STAT305, Problems/definitions on Poisson Processes

1. Defn: A Poisson process with rate $\lambda > 0$ is defined to be a counting process$N(t)$ such that:

   (i) $N(0) = 0$

   (ii) The process has independent increments

   (iii) $N(t+k) - N(t) \sim$ Poisson (λk) for all $k > 0, t \geq 0$.

2. Defn: A Poisson process with rate $\lambda > 0$ is defined to be a counting process$N(t)$ such that:

   (i) $N(0) = 0$

   (ii) The process has independent and stationary increments

   (iii) $N(k) \sim$ Poisson (λk) for all $k > 0$.

3. Time to the first event T_1

$$P(T_1 > t) = P(N(t) - N(0) = 0) = e^{-\lambda t}$$

4. (a) Interarrival times of a Poisson process with rate λ are independent and follow an exponential distribution with mean $\frac{1}{\lambda}$.

   (b) The nth waiting time of a Poisson processes with rate λ follows a Gamma distribution with mean $\frac{n}{\lambda}$ and variance $\frac{n}{\lambda^2}$

5. For a Poisson process show, for $s < t$, that

$$P\{N(s) = k | N(t) = n\} = \binom{n}{k}(\frac{s}{t})^k(1 - \frac{s}{t})^{n-k}, \quad k = 0, 1, \ldots, n.$$

6. Imagine that buses arrive at a particular stop according to a Poisson process with rate 10 per hour. I start waiting at the stop at 1 :00.

   a. What is the probability that no buses arrive in the next half-hour?

   b. How many buses are expected to arrive in 2 hours?

   c. What is the variance of the number of buses that arrive over a 5 hour period?

   d. What is the probability that no buses arrive in the next half-hour, given that one hasn't arrived for over two hours?

   e. If you wait until 2 o'clock and no buses have arrived, what is the probability that you still have to wait at least a further half-hour before one comes?

7. Imagine that buses arrive at a particular stop according to a Poisson process with rate 2 per hour. I start waiting at the stop at 1 :00. Given that the second bus arrives between 2:00 and 3 :00 what is the probability that the first bus arrived before 2:00? You may leave your answer as a formula–do not bother plugging the numbers into the calculator.

8. Cosmic rays arrive at a particle detector according to a Poisson Process with rate 2.

   a. Compute the probability that 4 cosmic rays arrived in a time period of length t=5 starting from time t=0 given that given that 8 cosmic rays arrived in the time period from t=0 to t=8.

   b. Now suppose an arriving cosmic ray is detected with probability 0.8 independent of all other cosmic rays and the time at which the cosmic ray arrives. Compute the joint probability mass function of the number which are detected in [0,5], the number which arrive and are not detected in [0,5], the number which hare detected in the interval (5,8] and the number which arrive and are not detected in the interval (5,8].

   c. Compute the probability that 4 cosmic rays arrived in a time period of length t=5 starting from time t=0 given that given that 8 cosmic rays were detected in the time period from t=0 to t=8.

9. Cars pass a point on the highway at a Poisson rate of one per minute. If five percent of the cars on the road are Dodges, then

   (a) what is the probability that at least one Dodge passes by during a hour?

   (b) given that ten Dodges have passed by in an hour, what is the expected number of cars to have passed by in that time?

   (c) if 50 cars have passed by in an hour, what is the probability that five of them were Dodges?

10. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the nth event. Find

    (a) $\mathrm{E}(S_4)$,

    (b) $\mathrm{E}[S_4 | N(1) = 2]$,

    (c) $\mathrm{E}[N(4) - N(2) | N(1) = 3]$.

11. Telephone calls arrive at a switchboard in a Poisson process at the rate of 2 per minute. A random one-tenth of the calls are long distance.

    (a) What is the probability that no call arrives between 9:00-9:05am?

    (b) What is the probability that at least 2 calls arrive between 10:00-10:02am?

(c) What is the probability of at least one long distance call in a ten minute period?

(d) Given that there have been 8 long distance calls in an hour, what is the expected number of calls to have arrived in the same period?

(e) Given that there were 90 calls in an hour, what is the probability that 10 were long distance?

12. Two individuals, A and B, both require kidney transplants. If she does not receive a new kidney, then A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the first kidney will go to A (or to B if B is alive and A is not at that time) and the next one to B (if still living).

    (a) What is the probability A obtains a new kidney?

    (b) What is the probability B obtains a new kidney?