MATHEMATICAL MODEL OF ICE MELTING ON TRANSMISSION LINES

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Abstract. During ice storms, ice forms on high voltage electrical lines. This ice formation often results in downed lines and has been responsible for considerable damage to life and property as was evidenced in the catastrophic ice storm of Quebec recently. There are two main aspects, viz., the formation of ice and its timely mitigation. In this paper, we mathematically model the melting of ice due to a higher current applied to the transmission wire. The two dimensional cross section contains four layers consisting of the transmission wire, water due to melting of ice, ice, and the atmosphere. The model includes heat equations for the various regions with suitable boundary conditions. Heat propagation and ice melting are expressed as a Stefan-like problem for the moving boundary between the layers of ice and water. The model takes into account gravity which leads to downward motion of ice and to forced convection of heat in the water layer. In this paper, the results are applied to the case when the cross sections are concentric circles to yield melting times for ice dependent on the increase in intensity of the electrical flow in the line.

1. Introduction. Ice formation on electrical transmission line is a major problem leading to high casualties in terms of people and property. The ice storm of 1998 in Montreal, Canada left four million people without heat and electricity because of a destroyed electrical network. Other losses include millions of trees, 120,000 km of power lines, 130 major transmission towers each worth about $100,000 and 30,000 electrial poles each worth $3000. To avoid such catastrophy, the ice formed on the transmission lines should be melted as soon as it forms, by conduction of heat from a temporary increase in electrical power.

In spite of the seriousness of the problem, there appears to be no satisfactory solutions discussed in the literature, mathematical or otherwise. Some crude practices like using long poles to break the ice are used. We hope the present model can serve as a first step toward the solution of the problem by controlling the intensity of electricity in the lines. Modeling is needed as insufficient electricity does not solve the problem while even a little excess will expand the wire, making it sag and break, and possibly overload transformers. Hence the selection of optimum levels of electricity accompanied by (ususally) short periods of increased electricity play a pivotal role, thus avoiding guesswork. We expect the model to be tested by Manitoba Hydro under field conditions.

The object of this paper is to develop a mathematical model of ice melting and to solve a simplified case numerically. The design of the mathematical model includes various factors relating to ice melting by analyzing the boundaries between the different material regions and also the heat transfer equations that govern the melting process. The problem draws upon principles from fluid mechanics, heat transfer, and gravitational effects. In this paper we consider the models under simple conditions without taking into account the effects of humidity and wind. We develop the modeling aspect of the problem incrementally and systematically so as to keep it tractable.

Since the full model consists of complex coupled equations which cannot be solved in closed form, numerical solutions are sought. A finite difference scheme is used to

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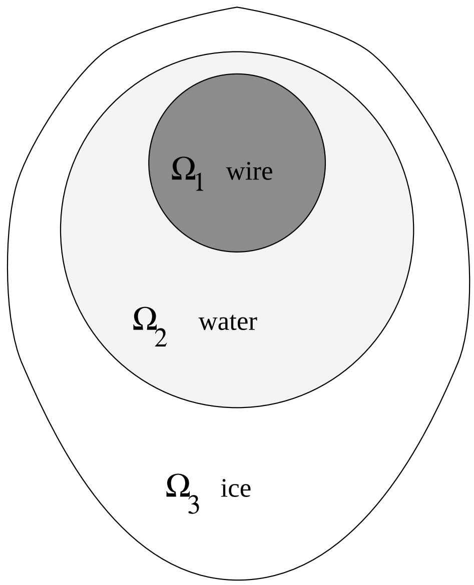

FIG. 2.1. *Different regions in the ice melting problem.*

discretize the governing equations. The present numerical simulation considers the simplest case of axially symmetric solution without gravity.

2. Formulation of the Full Model. We consider the transmission wire to be a homogeneous horizontal cylinder of constant cross-section.

The ice-covered wire is also assumed to have a constant cross-section. Thus we can describe the ice-melting process in two dimensions.

We distinguish four material regions in the cross-section (Figure 2.1):

(i) Ω_1 metal (electric wire),
(ii) Ω_2 water melted off from ice,
(iii) Ω_3 ice,
(iv) Ω_4 atmosphere.

Boundaries between adjacent regions are denoted Γ_{12}, Γ_{23} and Γ_{34}. In this model it is assumed that boundary between ice and water is sharply defined.

The coordinate system is attached to the wire with its origin at the centre of Ω_1 so the boundary Γ_{12} does not change its position in time.

At the initial moment, $\Gamma_{12}|_{t=0} = \Gamma_{23}|_{t=0}$ and the region Ω_2 (water) is empty. The following *process termination criterion* is adopted*:* Γ_{23} touches Γ_{34} at some point, i.e., the water region reaches the external boundary at some point. In particular, not all ice needs to be completely melted.

The given information for this problem includes the two-dimensional cross-section with two initial regions: wire and ice, with regions and boundaries as designated above. The wire region has a heat source of constant intensity q. It is assumed values are available for the material parameters (specific heat c_i, constant density of the media ρ_i, thermal conductivity of the media k_i, latent heat of melting ice λ, wire resistivity ρ and current density in wire j), the initial and external temperatures and electric current.

The unknowns for this problem are

1. boundary between water and ice $\Gamma_{23}(t)$,
2. relative position of Γ_{12} and Γ_{34} in terms of the velocity of the ice $v_{ice}(t)$, which is assumed to be a vector in the vertical direction only,
3. temperature fields in Ω_1, Ω_2, and Ω_3,
4. velocity field $v(x, y, t)$ in the water region Ω_2,
5. pressure field $P(x, y, t)$ in Ω_2,
6. t_{end}, the instant the process termination criterion described above occurs in the case where sufficient heat is available.

The mathematical model is described by heat transfer equations within the regions Ω_1, Ω_2, and Ω_3 and across the boundaries Γ_{12}, Γ_{23} and Γ_{34} of the regions. Let

$$T = \begin{cases} T_1 & \text{for points in } \Omega_1 \\ T_2 & \text{for points in } \Omega_2 \\ T_3 & \text{for points in } \Omega_3. \end{cases}$$

Within the regions, the heat transfer equations are given by

$$\rho_i c_i \frac{\partial T_i}{\partial t} = k_i \Delta T_i + q_i, \qquad i = 1, 3 \tag{2.1}$$

$$\rho_2 c_2 \left(\frac{\partial T_2}{\partial t} + v \cdot \nabla T_2 \right) = k_2 \Delta T_2 + q_2.$$

Notice that $q_1 = q = j^2 \rho \neq 0$ whereas $q_2 = q_3 = 0$ since there are no internal heat sources in water and ice. In Ω_1, it is assumed the solution must be a form in which the temperature is finite everywhere - this assumption is essential in solving the boundary value problem.

The water region is modeled by the incompressible viscous Navier-Stokes in the Boussinesq approximation:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla P}{\rho_2} + \nu \Delta v - g(1 - \mu T_2)\hat{y}.$$

Here P is the pressure, μ is the coefficient of thermal expansion of water, ν is the kinematic viscosity of water, g is the gravitational acceleration and $\hat{y}$ is the unit vector in the vertical direction. The last term in the above equation is the buoyancy term which pushes hot water (relative to the reference temperature 0 degree Celsius) upward. The fluid is assumed to be incompressible:

$$\nabla \cdot v = 0.$$

Finally, Newton's second law applied to the ice bulk is

$$|\Omega_3| \frac{dv_{ice}}{dt} = -g|\Omega_3|\hat{y} + \frac{1}{\rho_2} \int_{\Gamma_{23}} P(x, y)\mathbf{n} ds, \tag{2.2}$$

where $|\Omega_3|$ is the volume (i.e., two-dimensional area) of the ice region, and $\mathbf{n}$ is the unit outward normal vector along Γ_{23}. This equation holds whenever ice does not touch the wire (otherwise, we stop by the process termination criterion).

The boundary conditions for the water region are:

$$v|_{\Gamma_{12}} = 0, \quad \text{and} \quad v|_{\Gamma_{23}} = v_{ice}.$$

To describe the boundary conditions at Γ_{12}, we assume the temperature must be continuous across the boundary and that

$$\left[k_1 \frac{\partial T_1}{\partial n} - k_2 \frac{\partial T_2}{\partial n}\right]_{\Gamma_{12}} = 0. \tag{2.3}$$

To describe the boundary conditions at Γ_{23}, we assume

$$T_2|_{\Gamma_{23}} = T_3|_{\Gamma_{23}} = 0 \tag{2.4}$$

(temperature is measured in degrees Celsius) and that the *Stefan condition*

$$\left[k_2 \frac{\partial T_2}{\partial n} - k_3 \frac{\partial T_3}{\partial n}\right]_{\Gamma_{23}} = -\rho_3 \lambda \left(\mathbf{v}_\Gamma \left(x\right), \mathbf{n}\right) \tag{2.5}$$

holds where $\mathbf{v}_\Gamma \left(x\right)$ is the velocity of ice melting in the normal direction to the phase boundary and $\left(\mathbf{v}_\Gamma \left(x\right), \mathbf{n}\right)$ denotes the inner product of the velocity and normal vectors. This velocity is defined by

$$\mathbf{v}_\Gamma \left(x\right) = \left|\frac{\partial \gamma \left(B\right)}{\partial n}\right| \mathbf{n} - v_{ice} \tag{2.6}$$

where $\gamma\left(B\right)$ is the position of a point on Γ_{23}.

The boundary condition at Γ_{34} is assumed to be described by *Newton's law of cooling*

$$k_3 \frac{\partial T_3}{\partial n}\bigg|_{\Gamma_{34}} = h \left(T_{env} - T_3|_{\Gamma_{34}}\right) \tag{2.7}$$

where T_{env} is the external temperature.

To obtain rough estimations of the melting time, simple methods based on energy considerations are available. Besides their usefulness as benchmarks, these estimations show a crucial role of the heat transfer parameter h in Equation (2.7). Unfortunately, this parameter is not apriori known or directly measurable.

It would be a formidable task to prove existence and uniqueness results. We are content with supplying a few remarks.

In the Navier-Stokes equations, pressure is only determined up to a constant. In (2.2), the pressure term appears. It is easy to check that the integral in this equation does not change when a constant is added to P.

Let us discuss the solvability of the Navier-Stokes equations assuming that the fluid is irrotational. Introduce the hydrodynamical potential $\phi(x, y)$ so that

$$v = \left[\begin{array}{c} -\phi_y \\ \phi_x \end{array}\right].$$

This potential satisfies $\triangle\phi = 0$ in Ω_2 with Neumann boundary conditions. It can be checked that the circulation of v along both boundaries is zero, hence ϕ is well defined up to an additive constant. Since the first derivatives of ϕ vanish on Γ_{12}, we may take

$$\phi|_{\Gamma_{12}} = 0.$$

It is possible to reduce the boundary value problem for ϕ in a doubly connected region to a Dirichlet problem in a simply connected domain. For simplicity, assume the problem is symmetric about the line $x = 0$. Along this line $\phi_y = 0$. Thus $\phi = 0$ at points with $x = 0$ on Γ_{23}. From the boundary condition at Γ_{23}, one sees that $\phi|_{\Gamma_{23}} = -|v_{ice}|x+$ constant. We can now conclude the constant must be zero and so ϕ is known on the entire boundary of each of two symmetric halves of Ω_2.

3. Setup For a Model with No Gravity. We now simplify these systems of differential equations and their boundary conditions to correspond to the simple (and admittedly unrealistic) geometry we are interested in.

The model to be considered neglects the effect of gravity. The absence of gravity means the external boundary Γ_{34} does not move, so there is only one moving boundary in the problem which is the phase interface Γ_{23} between ice and water. This is an example of the *classical Stefan model.* The assumption of no gravity means Γ_{12}, Γ_{23} and Γ_{34} are all circles at all times (the effect of gravity is to cause the ice to "sag" so the boundary Γ_{34} is changing with time, a somewhat more difficult problem).

Let the boundaries Γ_{12}, $\Gamma_{23}(t)$ and Γ_{34} be concentric circles of radii R_1, $R_2(t)$ and R_3 respectively so that $R_2(0) = R_1$ and $R_2(t_{end}) = R_3$ (in the case of sufficiently large q). In polar coordinates (r, ϕ), it is assumed the temperature is a function of r only (angular symmetry). The problem reduces to a one-dimensional form. The heat transfer described in Equation (2.1) takes the form

$$\rho_i c_i \frac{\partial T}{\partial t} = k_i \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + q_i \text{ where } i = 1, 2, 3. \tag{3.1}$$

The function $T(r, t)$ is continuous and its derivative with respect to r, denoted by T', is piecewise continuous and subject to boundary conditions

$$k_1 T'(R_1 - 0) = k_2 T'(R_1 + 0), \tag{3.2}$$

$$k_2 T'(R_2(t) - 0) = k_3 T'(R_2(t) + 0) - \rho_3 \lambda \frac{dR_2}{dt}, \tag{3.3}$$

$$T(R_2(t)) = 0, \tag{3.4}$$

$$k_3 T'(R_3) = h(T_{env} - T(R_3)). \tag{3.5}$$

Note the dependence on t has been suppressed in the notation of Equations (3.2) through (3.5).

3.1. Solutions for Incomplete Melting. If the generated heat is insufficient for complete melting of the ice shell, then there exists a stationary state. Equating time derivatives from Equations (3.1) and (3.3) to zero, it is possible to solve this problem as closed form solutions. There are two possible cases.

3.1.1. Case 1: No Water Layer. If there is no water layer, i.e. the Joulean heat is insufficient to start the melting process, then in this case, $R_1 = R_2(t)$ for all t so the boundaries Γ_{12} and Γ_{23} coincide. For clarity, let Γ_{13} denote this coincident boundary. The boundary condition at Γ_{13} is analogous to that of Equation (3.2) with the index 2 replaced by index 3:

$$\begin{aligned} k_1 \left. \frac{\partial T_1}{\partial r} \right|_{\Gamma_{13}} &= k_3 \left. \frac{\partial T_3}{\partial r} \right|_{\Gamma_{13}}, \\ T_1|_{\Gamma_{13}} &= T_3|_{\Gamma_{13}}. \end{aligned} \tag{3.6}$$

Taking into account the external boundary condition given by Equation (3.5), we find the solution

$$T_1(r) = T_{env} + \frac{(R_1^2 - r^2) q}{4k_1} + \frac{R_1^2 q}{2} \left(\frac{1}{R_3 h} + \frac{1}{k_3} \ln \frac{R_3}{R_1} \right) \tag{3.7}$$

in Ω_1 and

$$T_3(r) = T_{env} + \frac{R_1^2 q}{2}\left(\frac{1}{R_3 h} + \frac{1}{k_3}\ln\frac{R_3}{r}\right) \tag{3.8}$$

in Ω_3. Case 1 occurs if the temperature at Γ_{13} is below the melting temperature which is assumed to be zero degrees Celsius. Note that

$$T_1(R_1) = T_3(R_1) = T_{env} + \frac{R_1^2 q}{2}\left(\frac{1}{R_3 h} + \frac{1}{k_3}\ln\frac{R_3}{R_1}\right)$$

from which it follows that

$$T_{env} + \frac{R_1^2 q}{2}\left(\frac{1}{R_3 h} + \frac{1}{k_3}\ln\frac{R_3}{R_1}\right) < 0. \tag{3.9}$$

3.1.2. Case 2: Stationary Water Layer. The water layer Ω_2 exists in a stationary state; thus

$$R_1 < R_2 < R_3 \text{ and } R_2(t) \text{ is a constant.} \tag{3.10}$$

Using all the boundary conditions except Equation (3.5), we find the solution

$$T_1(r) = \frac{R_1^2 q}{2k_2}\ln\frac{R_2}{R_1} + \frac{q}{4k_1}\left(R_1^2 - r^2\right) \tag{3.11}$$

in Ω_1,

$$T_2(r) = \frac{R_1^2 q}{2k_2}\ln\frac{R_2}{r} \tag{3.12}$$

in Ω_2 and

$$T_3(r) = -\frac{R_1^2 q}{2k_3}\ln\frac{r}{R_2} \tag{3.13}$$

in Ω_3. The stationary position of Γ_{23} where $r = R_2$ is not apriori known. Substituting the solution in Equation (3.13) into the boundary condition described by Equation (3.5) gives the equation

$$T_{env} + \frac{R_1^2 q}{2}\left(\frac{1}{R_3 h} + \frac{1}{k_3}\ln\frac{R_3}{R_2}\right) = 0. \tag{3.14}$$

Since $R_2 < R_3$, this implies

$$T_{env} + \frac{R_1^2 q}{2R_3 h} < 0, \tag{3.15}$$

a condition which is useful in considering the case where complete melting occurs.

3.2. Complete Melting. Suppose that initially the water layer does not exist, and then such current is applied so the negation of Inequality (3.15) is true. The ice shell will finally melt. It is desirable to calculate the melting time t_{end}.

3.2.1. Time dependence. First estimations. The simplest lower bound for t_{end} may be derived from the energy needed to melt the bulk of the ice layer. The mass of ice per unit length of wire is

$$m_{ice} = \rho_3 \pi \left(R_3^2 - R_1^2\right).$$

The melting energy per unit length is

$$E_{melt} = \lambda m_{ice} = \lambda \rho_3 \pi \left(R_3^2 - R_1^2\right) \tag{3.16}$$

and the heat power per unit length equals $q\pi R_1^2$. The ratio of the melting energy per unit length to the heat power per unit length provides a lower bound to the melting time. Hence

$$t_{end} \geq \frac{\lambda \rho_3}{q} \frac{R_3^2 - R_1^2}{R_1^2}. \tag{3.17}$$

This estimation doesn't take into account the energy dissipation for heating metal, ice and water.

A slightly more advanced estimation is based on the full enthalpy increment, which includes an increment of the internal energy not only due to phase change, but also due to temperature increase. The initial enthalpy of the region Ω_1 is given by

$$H_1(0) = \rho_1 c_1 \int_0^{R_1} T_1(r, 0) \cdot 2\pi r \, dr.$$

Substituting the stationary solution from Equation (3.7) with heat flow q and integrating, we obtain

$$H_1(0) = \rho_1 c_1 \pi R_1^2 \left[T_{env} + \frac{R_1^2 q}{2} \left(\frac{1}{4k_1} + \frac{1}{R_3 h} + \frac{1}{k_3} \ln \frac{R_3}{R_1}\right)\right].$$

Similarly, to find the initial enthalpy of the region Ω_3 we substitute the solution from Equation (3.8) as the integrand and perform integration from R_1 to R_3:

$$H_3(0) = \rho_3 c_3 \pi \left[\left(R_3^2 - R_1^2\right)\left(T_{env} + \frac{R_1^2 q}{2}\left(\frac{1}{2k_3} + \frac{1}{R_3 h}\right)\right) - \frac{R_1^4 q}{2k_3} \ln \frac{R_3}{R_1}\right].$$

Let us evaluate the enthalpy of the final state, when Γ_{23} and Γ_{34} coincide. To obtain a lower bound, we take the minimal heat power q that guarantees the complete melting, i.e. q that turns Inequality (3.15) into equality, which is

$$q_{melt} = -\frac{2R_3 T_{env} h}{R_1^2}.$$

Integrating Equation (3.11) from 0 to R_1 and using $q = q_{melt}$ and $R_2(t_{end}) = R_3$, we get

$$H_1(t_{end}) \geq -\frac{\rho_1 c_1 \pi R_1^2 R_3 T_{env} h}{k_2} \left(\ln \frac{R_3}{R_1} + \frac{1}{4}\right).$$

Note that $T_{env} < 0$ (otherwise, the ice layer would be melting from the outside) so $q_{melt} > 0$ and $H_1(t_{end}) \geq 0$.

The enthalpy of Ω_3 is obtained by integration of the solution given by Equation (3.12) from R_1 to R_3 and adding the melting energy described by Equation (3.16):

$$H_2(t_{end}) \geq \rho_2 c_2 \pi \frac{R_3 T_{env} h}{2k_2} \left(2R_1^2 \ln \frac{R_3}{R_1} + R_1^2 - R_3^2 \right) + E_{melt}.$$

Once again, we have used $q = q_{melt}$ and $R_2(t_{end}) = R_3$. We obtain the following estimate of the melting time:

$$t_{end} \geq \frac{H_1(t_{end}) + H_2(t_{end}) - H_1(0) - H_3(0)}{q}.$$

The main error in the enthalpy estimation is caused by the ignored heat transfer through the external boundary Γ_{34}. This error increases as h becomes larger.

3.3. Quasi-stationary Approximation. The quasi-stationary approximation (QSA) [1,Ch.3] gives the most precise lower bound for melting time based on the available analytical solution of the stationary problem in case when the latent heat is much greater than specific heats. Like the enthalpy approach, the QSA is more precise than the simplest estimation in Inequality (3.17) since it takes into account additional heat drainages. But unlike the enthalpy approach, the QSA accounts for heat exchange via the external boundary, while ignoring heat consumed for temperature increase. So the two approaches are complementary in a sense and both underestimate t_{end}.

In the QSA we equate specific heats to zero and at each moment of time we consider the stationary problem. Dynamics enters via the Stefan boundary condition, which is not considered when solving the stationary problems; when the solution is known it defines the interface velocity.

Equating left hand sides of Equation (3.1) to zero and solving the stationary equations with boundary conditions described by Equations (3.2) to (3.5) excluding Equation (3.3), we find that in Ω_1 and Ω_2 the solution is given by Equations (3.11) and (3.12) respectively and in Ω_3 it has the form

$$T_3(r) = hT_{env} \left(\frac{k_3}{R_3} + h \ln \frac{R_3}{R_2} \right)^{-1} \ln \frac{r}{R_2}. \tag{3.18}$$

Substituting Equations (3.12) and (3.18) in the boundary condition of Equation (3.3), we find

$$\frac{dR_2}{dt} = \frac{1}{\rho_3 \lambda R_2} \left[k_3 h T_{env} \left(\frac{k_3}{R_3} + h \ln \frac{R_3}{R_2} \right)^{-1} + \frac{k_2 R_1^2 q}{2k_1} \right].$$

4. Numerical Results. We have performed a numerical simulation of a simplified model of the axially symmetric case without gravity, neglecting the variation of temperature inside the wire. More precisely, the problem is:

$$T_t = \frac{\kappa_j}{r}(rT_r)_r, \qquad j = 2, 3$$

where $\kappa_j = \dfrac{k_j}{\rho_j c_j}$. The boundary conditions are

$$T(R_1) = T_{wire}, \quad T(R_2(t)) = 0, \quad k_3 T_r(R_3) = h(T_{env} - T(R_3))$$

supplemented by the Stefan condition

$$k_2 T_r(R_2(t)-) - k_3 T_r(R_2(t)+) + \rho_3 \lambda \frac{dR_2(t)}{dt} = 0.$$

First, we cast the problem in non-dimensional form. Let

$$\tilde{t} = \frac{t\kappa_3}{R_1^2}, \quad \tilde{r} = \frac{r}{R_1}, \quad \tilde{T} = \frac{T}{T_{wire}}, \quad \tilde{R}_2 = \frac{R_2}{R_1}, \quad \tilde{R}_3 = \frac{R_3}{R_1}.$$

The PDE in the water region becomes (dropping the tilde)

$$T_t = \frac{\kappa_2}{\kappa_3 r}(rT_r)_r$$

with boundary conditions

$$T(1) = 1, \quad T(R_2(t)) = 0.$$

The PDE in the ice region is

$$T_t = \frac{1}{r}(rT_r)_r$$

with boundary conditions

$$T(R_2(t)) = 0, \quad \frac{k_3}{R_1 h} T_r(R_3) = \frac{T_{env}}{T_{wire}} - T(R_3)$$

and the Stefan condition

$$0 = \frac{k_2 \rho_2 c_2 T_{wire}}{k_3 \rho_3 \lambda} T_r(R_2(t)-) - \frac{c_3 T_{wire}}{\lambda} T_r(R_2(t)+) + \frac{dR_2(t)}{dt}.$$

The values of the parameters in metric units (kg, m, s, etc.) are

$$\rho_2 = 1000, \ k_2 = .60, \ c_2 = 4190,$$

$$\rho_3 = 1000, \ k_3 = 2.034, \ c_3 = 2054, \ \lambda = 3.34 \times 10^5, \ h = 1,$$

and $R_1 = .01, \ R_2 = .02, \ T_{wire} = 30, \ T_{env} = -5.$

A second-order finite difference scheme for the spatial variable and backward Euler scheme for the temporal variable were used. The spatial domain was divided into 100 intervals while the time step was .01. The evolution of $R_2(t)$ is shown in Figure 4.1. It follows the classical $t^{1/2}$ growth rate. More precisely,

$$R_2(t) = R_1 + \beta\sqrt{t}$$

for some positive β. For the current simulation, $\beta = 3.79 \times 10^{-4}$ and the time of melting is 697 seconds. The temperature distributions at $t = 20$ seconds and at the time of melting are shown in Figures 4.2 and 4.3. Note the discontinuity in the slope of the temperature at the interface in Figure 4.2. Taking $h = .1$, the solution is almost identical with time of melting also equal to 697 seconds.

There is a critical h approximately equal to 5 such that for all values of h larger than the critical value, the solution reaches a steady state with not all ice melted. In this steady case, the PDEs can be solved analytically and we have the following relationship between the steady interface R_2 and h:

$$h = \frac{k_3}{R_1 R_3} \frac{-1}{\frac{k_3 \rho_3 c_3 T_{env}}{k_2 \rho_2 c_2 T_{wire}} \ln R_2 + \ln \frac{R_3}{R_2}}$$

with R_i in non-dimensional form, $i = 1, 2, 3$. In figure 4.4, we plot R_2 versus h. For large h, the steady interface R_2 approaches approximately 1.73.

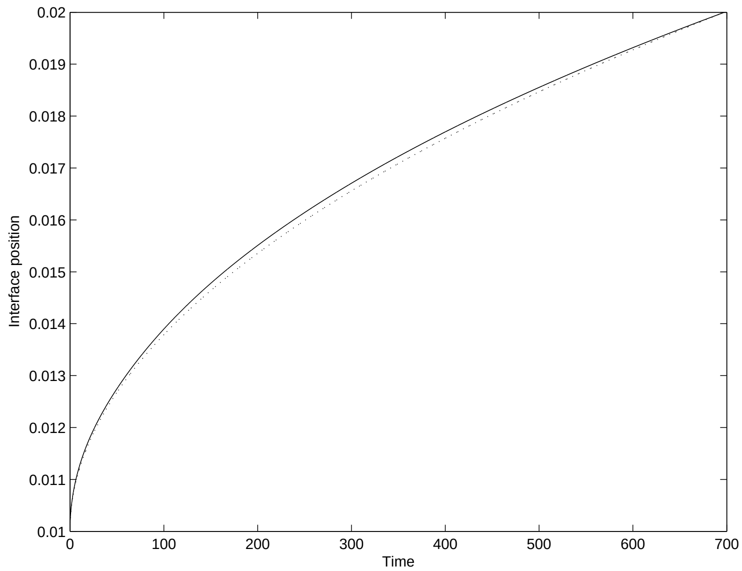

FIG. 4.1. *Graph of R_2 (solid line) and $R_1 + \beta\sqrt{t}$ (dotted line).*

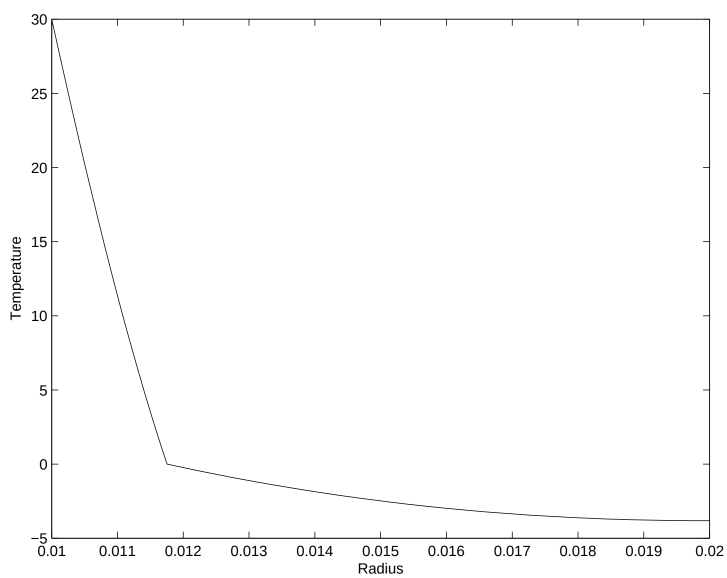

FIG. 4.2. *Temperature distribution at $t = 20s$.*

5. Discussion. A two-dimensional cross-sectional model of melting ice on an electrical transmission line due to an applied current has been presented. The model includes both conduction and convection heat transfer mechanisms.

Of course, the real phenomenon of ice melting is more complex. Some factors that are not taken into consideration in our simplistic model are:

(i) effects of gravity;

(ii) formation of cracks in ice due to structure of ice, difference of densities of water

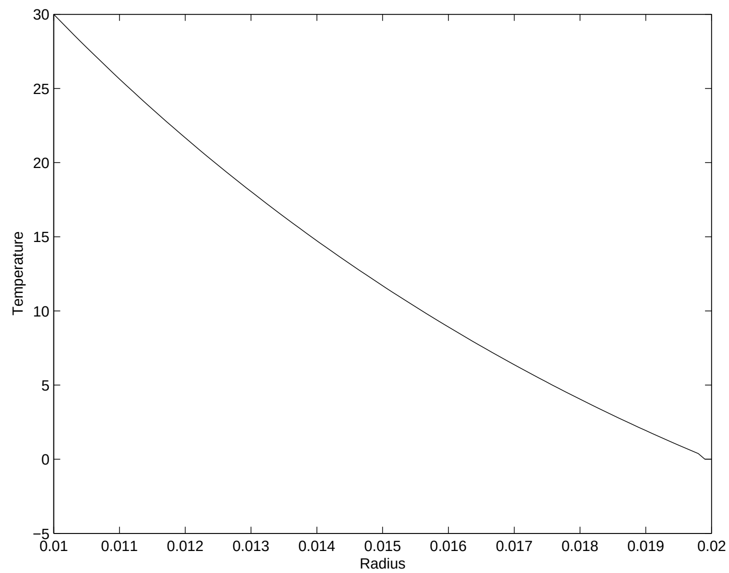

FIG. 4.3. *Temperature distribution at the time of melting.*

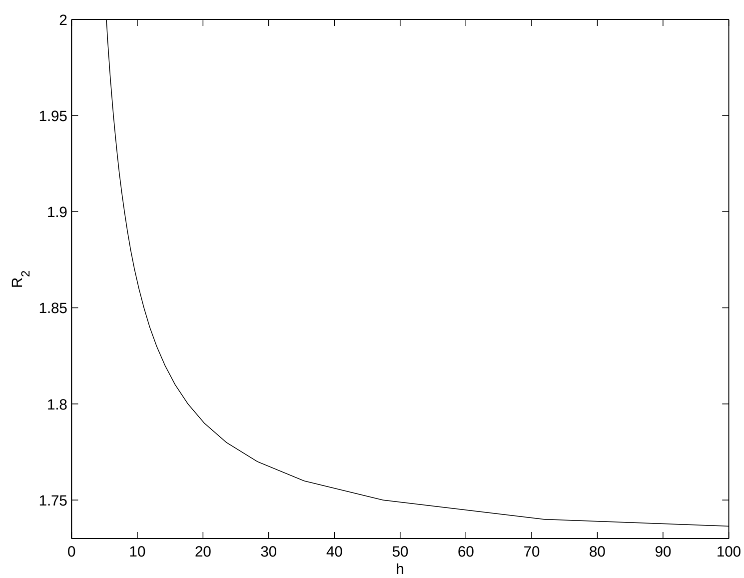

FIG. 4.4. *Relationship between R_2 and h in the case of incomplete melting.*

and ice and the longitudinal bent or twist of the cable;

(iii) other thermal phenomena such as solar radiation, sublimation;

(iv) atmosphere factors such as wind and precipitation.

A serious obstacle in practical application of the proposed model is an inherent uncertainty of the parameter h in the external boundary condition. An extensive experimental study is needed in order to establish a relationship of the coefficient in Newton's law of cooling (or the Biot number) to observable or measurable data.

A more realistic model accounts for gravity. In this case, additional equations are introduced to describe the evolution of the boundary Γ_{23}. This additional complexity in the model requires numerical methods to obtain solutions. The model is quite complex and difficult for numerical investigation. Especially difficult is a question of the thickness of the water layer near the wire's top, where the most intensive melting occurs.

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