STATISTICS 5.305 Practice Problems

1. (a) Define standard Brownian motion $\{B_t, t \geq 0\}$ and give its transition probability density.

   (b) Write down the transition probability density of general Brownian motion $W_t = \sigma B_t + \mu t$

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2. Let $S_t$be the stock price at time t. $\frac{S_t}{S_0}$ changes according to a Geometric Brownian Motion

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t},$$

   where $\{B_t : t \geq 0\}$ is a standard Brownian motion.

   Show that the probability the stock price will exceed the value b in t (time units) given that the current price is a is

$$P[S_t > b | S_0 = a] = 1 - \Phi(\frac{log\frac{b}{a} - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}).$$

   Calcute, given the parameters $\mu = 25$ % p.a., $\sigma = 20$ % on an annual basis, the probability that the share price will exceed 45 in four months' time given that its current price is 38.

3. Let $\{B(t), t \geq 0\}$ be a Standard Brownian motion and let $Y(t) = tB(1/t), t > 0$ and $Y(0) = 0$.

   (a) What is the distribution of $Y(t)$?

   (b) Find $Cov(Y(s),\ Y(t))$.

   (c) Argue that $\{Y(t), t \geq 0\}$ is a standard Brownian motion process.

4. Let $\{B(t), t \geq 0\}$ be a Standard Brownian motion and let $Y(t) = B(t+1) - B(t)$.

   Fid the $Cov[Y(t), Y(t+s)]$.

5. Suppose you own one share of a stock whose price changes according to a Standard Brownian motion process. Suppose that you purchased the stock at a price $b+c,\ c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b+c$ or when an additionaltime t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

6. Let $\{B(t), t \geq 0\}$ be a Standard Brownian motion.

   (i) Find $P(-3 \leq 2B(2) - 3B(5) \leq 5)$.

   (ii) Find the variance of $B(2) - 3B(3) + 2B(5)$.

   (iii) Find $Cov(3 + B(2) - 3B(3) + 2B(5), 9 + 3B(3) - 3B(9))$.

   (iv) Find $E[3 + B(2) - 3(B(4))^2 + 2(B(7))^4]$.

   (v) Find $P(-1 \leq B(2) \leq 3|B(1) = 1)]$.

   (vi) Find $P(-2 \leq B(3) - B(4) \leq 1|B(1) = 1)$.

   (vii) Find $P(0 \leq B(3) \leq 4|B(5) = 3)$.

   (viii)Find $Cov(B(s), B(t))$ and $Cov(B(s), B^2(t))$.

7. Suppose a person owns 1 share of stock currently worth 102 dollars. Assume that the change in value of this share over time follows a standard Brownian motion process where time is measured in months.

   (a)What is the probability that the price three months from now is greater than 105?

   (b)What is the probability that the price of the stock will be 105 or greater at some time between $t = 0$ and $t = 3$?

8. For $0 \leq s < t$, the distribution of $B(s)$ given $B(t)$ is normal with mean $\frac{s}{t}B(t)$ and variance $\frac{s(t-s)}{t}$.

9. Let $\{B(t), t \geq 0\}$ be a Standard Brownian motion and let T_a be the first time that the Brownian motion hits a . For $t > 0$ let $M_t = sup_{0 \leq s \leq t} B(s)$ be the maximum process over $[0, t]$.

   Find the following

(i) $P(M_t \leq 5)$.

(ii) The density of M_t.

(iii) The median of M_t.

(iv) The first and the third quartile of M_t.

(v) The mean and variance of M_t.

(vi) The density and the median of T_a.

(vii) The probabilities $P(T_2 \leq 4)$ and $P(2 \leq T_{-3} \leq 5)$.

(viii)The first and the third quartile of T_{-a}.