phi_v')\right) \right) + O(s^2). \tag{4.10}$$

Definition 4.2. We define the modified section $\Phi_v^*(g, s) \in I_v(g, s)$ by the formula

$$\Phi_v^*(g, s) := \Phi_v(g, s, \lambda_{\mathcal{V}_v}(\varphi_{\mathcal{L}, v})) - s \cdot \left(\Phi\left(g, s, \lambda_{\mathcal{V}_v}(\phi_v)\right) + \Phi\left(g, s, \lambda_{\mathcal{V}_v'}(\phi_v')\right) \right). \tag{4.11}$$

Note that in general, the value at $s = 0$ is given by

$$\Phi_v^*(g, 0) = \Phi_v(g, 0, \lambda_{\mathcal{V}_v}(\varphi_{\mathcal{L}, v})). \tag{4.12}$$

Moreover, for almost all v, we have that $\Phi_v^*(g, s)$ is the standard spherical section, as per Remark 4.1.

For future use, we will gather some information about the Whittaker functionals $W_{m,v}(g, s, \Phi_v^*)$ near $s = 0$.

Lemma 4.3. *Suppose $\Phi(g, s) \in I_v(s, \chi)$ is an arbitrary section, $m \in F_v^\times$ and $t \in E_v^\times$. Then*

$$W_{m,v}\left(m(t)^{-1}, s, \Phi\right) = |N(t)|_v^{s-\frac{1}{2}}\chi(\bar{t})W_{N(t)^{-1}m,v}(e, s, \Phi). \tag{4.13}$$

where $N(t) = N_{E_v/F_v}(t)$.

Proof. By definition, we have

$$W_{m,v}\left(m(t)^{-1}, s, \Phi\right) = \int_{b \in F_v} \Phi\left(wn(b)m(t)^{-1}, s\right) \psi_p(-mb)\, db. \tag{4.14}$$

A straightforward computation shows that

$$w\,n(b)\,m(t^{-1}) \;=\; m(\bar{t})\,w\,n(bN(t)). \tag{4.15}$$

so

$$\Phi\left(wn(b)m(t)^{-1}, s\right) = \Phi\left(m(\bar{t})\,w\,n(bN(t)), s\right) = |N(t)|_v^{s+\frac{1}{2}}\chi(\bar{t})\Phi\left(w\,n(bN(t)), s\right). \tag{4.16}$$

Substituting this equation into (4.14) yields

$$
\begin{aligned}
W_{m,v}\left(m(t)^{-1}, s, \Phi\right) &= |N(t)|_v^{s+\frac{1}{2}}\chi(\bar{t})\int \Phi\left(wn(N(t)b), s\right)\psi_p(-mb)\, db \\
&= |N(t)|_v^{s+\frac{1}{2}}\chi(\bar{t})\int \Phi\left(wn(b), s\right)\psi_p(-mb/N(t))\,\frac{db}{|N(t)|_v} \\
&= |N(t)|_v^{s-\frac{1}{2}}\chi(\bar{t})W_{N(t)^{-1}m,v}(e, s, \Phi)
\end{aligned}
\tag{4.17}
$$

where in the second line we applied the change of variables $b \mapsto N(t)^{-1}b$. $\qquad\square$

Proposition 4.4. *Suppose v is a non-split non-archimedean place of F, and let $m \in F_v^\times$.*

(i) $W_{m,v}(e, 0, \Phi_v^*) = W_{m,v}(e, 0, \lambda_{\mathcal{V}_v}(\varphi_{\mathcal{L},v}))$.

(ii) If V_v does not represent m, then
$$W_{m,v}(e, 0, \Phi_v^*) = 0 \tag{4.18}$$

(iii) Suppose V_v does not represent m. Then

$$W'_{m,v}(e, 0, \Phi_v^*) - W'_{N(\varpi)^{-1}m,p}(e, 0, \Phi_v^*) = \; -\frac{1}{2}\log N(\mathfrak{p}_E) \cdot W_{m,v}(e, 0, \lambda_{V_v'}(\varphi_{\mathcal{L}})) \tag{4.19}$$

where $\mathfrak{p}_E$ is the prime ideal of E inducing the valuation v and ϖ is a uniformizer of E_v.

Proof. Part *(i)* follows immediately from (4.12). Part *(ii)* follows from part *(i)* and Proposition 2.2. For the third part, using (4.6) and (4.7) , we have

$$W_{m,v}\left(e, 0, \lambda_{V_v'}(\varphi_L)\right) = |N(t)|_v^{\frac{1}{2}}\chi(t)W_{m,v}\left(m(t)^{-1}, 0, \lambda_{V_v^{-\epsilon}}(\varphi_v^0)\right) \tag{4.20}$$

where $\Lambda = t\mathcal{O}_{E,v}$ as in the argument leading to (4.6). Hence, by Lemma 4.3, we conclude

$$
\begin{aligned}
W_{m,v}\left(e, 0, \lambda_{V_v'}(\varphi_L)\right) &= \chi(t\bar{t})W_{N(t)^{-1}m,v}\left(e, 0, \lambda_{V_v^{-\epsilon}}(\varphi_v^0)\right) \\
&= W_{N(t)^{-1}m,v}\left(e, 0, \lambda_{V_v^{-\epsilon}}(\varphi_v^0)\right);
\end{aligned}
\tag{4.21}
$$

here we use the fact $\chi(t\bar{t}) = 1$.

On the other hand, applying Lemma 4.3 again and the definition of $\Phi^*(g, s)$, we find

$$W_{m,v}(e, s, \Phi_v^*) = W_{N(t)^{-1}m,v}\left(e, s, \lambda_{V_v^\epsilon}(\varphi_v^0)\right) + O(s^2) \tag{4.22}$$

The Whittaker functions appearing on the right hand sides of (4.21) and (4.22) were computed explicitly by Yang [Yan05, §2], and the desired relations can be easily extracted from these formulas.
$\square$

The archimedean analogue is as follows. For later purposes, it will be convenient to introduce an additional parameter $y_v \in F_v = \mathbb{R}$ with $y_v > 0$, which will ultimately represent the imaginary part of a variable in a Hilbert upper half-space on which our modular forms are defined.

Proposition 4.5. *Suppose $v|\infty$, and let*

$$\Phi_v^+(g, s) = \Phi(g, s, \lambda_{\mathcal{V}_v}(\varphi_v)) \qquad and \qquad \Phi_v^-(g, s) = \Phi(g, s, \lambda_{\mathcal{V}_v'}(\varphi_v)). \tag{4.23}$$

where $\varphi_v(x) = e^{-\pi\langle x, x\rangle_{\mathcal{V}_v}}$. Recall here that $\mathcal{V}_v$ and $\mathcal{V}_v'$ have signatures $(1, 0)$ and $(0, 1)$, respectively.
For $m \in F^\times$ and $y_v \in F_v \simeq \mathbb{R}$ with $y_v > 0$, we set

$$\mathcal{W}_{m,v}(y_v, s, \Phi^\pm) := y_v^{-1/2}W_{m,v}(m(\sqrt{y_v}), s, \Phi_v^\pm) \cdot e^{2\pi y_v m_v}. \tag{4.24}$$

where $m_v = \sigma_v(m)$ with $\sigma_v\colon F \to \mathbb{R}$ denoting the real embedding corresponding to the place v.
Suppose that $m_v < 0$. Then for $t \in \mathbb{R}_{>0}$, we have

$$\frac{d}{dt}\mathcal{W}_{m,v}'(ty_v, 0, \Phi^+) = t^{-1}\mathcal{W}_{m,v}(ty_v, 0, \Phi^-) \tag{4.25}$$

and

$$\lim_{t\to\infty} \mathcal{W}_{m,v}(ty, 0, \Phi^+) = 0. \tag{4.26}$$

Proof. This follows from the explicit formulas in [KRY04, Proposition 14.1]. $\square$

At this stage, we have defined a local section $\Phi_v^*(s)$ for all non-split non-archimedian primes v. If $v|\infty$, we set $\Phi_v^*(g, s) = \Phi_v^+(g, s)$ as in the previous proposition. Finally, if v is a split prime of E, we set $\Phi_v^*(g, s) = \Phi(g, s, \lambda_{V_v}(\varphi_L))$

Putting everything together, we make the following definition.

Definition 4.6. Let

$$\Phi^*(s) := \otimes\Phi_v^*(s) \in I(s, \chi). \tag{4.27}$$

Remark 4.7. Let $\Sigma^{\text{bad}} = \{v_1, \dots, v_r\}$ denote the set of non-archimedean places of E such that either E_v/F_v is ramified, or E_v/F_v is an unramified field extension but $\mathcal{L}_v$ is not self-dual. Recall that $\Phi_v^*(g, s) = \Phi(g, s, \lambda_{\mathcal{V}_v}(\varphi_\mathcal{L}))$ if $v \notin \Sigma^{\text{bad}}$.

For each $v_i \in \Sigma^{\text{bad}}$, consider the collection $\{\mathcal{V}_w \mid w \neq v_i\} \cup \mathcal{V}_{v_i}'$; this is a coherent collection, in the sense of Section 2.2, so there exists an E-Hermitian space W_i such that $W_{i,v_i} \simeq \mathcal{V}_{v_i}'$ and $W_{i,w} \simeq \mathcal{V}_w$ at all other places.

A moment's contemplation of the definition of the local section $\Phi_v^*(g, s)$, cf. Definition 4.2, shows that there exists Schwartz functions

$$\varphi_i \in S(W_i), \qquad \varphi_i' \in S(\mathcal{V}) \tag{4.28}$$

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such that

$$\Phi^*(g,s) = \Phi\left(g,s,\lambda_{\mathcal{V}}\left(\varphi_L\right)\right) + s\left[\sum_{v_i \in \Sigma^{\mathrm{bad}}} \Phi(g,s,\lambda_{W_i}(\varphi_i)) + \Phi(g,s,\lambda_{\mathcal{V}}(\varphi_i'))\right] \\ + O(s^2) \tag{4.29}$$

Taking Eisenstein series, we find

$$E(g,s,\Phi^*) = E\left(g,s,\lambda_{\mathcal{V}}\left(\varphi_L\right)\right) + s\left[\sum_{v_i} E(g,s,\lambda_{W_i}(\varphi_i)) + E(g,s,\lambda_{\mathcal{V}}(\varphi_i'))\right] + O(s^2) \tag{4.30}$$

The terms $E(g,s,\lambda_{\mathcal{V}}(\varphi_i'))$ all vanish at $s = 0$, since they are incoherent. Thus, at $s = 0$ we have

$$E'(g,0,\Phi^*) = E'\left(g,0,\lambda_{\mathcal{V}}\left(\varphi_L\right)\right) + \sum_{v_i} E(g,0,\lambda_{W_i}(\varphi_i)). \tag{4.31}$$

In other words $E'(g,0,\Phi^*)$ can be expressed as sum of the central derivative of the incoherent Eisenstein series $E'\left(g,0,\lambda_{\mathcal{V}}\left(\varphi_L\right)\right)$, plus the central *values* of coherent Eisenstein series attached to totally positive definite spaces. $\diamond$

4.2 Green functions and special cycles

We now turn our attention to the construction of a family of elements $\widehat{Z}(m,\tau) \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(\mathcal{M})$.

We begin with a local construction. Let v be a non-archimedean place of F that is non-split in E, and recall that we had fixed a local system $\mathbb{V}^{(v)}$ on $\mathcal{M}_{/E_v^{\mathrm{alg}}}$ of one-dimensional Hermitian spaces over E, as in Section 3.

As part of our data, there is an isometry

$$\beta_z^{(v)} \colon \mathbb{V}^{(v)}|_z \otimes \mathbb{A}_E \xrightarrow{\sim} \mathcal{V}^v \times \mathcal{V}_v', \tag{4.32}$$

at each point $z \in \mathcal{M}(E_v^{\mathrm{alg}})$. Pulling back the Schwartz function $\varphi_{\mathcal{L},\mathbb{A}}$ along this map, we obtain a Schwartz function

$$\varphi_z = \bigotimes_{w \leq \infty} \varphi_{z,w} \in S(\mathbb{V}^{(v)}|_z \otimes \mathbb{A}_E) \tag{4.33}$$

on the adelification of the fibre of $\mathbb{V}^{(v)}$ at z.

For $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ and $x \in \mathbb{V}^{(v)}|_z$, we define a function

$$g(x,z) := \int_{\substack{\lambda \in E_v^\times \\ |\lambda|_{E_v} \geq 1}} \varphi_{z,v}(\lambda x)\, d^\times \lambda \cdot \log N_{E/\mathbb{Q}}(\mathfrak{p}) \tag{4.34}$$

where $d^\times \lambda$ is the multiplicative Haar measure on $E_v^\times$ such that the volume of $\mathcal{O}_{E,v}^\times$ with respect to $d^\times \lambda$ is one, and $\mathfrak{p}$ is the prime ideal of E corresponding to the place v.

There are two main motivations for this construction. The first is that it strongly parallels Kudla's construction in the archimedean place, cf. (4.46) below. The more important point is that $g(x,z)$ naturally encodes the Arakelov degrees of divisors locally on $\mathcal{M}$.

We make the second point more precise. Suppose the point $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ is defined over a finite extension L of E_v, and admits an integral extension $\widetilde{z} \in \mathcal{M}(\mathcal{O}_L)$. Suppose further that there is a line bundle $\widetilde{\omega}$ over $\mathcal{O}_L$ and a map

$$s \colon \mathbb{V}^{(v)}|_z \otimes_{E_v} L \to \widetilde{\omega} \otimes_{\mathcal{O}_L} L; \tag{4.35}$$

we may normalize s to have the property that

$$s(x) \in \pi\,\widetilde{\omega} \iff \varphi_{z,v}(x) = 1 \tag{4.36}$$

for every $x \in \mathbb{V}^{(v)}|_z \otimes_E E_v$, where π is a uniformizer of L.

For any x with $s(x) \in \widetilde{\omega}$, its vanishing locus $Z(s(x))$ defines a divisor on $\operatorname{Spec} \mathcal{O}_L$. Let $I = (f)$ denote the (principal) ideal that cuts out $Z(s(x))$; in other words, I is the largest ideal such that the image of $s(x)$ in $\widetilde{\omega} \otimes_{\mathcal{O}_L} \mathcal{O}_L/I$ vanishes. We define the normalized *Arakelov degree* of the divisor $Z(s(x))$ to be

$$\widehat{\deg}\, Z(s(x)) := \frac{1}{[L : E_v]} \mathrm{length}(\mathcal{O}_L/I)\, \log|\kappa(L)| = \frac{1}{[L : E_v]} \mathrm{ord}_L(f)\, \log|\kappa(L)| \tag{4.37}$$

where $\kappa(L)$ is the residue field of L. Note that the Arakelov degree is unchanged if L is replaced by a larger field.

Proposition 4.8. *With the notation as in the previous paragraph, we have*

$$g(x, z) = \widehat{\deg}\, Z(s(x)) \tag{4.38}$$

for all $x \in \mathbb{V}^{(v)}|_z$ with $s(x) \in \widetilde{\omega}$.

Proof. If π is a uniformizer of L, and $I = (f)$ is the ideal defining $Z(s(x))$, then $\mathrm{ord}_L(f)$ is, by construction, the largest integer such that $\pi^{-\mathrm{ord}_L(f)} s(x) \in \widetilde{\omega}$.

On the other hand, suppose ϖ is a uniformizer of E_v such that $\varpi = \pi^e$, with $e = e(L/E_v)$ the ramification index. Then, using the normalization (4.36), we may write

$$
\begin{aligned}
g(x, z) &= \int_{\substack{\lambda \in E_v^\times \\ |\lambda|_v \geq 1}} \varphi_{z,v}(\lambda x)\, d^\times \lambda \cdot \log N(\mathfrak{p}) \\
&= \sum_{m \geq 0} \int_{\mathcal{O}_{E,v}^\times} \varphi_{z,v}(\varpi^{-m} u x)\, d^\times u \cdot \log N(\mathfrak{p}) \\
&= \max\left(m \mid \varpi^{-m} s_x \in \widetilde{\omega}\right) \cdot \log N(\mathfrak{p}) \\
&= \left(\mathrm{ord}_\pi(f)/e\right) \cdot \log N(\mathfrak{p}) \\
&= \frac{\mathrm{ord}_\pi(f)}{[L : E_v]} \cdot \log|\kappa(L)|
\end{aligned}
\tag{4.39}
$$

where in the last line, we used the identity $[L : E_v] = e(L/E_v) \cdot f(L/E_v)$. This last expression is $\widehat{\deg}\, Z(s(x))$, by definition. $\qquad\square$

In practice, one often has a global line bundle ω, for which the vanishing locus of a section represents a moduli or deformation-theoretic problem; some examples of this phenomenon are presented in the appendix.

We record another crucial property of this construction:

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Lemma 4.9. *Suppose $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ and $x \in \mathbb{V}^{(v)}|_z$. For every $t \in H(\mathbb{A}_F)$, there is an isometry $r(t) \colon \mathbb{V}^{(v)}|_z \to \mathbb{V}^{(v)}|_{t \cdot z}$ such that*

$$\varphi_z(x) = \varphi_{t \cdot z}(t^{-1} r(t) x) \tag{4.40}$$

Proof. This follows immediately from the equivariance property of $\beta^{(v)}$, as in (3.14). $\qquad\square$

Finally, for $\alpha \in F^\times$, we define a function $g_v(\alpha)$ on $\mathcal{M}(E_v^{\mathrm{alg}})$ by the formula

$$
\begin{aligned}
g_v(\alpha)(z) &:= \sum_{\substack{x \in \mathbb{V}^{(v)}|_z \\ Q(x)=\alpha}} \varphi^v(x) \cdot g_v(x, z) \\
&= \log N(\mathfrak{p}) \sum_{\substack{x \in \mathbb{V}^{(v)}|_z \\ Q(x)=\alpha}} \varphi_z^v(x) \int_{\substack{\lambda \in E_v \\ |\lambda|_v \geq 1}} \varphi_{z,v}(\lambda x) \, d^\times \lambda
\end{aligned}
\tag{4.41}
$$

where $\varphi_z^v(x) = \prod_{w \neq v} \varphi_{z,w}(x)$.

For later comparison to the Fourier expansion of a Hilbert modular form, it will be convenient to introduce an additional parameter $\tau \in \mathbb{H}_E$, where

$$\mathbb{H}_E := \{\tau = (\tau_v)_{v|\infty} \in \mathbb{C}^d \mid \mathrm{Im}(\tau_v) > 0 \text{ for all } v\} \tag{4.42}$$

is the Hermitian upper half space attached to E (here $[E : \mathbb{Q}] = 2d$). For $\tau = x + iy \in \mathbb{H}_E$, choose a totally positive element $a \in F \otimes_{\mathbb{Q}} \mathbb{R}$ such that $a^2 = y$, and for an archimedean place v, set

$$g_{\tau,v} = n(x_v) m(a_v) \in G(F_v) \tag{4.43}$$

and let

$$g_{\tau,\infty} = (g_{\tau,v_1}, \ldots, g_{\tau,v_d}) \in G(F_{\mathbb{R}}). \tag{4.44}$$

We then define the Green function

$$g_v(\alpha, \tau)(z) := N(y)^{-\frac{1}{2}} \sum_{\substack{x \in \mathbb{V}^{(v)}|_z \\ Q(x)=\alpha}} \left(\omega_{\mathbb{V}_z^{(v)}}(g_{\tau,\infty}) \varphi_{z,\infty}(x) \right) \varphi_{z,f}^v(x) \, g_v(x, z) \tag{4.45}$$

where $\varphi_{z,\infty} = \prod_{w|\infty} \varphi_{z,w}$ and $N(y) = \prod_{v|\infty} y_v$.

Now suppose v is an archimedean place and let $z \in \mathcal{M}(E_v^{\mathrm{alg}})$. By construction, the space $\mathbb{V}^{(v)}|_z$ is of signature $(0, 1)$ at v. For $x \in \mathbb{V}^{(v)}|_z$, we define

$$g_v(x, z) := \int_1^\infty \varphi_{z,v}(tx) \, e^{2\pi t^2 \sigma_v(Q(x))} \frac{dt}{t} = \int_1^\infty e^{4\pi t^2 \sigma_v(Q(x))} \frac{dt}{t} \tag{4.46}$$

where $Q(x) = \frac{1}{2}\langle x, x \rangle$ and $\sigma_v \colon F \to \mathbb{R}$ is the embedding induced by v. This definition coincides with the Green function introduced by Kudla in [Kud97].

We incorporate the variable $\tau = (\tau_w)_{w|\infty} \in \mathbb{H}_E$ as follows. Let

$$g_v(x, \tau_v)(z) := y_v^{\frac{1}{2}} \, g_v\left(\sqrt{y_v}\, x\right)(z) \, e^{2\pi \sigma_v(Q(x))\tau_v}, \tag{4.47}$$

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and for $\alpha \in F^\times$, define a function $g_v(\alpha, \tau) \colon \mathcal{M}(E_v^{\mathrm{alg}}) \to \mathbb{R}$ by the formula

$$g_v(\alpha, \tau)(z) = N(y)^{-\frac{1}{2}} \sum_{\substack{x \in \mathbb{V}^{(v)}|_z \\ Q(x) = \alpha}} g_v(x, \tau_v)(z) \left(\prod_{\substack{w | \infty \\ w \neq v}} \omega_{\mathbb{V}_w^{(v)}}(g_{\tau, w}) \varphi_{z, w}(x) \right) \varphi_{z, f}(x). \qquad (4.48)$$

At this stage, we have constructed a function $g_v(\alpha, \tau)$ for every non-split place v.

Lemma 4.10. *For a given $\alpha \in F^\times$, there is at most one place v such that $g_v(\alpha, \tau)$ is not identically zero.*

Proof. Suppose $g_v(\alpha, \tau)(z) \neq 0$ for some place v and $z \in \mathcal{M}(E_v^{\mathrm{alg}})$. Then there is a vector in $\mathbb{V}^{(v)}|_z$ of norm m, so $\mathbb{V}^{(v)}|_z$ is isometric to $(E, \alpha x \overline{y})$. But $\mathbb{V}^{(v)}|_z$ and $\mathbb{V}^{(w)}|_{z'}$ are pairwise non-isometric for any $v \neq w$ and geometric points $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ and $z' \in \mathcal{M}(E_w^{\mathrm{alg}})$, hence there can be at most one such place v. $\qquad \square$

Finally, we arrive at the definition of the special cycles:

Definition 4.11. Let $\alpha \in F^\times$, and define the special cycle $\widehat{Z}(\alpha, y)$ in terms of the family of Green functions:

$$\widehat{Z}(\alpha, \tau) = (g_v(\alpha, \tau))_{v \leq \infty} \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(\mathcal{M}_K) \qquad (4.49)$$

Of course, by the preceding lemma, there is at most one non-zero entry in the family.

4.3 Proof of the main theorem

We are now ready to formulate our main theorem.

Let $E(g, s, \Phi^*)$ denote the Eisenstein series attached to the section $\Phi^*(g, s)$ as defined in Section 4.1. It will be convenient to express $E(g, s, \Phi^*)$ in "classical" coordinates, as follows: let $\tau \in \mathbb{H}_E$, and set

$$g_\tau = (g_{\tau, \infty}, 1, \dots) \in G(\mathbb{A}_F) \qquad (4.50)$$

where $g_{\tau, \infty}$ is as in (4.44). We then define

$$E(\tau, s, \Phi^*) := N(y)^{-\frac{1}{2}} E(g_\tau, s, \Phi^*), \qquad (4.51)$$

where

$$N(y) = \prod_{v | \infty} y_v; \qquad (4.52)$$

then $E(\tau, s, \Phi^*)$ is a (non-holomorphic) Hilbert modular form of parallel weight one.

Write the Fourier expansion of $E(\tau, s, \Phi^*)$ as

$$E(\tau, s, \Phi^*) = \sum_{\alpha \in F} E_\alpha(\tau, s, \Phi^*). \qquad (4.53)$$

Theorem 4.12. *For all $\alpha \in F^\times$, we have*

$$\widehat{\deg}\, \widehat{Z}(\alpha, \tau) = -\#\mathcal{M}(\mathbb{C}) \cdot E_\alpha'(\tau, 0, \Phi^*). \qquad (4.54)$$

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Proof. By the product formula (2.34), we have

$$E'_\alpha(\tau, 0, \Phi^*) = N(y)^{-\frac{1}{2}} \cdot \sum_{v \leq \infty} W'_{\alpha,v}(g_{\tau,v}, 0, \Phi^*_v) \prod_{w \neq v} W_{\alpha,w}(g_{\tau,w}, 0, \Phi^*_w). \tag{4.55}$$

For $\alpha \in F^\times$, let

$$\mathrm{Diff}(\alpha) := \{v \leq \infty \mid \alpha \text{ is not represented by } \mathcal{V}_v\}. \tag{4.56}$$

We observe that $\#\mathrm{Diff}(\alpha) \geq 1$. Indeed, if $\mathcal{V}_v$ represents α, then $\mathcal{V}_v \simeq (E_v, \alpha xy')$. If $\mathrm{Diff}(\alpha) = \emptyset$, then this would hold true for all places, and so

$$\prod_{v \leq \infty} \mathrm{inv}_v(\mathcal{V}_v) = 1, \tag{4.57}$$

which contradicts the assumption that $\mathcal{V}$ is an incoherent collection.

If $\#\mathrm{Diff}(\alpha) \geq 2$, then the vanishing criterion in Proposition 2.2 implies that $E'_\alpha(\tau, 0, \Phi^*) = 0$. On the other hand, every space $\mathbb{V}^{(v)}|_z$ differs from $\mathcal{V}$ at exactly one place, so no such space can represent α; it follows that $\widehat{Z}(\alpha, \tau) = 0$ in this case is well.

Thus, it remains to consider the case $\mathrm{Diff}(\alpha) = \{v\}$ is a singleton. Note that v is necessarily non-split, so that $E_v = E \otimes_v F_v$ is a field.

First, suppose v is non-archimedean, and let $\mathfrak{p}$ denote the prime ideal of E inducing the valuation v of E. Then

$$N(y)^{\frac{1}{2}} E'_\alpha(\tau, 0, \Phi^*) = W'_{\alpha,v}(e, 0, \Phi^*_v) \cdot W_{\alpha,\infty}(g_{\tau,\infty}, 0, \Phi^*_\infty) \cdot \prod_{\substack{w < \infty \\ w \neq v}} W_{\alpha,w}(e, 0, \Phi^*_w) \tag{4.58}$$

where

$$W_{\alpha,\infty}(g_{\tau,\infty}, 0, \Phi^*_\infty) = \prod_{w \mid \infty} W_{\alpha,v}(g_{\tau,w}, 0, \Phi^*_w). \tag{4.59}$$

On the other hand,

$$\begin{aligned}
N(y)^{\frac{1}{2}} \widehat{\deg}\, \widehat{Z}(\alpha, \tau) &= N(y)^{\frac{1}{2}} \sum_{z \in \mathcal{M}(E_v^{\mathrm{alg}})} g_v(\alpha, \tau)(z) \\
&= \sum_{z \in \mathcal{M}(E_v^{\mathrm{alg}})} \sum_{\substack{x \in \mathbb{V}^{(v)}|_z \\ Q(x) = \alpha}} \left(\omega_{\mathbb{V}^{(v)}_z}(g_{\tau,\infty}) \varphi_{z,\infty}(x) \right) \varphi^v_{z,f}(x) \\
&\qquad \times \int_{\substack{\lambda \in E_v^\times \\ |\lambda|_v \geq 1}} \varphi_{z,v}(\lambda x)\, d^\times \lambda \, \log N(\mathfrak{p})
\end{aligned} \tag{4.60}$$

with notation as in (4.45). Next, for $\lambda \in E_\mathfrak{p}^\times$, let

$$\underline{\lambda} = (1, \dots, 1, \underbrace{\lambda}_{\mathfrak{p}\text{'th component}}, 1, \dots) \in \mathbb{A}_E^\times \tag{4.61}$$

and write

$$g = g_\tau\, m(\underline{\lambda}) \in G(\mathbb{A}_F) \tag{4.62}$$

so that

$$\widehat{\deg}\,\widehat{Z}(\alpha,\tau) = N(y)^{-\frac{1}{2}}\log N(\mathfrak{p})$$
$$\times \int_{|N(\lambda)|_v \geq 1} |N(\lambda)|_v^{-\frac{1}{2}}\chi(\lambda)^{-1} \sum_{z\in\mathcal{M}(E_v^{\mathrm{alg}})} \sum_{\substack{x\in\mathbb{V}^{(v)}|_z \\ Q(x)=\alpha}} (\omega_{\mathbb{V}_z^{(v)}}(g)\varphi_z)(x)\,d^\times\lambda. \quad (4.63)$$

We compute the double sum in the integrand using the Siegel-Weil formula, as follows.

Choose a subgroup $K_H \subset H(\mathbb{A}_F)$ of finite index in $H(F)\backslash H(\mathbb{A}_F)$ such that $\varphi_{\mathcal{L},\mathbb{A}}$ is K_H-stable, where $\varphi_{\mathcal{L},\mathbb{A}}$ is the Schwartz function (3.7). Assume furthermore that the action of $H(F)\backslash H(\mathbb{A}_F)$ on $\mathcal{M}(E_v^{\mathrm{alg}}) = \mathcal{M}(E^{\mathrm{alg}})$ factors through the class group

$$\mathcal{C} := H(F)\backslash H(\mathbb{A}_F)/K_H. \quad (4.64)$$

Fix a set of representatives

$$z_1,\ldots,z_r \quad (4.65)$$

for the orbits of the action of $\mathcal{C}$ on $\mathcal{M}(E^{\mathrm{alg}})$, and for each i, let c_i denote the order of the stabilizer of z_i with respect to this action.

Then, for any $g \in G(\mathbb{A}_F)$, and $i = 1,\ldots r$, we have

$$\sum_{z\in\mathcal{M}(E_v^{\mathrm{alg}})} \sum_{\substack{x\in\mathbb{V}^{(v)}|_z \\ Q(x)=\alpha}} (\omega_{\mathbb{V}_z^v}(g)\varphi_z)(x) = \sum_{i=1}^r \frac{1}{c_i} \sum_{t\in\mathcal{C}} \sum_{\substack{x\in\mathbb{V}^{(v)}|_{t\cdot z_i} \\ Q(x)=\alpha}} (\omega_{\mathbb{V}_{t\cdot z_i}^v}(g)\varphi_{t\cdot z_i})(x)$$
$$= \sum_{i=1}^r \frac{1}{c_i} \sum_{t\in\mathcal{C}} \sum_{\substack{x\in\mathbb{V}^{(v)}|_{z_i} \\ Q(x)=\alpha}} (\omega_{\mathbb{V}_{z_i}^v}(g)\varphi_{z_i})(t^{-1}x) \quad (4.66)$$
$$= \sum_{i=1}^r \frac{1}{c_i} \sum_{t\in\mathcal{C}} \Theta_\alpha(g,t,\varphi_{z_i})$$

where the second line follows from Lemma 4.9, and $\Theta_\alpha(g,t,\varphi_{z_i})$ is the α'th Fourier coefficient of the theta function defined in (2.38).

Now by the Siegel-Weil formula (Proposition 2.3), the product formula (2.34), and applying the definition of the Schwartz function φ_{z_i}, cf. (4.33), we can write

$$\sum_{t\in\mathcal{C}} \Theta_\alpha(g,t,\varphi_{z_i}) = \frac{1}{\mathrm{vol}(K_H,dh)} \int_{[H]} \Theta_\alpha(g,h,\varphi_{z_i})dh$$
$$= \frac{1}{2\mathrm{vol}(K_H)} E_\alpha(g,0,\lambda_{\mathbb{V}_{z_i}^v}(\varphi_{z_i})) \quad (4.67)$$
$$= \frac{1}{2\mathrm{vol}(K_H)} \cdot \prod_{\substack{w\leq\infty \\ w\neq v}} W_{\alpha,w}(g_w,0,\lambda_{\mathcal{V}_w}(\varphi_{\mathcal{L},w})) \cdot W_{\alpha,v}(g_v,0,\lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})).$$

Note that this is independent of i. Moreover, we observe that

$$\#\mathcal{M}(E^{\mathrm{alg}}) = \#\mathcal{C}\sum_{i=1}^r c_i^{-1} = \left(\frac{1}{\mathrm{vol}(K_H)}\int_{[H]}dh\right)\sum_{i=1}^r c_i^{-1} = \frac{1}{\mathrm{vol}(K_H)}\sum_{i=1}^r c_i^{-1}; \quad (4.68)$$

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thus, setting

$$C := \frac{\#\mathcal{M}(E^{\mathrm{alg}})}{2} \tag{4.69}$$

and taking $g = g_{\tau,\infty} \, m(\underline{\lambda})$, we have

$$\widehat{\deg} \, \widehat{Z}(\alpha, \tau) = C \cdot N(y)^{-\frac{1}{2}} \cdot W_{\alpha,\infty}(g_{\tau,\infty}, 0, \lambda_{\mathcal{V}_\infty}(\varphi_{\mathcal{L},\infty})) \prod_{\substack{w < \infty \\ w \neq v}} W_{\alpha,w}(g_w, 0, \lambda_{\mathcal{V}_w}(\varphi_{\mathcal{L},w}))$$

$$\times \log N(\mathfrak{p}) \int_{\substack{\lambda \in E_v^\times \\ |N(\lambda)|_v \geq 1}} |N(\lambda)|_v^{-\frac{1}{2}} \chi(\lambda)^{-1} W_{\alpha,v}(m(\lambda), 0, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})) \, d^\times \lambda. \tag{4.70}$$

Now applying Lemma 4.3, we may rewrite the second line above as

$$\log N(\mathfrak{p}) \int_{\substack{\lambda \in E_v^\times \\ |\lambda|_v \geq 1}} |N(\lambda)|_v^{-\frac{1}{2}} \chi(\lambda)^{-1} W_{\alpha,v}(m(\lambda), 0, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})) \, d^\times \lambda$$

$$= \log N(\mathfrak{p}) \int_{\substack{\lambda \in E_v^\times \\ |\lambda|_v \geq 1}} W_{N(\lambda)\alpha,v}(e, 0, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})) \, d^\times \lambda. \tag{4.71}$$

Note that since $\varphi_{\mathcal{L},v}$ is the characteristic function of a lattice,

$$W_{u\alpha,v}(e, 0, \Phi_v) = W_{\alpha,v}(e, 0, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})) \tag{4.72}$$

for all $u \in \mathcal{O}_{F,v}^\times$. Fixing a uniformizer ϖ of $E_\mathfrak{p}$ and applying Proposition 4.4(iii), we have

$$(4.71) = \log N(\mathfrak{p}) \sum_{k \geq 0} W_{N(\varpi)^{-k}\alpha,v}\left(e, 0, \lambda_{\mathcal{V}_v'}(\varphi_{\mathcal{L},v})\right)$$

$$= -2 \, W'_{\alpha,v}(e, 0, \Phi_v^*). \tag{4.73}$$

As for the remaining places, Proposition 4.4(i) gives

$$W_{\alpha,w}(e, 0, \lambda_{\mathcal{V}_w}(\varphi_{\mathcal{L},w})) = W_{\alpha,w}(e, 0, \Phi_w^*) \tag{4.74}$$

for non-archimedean places $w \neq v$, and, following definitions, we have

$$W_{\alpha,w}(g_{\tau,w}, 0, \lambda_{\mathcal{V}_w}(\varphi_{\mathcal{L},w})) = W_{\alpha,w}(g_{\tau,w}, 0, \Phi_w^*) \tag{4.75}$$

for $w | \infty$.

Substituting these expressions into (4.70) and comparing with (4.58) gives the result in this case.

The case that $\mathrm{Diff}(m) = \{v\}$ for an archimedean place v is entirely analogous, and is left to the reader. $\qquad\square$

Remark 4.13. The theorem above readily generalizes to the situation where the compact open subgroup $K \subset T(\mathbb{A}_{\mathbb{Q},f})$ is not neat. Indeed, fix a neat subgroup $K_0 \subset K$, let $\mathcal{M}_0$ denote the corresponding zero-dimensional Shimura variety, and let $\mathcal{M} = [\Gamma \backslash \mathcal{M}_0]$ denote the stack quotient, where $\Gamma = K/K_0$. Let $j \colon \mathcal{M}_0 \to \mathcal{M}$ denote the natural map.

Suppose $\widehat{\omega}$ is an adelic line bundle on $\mathcal{M}$, given by a family of Green functions $(g_v)_v$ as in Section 2.1. If we define its arithmetic degree on $\mathcal{M}$ to be

$$\widehat{\deg}\,\widehat{\omega} = \sum_{v \leq \infty} \sum_{z \in \mathcal{M}(E_v^{\mathrm{alg}})} \frac{1}{\#\operatorname{Aut}(z)} g_v(z), \tag{4.76}$$

then one may verify that

$$\frac{\widehat{\deg}\,j^*\widehat{\omega}}{\#\mathcal{M}_0(\mathbb{C})} = \frac{\widehat{\deg}\,\widehat{\omega}}{\#\mathcal{M}(\mathbb{C})}. \tag{4.77}$$

where the left hand side is the "stacky" cardinality. This in turn implies that the main theorem continues to hold in this case.

Alternatively, one can take the definition of the arithmetic degree above, and make the minor changes required in the proof of Theorem 4.12 directly. ◇

A Examples

A.1 The weight one Eisenstein series of Kudla-Rapoport-Yang

Suppose that $E \subset \mathbb{C}$ is an imaginary quadratic field, and let $\mathcal{O}_E \subset E$ denote the ring of integers. Let $\mathcal{M}$ denote the moduli stack over $\operatorname{Spec}(\mathcal{O}_E)$ whose S-points, for a connected $\mathcal{O}_E$-scheme S, parametrizes pairs (A, i) where A is an elliptic curve over S, and $i \colon \mathcal{O}_E \to \operatorname{End}(A)$ is an action of $\mathcal{O}_E$ such that the induced action on $\operatorname{Lie}(A)$ coincides with the structural morphism $\mathcal{O}_E \to \mathcal{O}_S$. Then $\mathcal{M}$ is a zero-dimensional Shimura variety with

$$\mathcal{M}(\mathbb{C}) \simeq \left[E^\times \backslash \mathbb{A}_{E,f}^\times / \widehat{\mathcal{O}_E}^\times \right]. \tag{A.1}$$

In [KRY99], the authors assume that $E = \mathbb{Q}(\sqrt{-D})$ for a prime D, construct a family of special cycles on $\mathcal{M}$ via a moduli problem, and prove that their arithmetic degrees coincide with the Fourier coefficients of an incoherent weight one Eisenstein series. In this section, we explain how their results can be viewed in the context of the present work.

To begin, we fix our incoherent collection $\mathcal{V} = (\mathcal{V}_v)_v$ as follows. If v corresponds to a rational prime p, we set $\mathcal{V}_v = (E \otimes_{\mathbb{Q}} \mathbb{Q}_p, -2x\overline{y})$. At the infinite place, we set $\mathcal{V}_\infty = (\mathbb{C}, 2x\overline{y})$; here we view $\mathbb{C} \simeq E \otimes_{\mathbb{Q}} \mathbb{R}$ via the fixed embedding $E \to \mathbb{C}$. As for the lattice $\mathcal{L}$, we set $\mathcal{L}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for each prime p.

Next, let p denote a rational prime that is non-split in E, and take a geometric point $z \in \mathcal{M}(E_p^{\mathrm{alg}})$ corresponding to an elliptic curve A_z with its $\mathcal{O}_E$-action. Let $A_{\overline{z}} \in \mathcal{M}(\mathbb{F}_p^{\mathrm{alg}})$ denote the reduction mod p, which is equipped with an induced $\mathcal{O}_E$-action $\iota \colon \mathcal{O}_E \to \operatorname{End}(A_{\overline{z}})$, and set

$$\mathbb{V}^{(p)}|_z := \{ x \in \operatorname{End}^0(A_{\overline{z}}) \mid \iota(a) \circ x = x \circ \iota(\bar{a}) \text{ for all } a \in \mathcal{O} \}; \tag{A.2}$$

this is the space of "special quasi-endomorphisms" of $A_{\overline{z}}$ as in [KRY99].

Since p does not split in E, the elliptic curve $A_{\overline{z}}$ is necessarily supersingular, and so $\operatorname{End}^0(A_{\overline{z}})$ is a rational quaternion algebra that is non-split precisely at ∞ and p. Since $\operatorname{End}^0(A_{\overline{z}})$ contains an embedded copy of E, there is a decomposition

$$\operatorname{End}^0(A_{\overline{z}}) = \iota(E) \oplus \iota(E)\theta \tag{A.3}$$

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where θ is an element satisfying $\theta\iota(a) = \iota(\bar{a})\theta$ for all $a \in \mathcal{O}_E$. In particular, $\mathbb{V}^{(p)}|_z = \iota(E)\theta$ is a one-dimensional vector space over E. We equip $\mathbb{V}^{(p)}|_z$ with the E-Hermitian form $\langle\cdot,\cdot\rangle_z$ such that $\frac{1}{2}\langle x,x\rangle_z = \mathrm{Nrd}(x) = -x^2$; here $\mathrm{Nrd}(x)$ is the reduced norm.

We need to show that as z varies, these spaces form an $H(\mathbb{A}_\mathbb{Q})$-equivariant local system, where $H(\mathbb{A}_\mathbb{Q}) = \{t \in \mathbb{A}_E^\times \mid N(t) = 1\}$. This is best seen via Serre's tensor construction as follows. For $t \in H(\mathbb{A}_\mathbb{Q})$, let $\mathfrak{a} = t\widehat{\mathcal{O}}_E \cap E$ denote the corresponding fractional ideal. Then we may identify $A_{t\cdot z} \simeq A_z \otimes_{\mathcal{O}_E} \mathfrak{a}$, and we obtain a natural E-linear quasi-isogeny

$$\varphi_t \colon A_z \to A_z \otimes_{\mathcal{O}_E} \mathfrak{a} \simeq A_{t\cdot z}, \qquad a \mapsto a \otimes 1. \tag{A.4}$$

This induces the desired isometry

$$r(t) \colon \mathbb{V}^{(p)}|_{t\cdot z} \to \mathbb{V}^{(p)}|_z, \qquad x \mapsto \varphi_t \circ x \circ \varphi_t^{-1} \tag{A.5}$$

of E-Hermitian spaces, where, abusing notation, we use the same symbol φ_t to denote the induced quasi-isogeny between the special fibres of A_z and $A_{t\cdot z}$.

Now let $z \in \mathcal{M}(E_p^{\mathrm{alg}})$ as above, and let

$$\mathbb{L}^{(p)}|_z := \mathbb{V}^{(p)}|_z \cap \mathrm{End}(A_{\bar{z}}). \tag{A.6}$$

At a prime $q \neq p$, we may identify the Tate module $\mathrm{Ta}_q(A_{\bar{z}}) \simeq \mathcal{O}_{E,q}$, and so we obtain isomorphisms

$$\mathrm{End}(A_{\bar{z}}) \otimes_\mathbb{Z} \mathbb{Z}_q \simeq \mathrm{End}(\mathrm{Ta}_q(A_{\bar{z}})) \simeq \mathrm{End}(\mathcal{O}_{E,q}). \tag{A.7}$$

The space of $\mathcal{O}_E$-antilinear endomorphisms of $\mathcal{O}_{E,q}$ is generated, as an $\mathcal{O}_{E,q}$-module, by the endomorphism $a \mapsto \bar{a}$; pulling this back through the above identifications, we obtain a generator $f \in \mathbb{L}^{(p)}|_z \otimes \mathbb{Z}_q$ with $f^2 = -1$. This choice of basis vector induces an isometry $\mathbb{L}^{(p)}|_z \otimes \mathbb{Z}_q \simeq \mathcal{L}_q$, or put differently, an isometry

$$\beta_{z,q}^{(p)} \colon \mathbb{V}^{(p)}|_z \otimes_\mathbb{Q} \mathbb{Q}_q \xrightarrow{\sim} \mathcal{V}_q \tag{A.8}$$

identifying $\mathbb{L}_z^{(p)} \otimes \mathbb{Z}_q$ with $\mathcal{L}_q$.

At the prime p, note that we may choose the element θ such that its image in $\mathbb{L}^{(p)}|_z \otimes \mathbb{Z}_p$ is a generator, so that

$$\mathbb{L}^{(p)}|_z \otimes \mathbb{Z}_p \simeq (\mathcal{O}_{E,p}, 2\mathrm{Nrd}(\theta)\, x\bar{y}) \tag{A.9}$$

On the other hand, because $\mathrm{End}^0(A_{\bar{z}}) \otimes \mathbb{Q}_p$ is a division algebra, $\mathbb{V}^{(p)}|_{z,p}$ cannot represent -2, so $2\mathrm{Nrd}(\theta) \equiv -2\varsigma_p \in \mathbb{Q}_p^\times/N(E_p^\times)$, where ς_p is the element chosen in Section 3.2. Moreover, $\mathrm{End}(A_{\bar{z}}) \otimes \mathbb{Z}_p$ is the unique maximal order, consisting precisely of those endomorphisms whose reduced norms have non-negative p-adic valuation. Hence $\mathrm{ord}_p(\mathrm{Nrd}(\theta)) = 0$ or 1, depending on whether p is ramified or inert, respectively, and we find $\mathrm{Nrd}(\theta) = -N(u)\varsigma_p$ for some $u \in \mathcal{O}_{E,p}^\times$. To summarize, we have an isometry

$$\beta_{z,p}^{(p)} \colon \mathbb{V}^{(p)}|_z \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathcal{V}_p' \tag{A.10}$$

identifying $\mathbb{L}^{(p)}|_z \otimes_\mathbb{Z} \mathbb{Z}_p$ with $\mathcal{L}_p$. Finally, at the archimedean place, we have that both $\mathbb{V}_{z,\infty}^{(p)}$ and $\mathcal{V}_\infty$ are positive definite one dimensional Hermitian spaces, and so we may fix an isometry $\beta_{z,\infty}^{(p)}$ between them.

Putting these local factors together, we obtain an isometry

$$\beta^{(p)}|_z \colon \mathbb{V}^{(p)} \otimes_E \mathbb{A}_E \xrightarrow{\sim} \mathcal{V}^p \times \mathcal{V}_p' \tag{A.11}$$

which at the non-archimedean places identifies $\mathbb{L}^{(p)}|_z \otimes \widehat{\mathcal{O}}_E$ with $\widehat{\mathcal{L}}$; note that this latter condition determines the finite part of $\beta_z^{(p)}$ up to multiplication by an element of $\widehat{\mathcal{O}}_E^\times$ of norm one. One can then check that for $t \in H(\mathbb{A}_\mathbb{Q})$, the corresponding maps $\beta_z^{(p)}$ and $\beta_{t \cdot z}^{(p)}$ satisfy the required equivariance property (3.14).

Now consider the archimedean place $v = \infty$. For a point $z \in \mathcal{M}(E_v^{\mathrm{alg}}) = \mathcal{M}(\mathbb{C})$, the homology group $H_1(A_z(\mathbb{C}), \mathbb{Z})$ has an induced $\mathcal{O}_E$ action, and we define

$$\mathbb{L}^{(\infty)}|_z = \{x \in \mathrm{End}\,(H_1(A_z(\mathbb{C}), \mathbb{Z})) \mid x \circ \iota(a) = \iota(\overline{a}) \circ x\} \tag{A.12}$$

and

$$\mathbb{V}^{(\infty)}|_z = \mathbb{L}^{(\infty)}|_z \otimes_\mathbb{Z} \mathbb{Q}, \tag{A.13}$$

equipped with the E-hermitian form determined by $\frac{1}{2}\langle x, x\rangle_z = -x^2$. A similar argument to the non-archimedean case reveals that there is an isometry

$$\beta^{(\infty)} \colon \mathbb{V}^{(\infty)}|_z \otimes_E \mathbb{A}_E \xrightarrow{\sim} \mathcal{V}^\infty \times \mathcal{V}'_\infty \tag{A.14}$$

satisfying the equivariance (3.14).

At this point, we have specified the data $(\mathcal{V}, \widehat{\mathcal{L}}, (\mathbb{V}^{(v)}, \beta^{(v)}))$ as set out in Section 3. Carrying out the construction in Section 4.2, we obtain a family of Green functions $g_v(\alpha, \tau) \colon \mathcal{M}(E_v^{\mathrm{alg}}) \to \mathbb{R}$, and consequently a family of special cycles $\widehat{Z}(\alpha, \tau) \in \widehat{\mathrm{Pic}}_\mathbb{A}(\mathcal{M})$ for $\alpha \in \mathbb{Q}^\times$ and $\tau \in \mathbb{H}$.

On the other hand, in [KRY99], the authors define a family of special cycles $\mathscr{Z}(\alpha)$ via the following moduli problem. Suppose $\alpha > 0$; for a scheme S over $\mathrm{Spec}(\mathcal{O}_E)$, the S-points $\mathscr{Z}(\alpha)(S)$ parametrize tuples (A, i, x) where (A, i) is a point of $\mathcal{M}(S)$ and $x \in \mathrm{End}_S(A)$ is an endomorphism satisfying $x \circ i(a) = i(\overline{a}) \circ x$ for all $a \in \mathcal{O}_E$ and $-x^2 = \alpha$.

These cycles are related to the cycles $\widehat{Z}(\alpha, \tau)$ as follows. Fix $\alpha \in \mathbb{Z}$ with $\alpha > 0$, and suppose $\mathrm{Diff}(\alpha) = \{v\}$. Suppose $\overline{y} \in \mathscr{Z}(\alpha)(\kappa(v)^{\mathrm{alg}})$ is a geometric point corresponding to a triple (A, i, x), lying over the point $\overline{z} = (A, i) \in \mathcal{M}(\kappa(v)^{\mathrm{alg}})$; in particular, we view x as an element of $\mathbb{L}^{(v)}|_z$, where $z \in \mathcal{M}(E_v^{\mathrm{alg}})$ lifts $\overline{z}$. Then Gross' computation [Gro86] for the length of the deformation ring of x gives

$$\begin{aligned}
\mathrm{length}\,\mathcal{O}_{\mathscr{Z}(\alpha), \overline{y}} &= \max\left\{k \mid \varpi^{-k}x \in \mathbb{L}_z^{(v)} \otimes_{O_E} O_{E,v}\right\} + 1 \\
&= \int_{\substack{\lambda \in E_v^\times \\ |N(\lambda)|_v \geq 1}} \varphi_{z,v}(\lambda x)\, d^\times x
\end{aligned} \tag{A.15}$$

where ϖ is a uniformizer of E_v. It follows that

$$\log \#\kappa(v) \cdot \mathrm{length}\mathcal{O}_{\mathscr{Z}(\alpha), z} = g_v(\alpha, z)$$

and hence

$$\widehat{\deg}\, \mathscr{Z}(\alpha)\, q^\alpha \;=\; \widehat{\deg}Z(\alpha, \tau). \tag{A.16}$$

When $\alpha < 0$, a direct computation shows that the above identity continues to hold, where $\mathscr{Z}(\alpha) = \mathscr{Z}(\alpha, \tau)$ is defined in [KRY99, §6].

Thus, our main theorem Theorem 4.12 specializes to the following identity, which is due to Kudla, Rapoport and Yang in the case that the discriminant of E is prime, and generalized to arbitrary discriminant by Kudla and Yang.

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Theorem A.1 ([KRY99, KY13]). *For any $\alpha \neq 0$, we have*

$$\widehat{\deg}\,\mathscr{Z}(\alpha)\,q^\alpha = -\#\mathcal{M}(\mathbb{C}) \cdot E'_\alpha(\tau, 0, \Phi^*).$$

$\square$

It should be noted that the proofs of this theorem appearing in [KRY99] and [KY13] amounted to an explicit computation of both sides. The explicit computation for the left hand side reduces to the length computation (A.15), which also formed the basis for the identity (A.16). In particular, the methods we have developed in this paper do not offer a shortcut around this computation; however, and this is the point to emphasize, our framework offers a conceptual reason for the appearance of the Eisenstein series on the right hand side.

A.2   The Bruinier-Kudla-Yang formula

In this section, we revisit some aspects of a paper by Andreatta, Goren, Howard, and Madapusi Pera [AGHMP18] in which they prove an averaged version of Colmez' conjecture; at the heart of their proof is a formula, originally conjectured by Bruiner, Kudla, and Yang [BKY12], for the arithmetic intersection number of a special divisor and a "big CM cycle" on an orthogonal Shimura variety. Our aim is to highlight the relevant parts of their setup in the context of our discussion.

Suppose that E is a CM field with $[E : \mathbb{Q}] = 2d$ and let $F \subset E$ the maximal totally real subfield. We define a zero-dimensional Shimura variety as follows; note that our notation here will differ slightly from previous sections, in order to align more closely with *loc. cit.* Consider the tori

$$T_E := \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m), \qquad T_F := \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m), \tag{A.17}$$

and let

$$T := T_E / \ker(N_{F/\mathbb{Q}} \colon T_F \to \mathbb{G}_m). \tag{A.18}$$

We have $T_E(\mathbb{C}) \simeq \prod_{\sigma \colon E \to \mathbb{C}}(\mathbb{C}_\sigma)^\times$, where $\mathbb{C}_\sigma := E \otimes_{E,\sigma} \mathbb{C}$. Choose an embedding $\sigma_1 \colon E \to \mathbb{C}$, and let $h_E \colon \mathbb{S} \to T_{E,\mathbb{C}}$ denote the homomorphism given on $\mathbb{C}$-points by the formula $h_E(z_1, z_2) = (z_1, z_2, 1, \ldots, 1)$ where $z_1 \in \mathbb{C}_{\sigma_1}$ and $z_2 \in \mathbb{C}_{\overline{\sigma}_1}$, and all other components are one; we then define $h \colon \mathbb{S} \to T_\mathbb{R}$ to be the composition of h_E with the projection to $T_\mathbb{R}$. The reflex field of the pair (T, h) is the field $\sigma_1(E) \subset \mathbb{C}$.

Given a compact open subgroup $K \subset T(\mathbb{A}_f)$, the Shimura datum (T, h) described above gives rise to a zero-dimensional Shimura variety Y_K defined over E, with

$$Y_K(\mathbb{C}_{\sigma_1}) = T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K. \tag{A.19}$$

Our next step is to specify the requisite data, as in Section 3. Let $\tau_1, \ldots, \tau_d \colon F \to \mathbb{R}$ denote the set of real embeddings. Fix an element $\xi \in F^\times$ such that $\tau_1(\xi) < 0$ and $\tau_i(\xi) > 0$ for $i = 2, \ldots, d$.

We define our incoherent family $\mathcal{V} = (\mathcal{V}_v)_v$ as follows: if v corresponds to the embedding τ_1, we set $\mathcal{V}_v = (E_v, -\xi x \overline{y})$, while at every other place v, we set $\mathcal{V}_v = (E_v, \xi x \overline{y})$. In addition, we fix an $\widehat{\mathcal{O}}_E$-stable, K-invariant lattice $\widehat{\mathcal{L}} \subset \prod_{v < \infty} \mathcal{V}_v$.

Next, we need to construct the family of local systems $\{(\mathbb{V}^{(v)}, \beta^{(v)})\}$, which will require a little bit of preparation. Let (V, Q) denote the rational quadratic space of signature $(2d - 2, 2)$ whose underlying space is $V = E$ with quadratic form $Q(x) = tr_{F/\mathbb{Q}}(\xi x \overline{x})$. As explained in [AGHMP18, §5], this data gives rise to an embedding

$$j_{/\mathbb{Q}} \colon Y_K \to \mathcal{M} \tag{A.20}$$

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where $\mathcal{M}$ is (the base change to $\mathrm{Spec}(\mathcal{O}_E)$ of) an integral model of the GSpin Shimura variety associated to V; note that this model depends on the choice of a maximal lattice $\widehat{L}^{\max} \subset V$ containing $\mathscr{L}$, which we suppress from the notation. This extends to a morphism

$$j \colon \mathcal{Y}_K \to \mathcal{M} \tag{A.21}$$

where $\mathcal{Y}_K$ is the normalization of Y_K in $\mathcal{M}$.

The variety $\mathcal{M}$ comes with an abelian scheme $\mathcal{A}^{KS} \to \mathcal{M}$, called the Kuga-Satake abelian scheme, which is of relative dimension 2^{d-1}. Pulling back along j, we obtain an abelian scheme

$$\mathcal{A} := j^* \mathcal{A}^{KS} \tag{A.22}$$

over $\mathcal{Y}_K$. For a connected $\mathcal{Y}_K$-scheme S, let

$$L(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \tag{A.23}$$

denote the space of *special endomorphisms*, as defined in [AGHMP18, §4.3, §4.5] The precise definition is quite technical, so we content ourselves here with a summary of the relevant properties, beginning with the fact that $L(\mathcal{A}_S)$ admits the structure of an $\mathcal{O}_E$-module, together with a positive definite Hermitian form.

It will be convenient to work with "good" places, as in [AGHMP18, Definition 5.3.3]. Let v denote a non-split, non-archimedean place of F. We say that v is good if the level structure K is maximal at v and either *(a)* the extension E_v/F_v is unramified, the lattice $\mathcal{L}_v$ is self-dual, and $L_v^{\max} = \mathcal{L}_v \subset E_v$; or *(b)* the extension E_v/F_v is ramified and $\mathcal{L}_v \subset L_v^{\max} \subsetneq \mathfrak{d}_{E_v/F_v}^{-1} \mathcal{L}_v$. Note all but finitely many places are good.

Now suppose v is a good (non-archimedean) place, and let $y \in \mathcal{Y}_K(E_v^{\mathrm{alg}})$ denote a geometric point; by construction, there is unique lift $\tilde{y} \in \mathcal{Y}_K(\mathcal{O}_v^{\mathrm{alg}})$, and a reduction $\overline{y} \in \mathcal{Y}_K(\kappa(v)^{\mathrm{alg}})$, where $\kappa(v)^{\mathrm{alg}}$ is an algebraic closure of residue field κ_v of E_v. We define the local system $\mathbb{V}^{(v)}$ by setting

$$\mathbb{V}^{(v)}|_y := L(\mathcal{A}_{\overline{y}}) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{A.24}$$

By [AGHMP18, Corollary 5.4.6, Proposition 7.6.2], this space is one-dimensional over E, and there is an isometry

$$\beta_y^{(v)} \colon \mathbb{V}^{(v)}|_y \otimes_E \mathbb{A}_E \xrightarrow{\sim} \mathcal{V}^v \times \mathcal{V}_v' \tag{A.25}$$

that identifies $L(\mathcal{A}_{\overline{y}})_w$ with $L_w^{\max}$ at all non-archimedean places w. In addition, [AGHMP18, Lemma 7.6.3] implies that the map $\beta^{(v)} \colon \mathbb{V}^{(v)} \otimes \mathbb{A}_E \to \underline{\mathcal{V}^v \times \mathcal{V}_v'}$ of local systems is $H(\mathbb{A}_F)$-equivariant in the sense of eq. (3.14).

Applying the construction in Section 4.2, we obtain a Green function $g_v(\alpha, \tau)$ for each good non-archimedean place v, and consequently a special cycle $\widehat{Z}(\alpha, \tau) \in \widehat{\mathrm{Pic}}_{\mathbb{A}}(Y_K)$ for $\alpha \in F^\times$ for any α with $\mathrm{Diff}(\alpha) = \{v\}$ for some good place v, cf. eq. (4.56).

As with the previous example, the cycles in [AGHMP18] are defined in terms of a moduli problem: for $\alpha \in F^\times$, consider the moduli problem $\mathscr{Z}(\alpha)$ over $\mathcal{Y}_K$ whose points, for a connected $\mathcal{Y}_K$-scheme S, are given by

$$\mathscr{Z}(\alpha)(S) = \left\{ x \in L(\mathcal{A}_S) \mid Q(x) = \frac{1}{2} \langle x, x \rangle = \alpha \right\}. \tag{A.26}$$

One has that $\mathscr{Z}(\alpha) = 0$ if α is not totally positive or $\#\mathrm{Diff}(\alpha) \geq 2$, and that when $\mathrm{Diff}(\alpha) = \{v\}$, the natural map $\mathscr{Z}(\alpha) \to \mathcal{Y}_K$ defines a zero-cycle that is contained in the special fibre of $\mathcal{Y}_K$ at v, see [AGHMP18, §7.6] for details.

Suppose $\alpha \gg 0$ with $\text{Diff}(\alpha) = \{v\}$ for a good place v, and for simplicity, assume that the residue characteristic of v is not 2. Fix a point $\overline{y} \in \mathscr{Z}(\alpha)(\kappa(v)^{\text{alg}})$, let $\overline{z} \in \mathcal{Y}_K(\kappa(v)^{\text{alg}})$ denote the point above p. Then $\overline{y}$ corresponds to a special endomorphism $x \in \mathbb{L}^{(v)}|_z$. A central result of [AGHMP18] is the determination of the deformation ring $\mathcal{O}_{\mathscr{Z}(\alpha),z}$; fixing a uniformizer ϖ_v of E_v, this computation can be expressed (in the special case at hand) as the identity

$$\text{length}\,\mathcal{O}_{\mathscr{Z}(\alpha),z} = \max\{k \mid \varpi_v^{-k}x \in \mathbb{L}^{(v)}|_z\} + 1 = \int_{\substack{\lambda \in E_v^\times \\ |N(\lambda)|_v \geq 1}} \varphi_{z,v}(\lambda x)\, d^\times \lambda, \qquad (A.27)$$

cf. [AGHMP18, Proposition 7.7.3]. The above identity continues to hold at dyadic places if we replace $\mathbb{L}^{(v)}$ with $\mathfrak{a} \cdot \mathbb{L}^{(v)}$, where $\mathfrak{a}$ is a fractional ideal of F such that for every place $\mathfrak{p}$ of F, we have $\text{ord}_{\mathfrak{p}}(\mathfrak{a}) = 1 - \text{ord}_{\mathfrak{p}}\,\partial_{E/F}$

On the automorphic side, we need to modify the construction of our Eisenstein series slightly to account for the bad places. Define a section $\widetilde{\Phi}(g,s) = \otimes_{v \leq \infty}\widetilde{\Phi}_v(g,s) \in I(s,\chi)$ as follows:

- If v is archimedean, we set $\widetilde{\Phi}_v(g,s) = \Phi_v^*(g,s)$ as in Section 4.1.

- If v is a non-archimedean place lying over the rational prime p, let $\varphi_{L^{\max},v}$ denote the characteristic function of $L_p^{\max} \cap \mathcal{V}_v$, and let $\widetilde{\Phi}_v(g,s) = \Phi(g,s,\lambda_{\mathcal{V}}(\varphi_{L^{\max},v}))$ denote the corresponding section.

Note that for good places v, we have that $\widetilde{\Phi}_v(g,s) = \Phi_v^*(g,s)$, where $\Phi_v^*(g,s)$ is defined as in Section 4.1.

Then, after a mild modification of the proof of Theorem 4.12 to account for the bad places, we have the following result, which appears as a special case of Theorem 7.8.1 in [AGHMP18], and is the main ingredient in the proof of the Bruinier-Kudla-Yang formula, as in [BKY12].

Theorem A.2 ([AGHMP18]). *Let $\alpha \in F^\times$, and suppose that $\text{Diff}(\alpha) = \{v\}$ for a good non-archimedean place v. Then*

$$\widehat{\deg}\,\mathscr{Z}(\alpha)q^\alpha = -\deg_{\mathbb{C}}(Y_K)\,E_\alpha'(\tau,0,\widetilde{\Phi}). \qquad (A.28)$$

$\square$

Remark A.3. As with the previous example, the identity (A.28) is derived as a consequence of the explicit formula (A.27), and our methods do not offer an alternative method to arrive at this identity in general.

Having said this, in cases of low ramification, it is possible to interpret the identity (A.28) in the spirit of Proposition 4.8. To illustrate this, suppose that $\text{Diff}(\alpha) = \{v\}$ where v is a good non-archimedean place for which the total ramification $e = e(E_v/\mathbb{Q}_p)$ satisfies $e \leq p - 1$. We further assume that v is not dyadic for simplicity.

We first observe that the space V (viewed as a representation T) together with the choice of lattice $L^{\max}$, gives rise to a filtered vector bundle $(V_{dR}, \text{Fil}^\bullet(V_{dR}))$ over $\mathcal{Y}_K$, equipped with an E-action and an E-Hermitian pairing. Sections of V_{dR} can be interpreted as "special" endomorphisms of the first de Rham homology group $H_1^{dR}(\mathcal{A})$ of the universal abelian scheme $\mathcal{A} \to \mathcal{Y}_K$. Moreover

$$\omega := \text{Fil}^1(V_{dR})^\vee \qquad (A.29)$$

is a line bundle; for details, see [AGHMP18, Proposition 3.8.1].

Now suppose that $\mathfrak{q}$ is a prime of E corresponding to the place v with uniformizer ϖ, and $\overline{y} \in \mathcal{Y}_K(\mathbb{F}_p^{\mathrm{alg}})$ is a geometric point lying above $\mathfrak{q}$. Let $\mathcal{O}_{\overline{y}}$ denote the completed local ring at $\overline{y}$ which is an unramified extension of $\mathcal{O}_{E,\mathfrak{q}}$. We denote this extension by $\iota \colon \mathcal{O}_{E,\mathfrak{q}} \to \mathcal{O}_{\overline{y}}$, so that in particular $\varpi_y = \iota(\varpi)$ is a uniformizer for y. Now for any $n \geq 1$, we let $\mathcal{O}_n := \mathcal{O}_{\overline{y}}/(\varpi_{\overline{y}})$. By our assumption $e < p - 1$ on the total ramification of $E_{\mathfrak{q}}$, it follows that the surjection

$$\mathcal{O}_n \to \mathcal{O}_1 = \kappa(\overline{y})$$

is a PD extension.

Suppose that $x \in L(\mathcal{A}_{\overline{y}}) \otimes \mathbb{Q}$ is a special quasi-endomorphism. By Grothendieck-Messing theory, x induces a special quasi-endomorphism of $H_1^{dR}(\mathcal{A}_n)$; by abuse of notation, we denote this element as $x \in V_{dR,\mathcal{O}_n}$. Finally, we define an element $s_n(x) \in \omega_{\mathcal{O}_n} = (\mathrm{Fil}^1 V_{dR,\mathcal{O}_n})^\vee \otimes \mathbb{Q}$ by pairing elements of $\mathrm{Fil}^1 V_{dR,\mathcal{O}_n}$ against x. Taking the limit in n, we get the desired element $s(x) \in \omega_{\mathcal{O}_{\overline{y}}} \otimes \mathbb{Q}$.

By [AGHMP18, Proposition 4.3.2], for a non-zero $x \in L(\mathcal{A}_{\overline{y}})$, the vanishing locus $Z(s(x))$ corresponds precisely to the locus on which x deforms. Moreover, it can be verified (using, for example, the coordinates described in [AGHMP18, §2]) that the map $x \mapsto s(x)$ is $\mathcal{O}_E$ linear, where we view $\omega_{\mathcal{O}_y}$ as an $\mathcal{O}_E$-module by restricting the structure map $\iota \colon \mathcal{O}_{E,\mathfrak{q}} \to \mathcal{O}_{\overline{y}}$ to $\mathcal{O}_E$.

Let $\mathcal{L}'_y := s^{-1}\left(\varpi \cdot \omega_{\mathcal{O}_{\overline{y}}}\right) \subset L(A_{\overline{y}}) \otimes \mathbb{Q}$, and let $\varphi'_{y,v}$ denote the characteristic function of $\mathcal{L}'_y$. Then Proposition 4.8 implies that

$$\widehat{\deg}\, Z(s(x)) = g'(x,y)$$

where

$$g'(x,z) = \int_{\substack{\lambda \in E_v^\times \\ |N(\lambda)|_v \geq 1}} \varphi'_{y,v}(\lambda x)\, d^\times \lambda \, \log N(\mathfrak{q}).$$

is the Green function attached to the characteristic function of $\mathcal{L}'_y$.

From here, one can apply the methods in the present paper, and conclude that under the above hypotheses, there is an identity

$$\widehat{\deg}\,\mathscr{Z}(\alpha)\, q^\alpha = -\deg_{\mathbb{C}}(Y_K) E'_\alpha(\tau, 0, (\Phi')^*)$$

where the local section $(\Phi')^*$ is determined at v by the lattice $\mathcal{L}'_v$. In other words, we may conclude at this stage that the desired formula holds for *some* Eisenstein series. With substantial additional effort, one can show $\mathcal{L}'_y = L(A_{\overline{y}})$ as in [AGHMP18, §2], which shows that the Eisenstein series is the expected one. $\diamond$

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