Mathematical and Computational Aspects of Infinite Matrices and Their Applications

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Abstract

Infinite matrices, the forerunner of many branches of classical mathematics (infinite quadratic forms, integral equations, summability etc.) and the modern operator theory, is revisited to demonstrate its deep influence on the development of many topics in mathematics and its applications. This review attempts to discuss its relationship with the theory of finite matrices such as its sections etc. The talk focusses on theoretical and computational aspects, complete with easily computable error bounds concerning linear infinite algebraic and differential systems as well as conformal mapping problems in doubly connected regions and solutions of differential equations defined on semi-infinite intervals.

CONTENTS

1. Introduction

Earlier researchers include:

- **Paul Appel** *(1884)*
- **G.W.Hill** *(1887)*
  Both used infinite matrices and infinite determinants without logical justification
- **Henri Poincare** *(1884)*
- **Helge von Koch** *(1893)* proved all the routine theorems on infinite matrices
- **David Hilbert** *(1906)* used infinite quadratic forms which are equivalent to infinite matrices to solve the integral equation

$$f(s) = \phi(s) + \lambda \int_a^b K(s,t)\,\phi(t)dt$$

- **Erhard Schmidt**
- **Ernst Hellinger**
- **Otto Toeplitz**
  Many theorems couched in terms of infinite matrices fundamental to the theory of most abstract operators were proved.
- **John von Neumann** *(1929)* discarded the infinite matrix approach and used an abstract approach.

Finite matrices correspond to the natural linear operators on finite dimensional spaces and this naturally lead to infinite matrices as an extension of infinite matrices and this became linear operators defined on sequence spaces.

Analysis is a main source for infinite matrices:

$$\frac{du}{dt} + uf(z) = 0$$

with $f(z) = \sum_{-\infty}^{\infty} f_n z^n$ (a Laurent series).

If $u = \sum_{-\infty}^{\infty} a_n z^n$, we get

$$(n+1)\, a_{n+1} + \sum_{k=-\infty}^{\infty} u_n f_{n-k} = 0\,,$$

$$(n = \ldots, -1, 0, 1, \ldots)$$

or

$$Mu = 0\,,$$

where $u = (\ldots, u_{-1}, u_0, u_1, \ldots)^T$ and

$$M = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & \cdots \\ \cdots & f_2 & f_1 & f_0 & 1+f_{-1} & f_{-2} & \cdots \\ \cdots & f_3 & f_2 & f_1 & f_0 & 2+f_{-1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

A source of infinite matrices is solution of Fredholm integral equation:

$$y(x) = F(x) + \lambda \int_a^b K(x,\xi) y(\xi) d\xi$$

$$K(x,\xi) = \sum_{n=1}^{\infty} f_n(x) g_n(\xi) \quad \text{(separable kernel)}$$

Using the moments $c_n = \int_a^b g_n(\xi) y(\xi) d\xi$,

we get $\quad y(x) = F(x) + \lambda \sum_{n=1}^{\infty} c_n f_n(x)$

$$\sum_{j=1}^{\infty} (\delta_{ij} - \lambda \alpha_{ij}) c_j = \beta_i \ , \ i = 1, 2, \ldots, \infty$$

$$\alpha_{ij} = \int_a^b g_i(\xi) f_i(\xi) d\xi$$

$$\beta_i = \int_a^b F(x) x^i dx$$

$$K(x,\xi) = \frac{1}{1 - x\xi} \text{ is Hilbert kernel}$$

Consider $y(s;a) = \lambda \int_0^a \frac{y(t;a)}{1-st}\, dt\ , \qquad 0 < a < 1$

$$y(s;a) = \lambda \sum_0^\infty s^n \int_0^a t^n y(t;a)\, dt$$

$$\int_0^a s^m y(s;a)\, ds = \lambda\, y_n^{(a)} \int_0^a s^{n+m}\, ds = y_m$$

$$y_m^{(a)} = \lambda \sum_{j=1}^\infty \frac{a^{m+n}}{i+j-1} y_n^{(a)}$$

If we formally let $a = 1$, and $\lambda = \frac{1}{\pi}$ is the spectral radius.

We also know that:

$$\text{(A)} \qquad y(s) = \lambda \int_0^1 \frac{y(t)}{1-st}\, dt$$

has an exact solution when $\lambda = \frac{1}{\pi}$ (kernel is not square integrable)

We get:

$$\text{(B)} \qquad y_m^{(1)} = \lambda \sum_{j=1}^\infty \frac{a^{m+n}}{i+j-1} y_n^{(1)}$$

In $S^2(0,\pi]$, S has no eigenvalue and L^2 spectrum of S is the interval $[0,\pi]$ and it is purely continuous.

Is it true that (A) and (B) with a replaced by 1 are equivalent?

[UTILITAS MATHEMATICA, PNS]

- **Karl Weierstrass, Vito Volterra, Giulio Ascoli, Ceasare Arzela, J.Hadamard, Salvatore Pincherle, E.H.Moore**

- **Maurice Frechet developed concepts of abstract metric spaces.**

In this talk we will concentrate on developing meaningful easily computable error estimators for solutions of linear algebraic systems, linear differential systems etc.
This analysis depends on extensions of algebraic results for finite matrices, in particular, the property of nonsingularity.

Reference:

M.Bernkopf - *A history of infinite matrices*.

2. Some topics

Theory of infinite matrices and related finite matrices (such as sections or truncations).

Relations to linear operator theory on separable spaces and sequence spaces.

Computational and numerical aspects.

Approximation theory.

Applications

- Linear algebraic and eigenvalue problem
- Linear differential systems
- Conformal mapping of doubly connected regions
- Mathieu equation
- Second order linear d.e.s. defined on semi-infinite intervals
- Iterative analysis
- LU and other factorizations
- Numerical inversion of the Laplace transform
- Reproducing kernel Hilbert spaces
- Biorthogonal infinite sets/bases
- Moments problems and generalized moment problem
- Integral equation
- Attenuation in transmission lines
- Porous media
- Interpolation
- Sobolev spaces
- Signal processing

- Time series
- Generalized inverses
- Linear programming
- Ill-posed problems
- Inverse eigenvalue problem
- Summability
- Compartmental analysis
- Birth and death problem
- Special cases:
  - Diagonally dominant
  - M-matrices
  - Toeplitz matrices
  - Hilbert matrix
  - Infinite determinants
  - Tridiagonal etc.

3. Nonsingularity of some finite matrices

$A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix.
$detA$ denotes determinant of A
$A^{-1} = \frac{1}{detA} \cdot (A_{ji})$ is the inverse of A

(a) Diagonal dominance

If $\sigma_i |a_{ii}| = \sum_{j\neq i} |a_{ij}|$, $0 \le \sigma_i < 1$,
then A is nonsingular and

$$\frac{1}{|a_{ii}|(1+\sigma_i)} \le \left|\frac{A_{ii}}{detA}\right| \le \frac{1}{|a_{ii}|(1-\sigma_i)}$$

[Ostrowski, Varga]

(b) Chain condition

$J = \{\, i \in N \mid |a_{ii}| > \sum_{j\neq i} \,\} \neq \emptyset$,
$N = \{\, 1, 2, \ldots, n \,\}$
$J = N$, $detA \neq 0$ by Gershgorin Theorem
A is irreducible , A is nonsingular [Taussky]

Theorem: Let A be such that for $i \notin J$ there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{ii_2}, \ldots, a_{i_r j}$ with $j \in J$.

Then $detA \neq 0$.

[PAMS, PNS+KHC]

(a) $z = \dfrac{c}{1-\zeta}$, $\zeta = \xi + i\eta$

(b) $z = c\,(\zeta + \dfrac{\lambda}{\zeta})$

(c) Combinations of (a) and (b)

(d) Finite element technique.

RESULTS

R_1	R_2	R_3	R_4
	two circles	two ellipses	circle/ellipse

$$R = \int\int w\,dxdy$$

[CQAM, CJ+PNS]
[JCAM, CJ+PNS]
[PIAS, KHC+PNS]

Applications

1. Flow of sap in trees
2. Leakage of electricity in coaxial cable
   (Attenuation problem)
3. Cholesterol problem in arteries [ML+SK+PNS]
4. Simultaneous flow of oil and gas [ASR+PMNS], [ML+SK+PNS]

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