Chapter 4

Markov Chains

4.1 Stochastic Processes

Often, we need to estimate probabilities using a collection of random variables. For example, an actuary may be interested in estimating the probability that he is able to buy a house in The Hamptons before his company bankrupt. To study this problem, the actuary needs to estimate the evolution of reserves of his company over a period of time. For example, consider the following insurance model. U_t is the surplus at time t. P_t is the earned premium up to time t. S_t is the paid losses until time t. The basic equation is

$$U_t = U_0 + P_t - S_t.$$

We may be interested in the probability of ruin, i.e. the probability that at some moment $U(t) < 0$. The continuous time infinite horizon survival probability is given by

$$\Pr\{U_t \geq 0 \text{ for all } t \geq 0 | U_0 = u\}.$$

The continuous time finite horizon survival probability is given by

$$\Pr\{U_t \geq 0 \text{ for all } 0 \leq t \leq \tau | U_0 = u\},$$

where $\tau > 0$. Probabilities like the one before are called ruin probabilities. We also are interested in the distribution of the time of the ruin.

Another model is the one used for day traders. Suppose that a day trader decides to make a series of investment. Wisely, he wants to cut his losses. So, he decides to sell when either the price of the stock is up 50% or down 10%. Let $P(k)$ be the price of the stock, k days after begin bought. For each nonnegative integer k, $P(k)$ is a r.v.. To study the considered problem, we need to consider the collection of r.v.'s $\{P(k) : k \in \{0, 1, 2, \dots\}$. We may be interested in the probability that the stock goes up 50% before going down 10%. We may be interested in the amount of time it takes the stock to go either 50% up or 10% down. To estimate these probabilities, we need to consider probabilities which depend on the collection of r.v.'s $\{P(k) : k \in \{0, 1, 2, \dots\}$.

Recall that ω is the sample space, consisting by the outcomes of a possible experiment. A probability P is defined in the (subsets) events of Ω. A r.v. X is a function from Ω into $\mathbb{R}$.

Definition 4.1.1. *A stochastic process is a collection of r.v.'s $\{X(t) : t \in T\}$ defined in the same probability space. T is a parameter set. Usually, T is called the time set.*
If T is discrete, $\{X(t) : t \in T\}$ is called a discrete-time process. Usually, $T = \{0, 1, \dots, \}$.
If T is an interval, $\{X(t) : t \in T\}$ is called a continuous-time process. Usually, $T = [0, \infty)$.

Let $\mathbb{R}^T$ be the collection of functions from T into $\mathbb{R}$. A stochastic process $\{X(t) : t \in T\}$ defines a function from Ω into $\mathbb{R}^T$ as follows $\omega \in \Omega \mapsto X(\cdot)(\omega) \in \mathbb{R}^T$. In some sense, we have that a r.v. associates a number to each outcome and a stochastic processes associates a function to each outcome. A stochastic process is a way to assign probabilities to the set of functions $\mathbb{R}^T$. To find the probability of ruin, we need to study stochastic processes.

A stochastic process can be used to model:
(a) the sequence of daily prices of a stock;
(b) the surplus of an insurance company;
(c) the price of a stock over time; (d) the evolution of investments;
(e) the sequence of scores in a football game;
(f) the sequence of failure times of a machine;
(g) the sequence of hourly traffic loads at a node of a communication network;
(h) the sequence of radar measurements of the position of an airplane.

4.2 Random Walk

Let $\{\epsilon_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with $\mathrm{P}(\epsilon_j = 1) = p$ and $\mathrm{P}(\epsilon_j = -1) = 1 - p$, where $0 < p < 1$. Let $X_0 = 0$ and let $X_n = \sum_{j=1}^n \epsilon_j$ for $n \geq 1$. The stochastic process $\{X_n : n \geq 0\}$ is called a random walk. Imagine a drunkard coming back home at night. We assume that the drunkard goes in a straight line, but, he does not know which direction to take. After giving one step in one direction, the drunkard ponders which way to take home and decides randomly which direction to take. The drunkard's path is a random walk.

Let $s_j = \frac{\epsilon_j + 1}{2}$. Then, $\mathrm{P}(s_j = 1) = p$ and $\mathrm{P}(s_j = 0) = 1 - p = q$. $S_n = \sum_{j=1}^n s_j$ has a binomial distribution with parameters n and p. We have that for each $1 \leq m \leq n$,

$$\begin{aligned}
&P(S_n = k) = \tbinom{n}{k} p^k (1-p)^k, \text{ for } 0 \leq k \leq n,\\
&E[S_n] = np, \mathrm{Var}(S_n) = np(1-p),\\
&S_m \text{ and } S_n - S_m \text{ are independent}\\
&\mathrm{Cov}(S_m, S_n - S_m) = 0, \mathrm{Cov}(S_m, S_n) = \mathrm{Cov}(S_m, S_m) = \mathrm{Var}(S_m) = mp(1-p),\\
&S_n - S_m \text{ has the distribution of } S_{n-m}
\end{aligned}$$

Since $S_n = \frac{X_n + n}{2}$ and $X_n = 2S_n - n$, for each $1 \leq m \leq n$,

$$\begin{aligned}
&P(X_n = j) = P(S_n = \tfrac{n+j}{2}) = \tbinom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} (1-p)^{\frac{n-j}{2}}, \text{ for } -n \leq j \leq n, \text{with } \tfrac{n+j}{2} \text{ integer}\\
&E[X_n] = E[2S_n - n] = 2E[S_n] - n = n(2p-1), \mathrm{Var}(X_n) = \mathrm{Var}(2S_n - n) = n4p(1-p),\\
&X_m \text{ and } X_n - X_m \text{ are independent}\\
&\mathrm{Cov}(X_m, X_n - X_m) = 0, \mathrm{Cov}(X_m, X_n) = \mathrm{Cov}(X_m, X_m) = \mathrm{Var}(X_m) = m4p(1-p),\\
&X_n - X_m \text{ has the distribution of } X_{n-m}
\end{aligned}$$

In general, a random walk can start anywhere, i.e. we will assume that $X_0 = i$ and $X_n = i + \sum_{j=1}^{n} \epsilon_j$.

Example 4.1. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right, find:*
(i) $\mathrm{P}\{X_4 = -2\}$
(ii) $\mathrm{P}\{X_3 = -1, X_6 = 2, \}$
(iii) $\mathrm{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\}$

Solution: (i)

$$\mathrm{P}\{X_4 = -2\} = \mathrm{P}\{\mathrm{Binom}(4, p) = (4 + (-2))/2\} = \binom{4}{1} p^1 q^3 = 4p^1 q^3.$$

(ii)

$$\begin{aligned} & \mathrm{P}\{X_3 = -1, X_6 = 2, \} = \mathrm{P}\{X_3 = -1, X_6 - X_3 = 3, \} = \mathrm{P}\{X_3 = -1\}\mathrm{P}\{X_6 - X_3 = 3\} \\ = \ & \binom{3}{1} p^1 q^2 \binom{3}{3} p^3 q^0 = 3p^4 q^2. \end{aligned}$$

(iii)

$$\begin{aligned} & \mathrm{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\} = \mathrm{P}\{X_5 = 1, X_{10} - X_5 = 3, X_{10} = 4, X_{16} - X_{10} = -2\} \\ = \ & \mathrm{P}\{X_5 = 1\}\mathrm{P}\{X_{10} - X_5 = 3\}\mathrm{P}\{X_{16} - X_{10} = -2\} = \binom{5}{3} p^1 q^4 \binom{5}{4} p^4 q^1 \binom{6}{2} p^2 q^4 \\ = \ & 750 p^7 q^9. \end{aligned}$$

Example 4.2. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right. Show that:*
(i) If $m \leq n$*, then* $\mathrm{Cov}(X_m, X_n) = m4p(1 - p)$*.*
(ii) If $m \leq n$*, then* $\mathrm{Var}(X_n - X_m) = (n - m)4p(1 - p)$*.*
(iii) $\mathrm{Var}(aX_n + bX_m) = (a + b)^2 n4p(1 - p) + b^2(m - n)4p(1 - p)$*.*

Solution: (i) Since X_m and $X_n - X_m$ are independent r.v.'s,

$$\begin{aligned} & \mathrm{Cov}(X_m, X_n) = \mathrm{Cov}(X_m, X_m + X_n - X_m) \\ = \ & \mathrm{Cov}(X_m, X_m) + \mathrm{Cov}(X_m, X_n - X_m) = \mathrm{Var}(X_m) = m4p(1 - p). \end{aligned}$$

(ii) Since $X_n - X_m$ has the distribution of X_{n-m},

$$\mathrm{Var}(X_n - X_m) = \mathrm{Var}(X_{n-m}) = (n - m)4p(1 - p).$$

(iii) Since X_m and X_n are independent r.v.'s,

$$\begin{aligned} & \mathrm{Var}(aX_m + bX_n) = \mathrm{Var}(aX_m + b(X_m + X_n - X_m)) \\ = \ & \mathrm{Var}((a + b)X_m + b(X_n - X_m)) = \mathrm{Var}((a + b)X_m) + \mathrm{Var}(b(X_n - X_m) \\ = \ & (a + b)^2 m4p(1 - p) + b^2(n - m)4p(1 - p) \end{aligned}$$

Example 4.3. *Suppose that* $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right, find:*
(i) $\mathrm{Var}(-3 + 2X_4)$
(ii) $\mathrm{Var}(-2 + 3X_2 - 2X_5)$*.*
(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$*.*

Solution: (i)

$$\mathrm{Var}(-3+2X_4) = (2)^2\mathrm{Var}(X_4) = (2)^2(4)p(1-p) = 16p(1-p).$$

(ii)

$$\begin{aligned} & \mathrm{Var}(-2+3X_2-2X_5) = \mathrm{Var}(3X_2-2X_5) = \mathrm{Var}((3-2)X_2 - 2(X_5-X_2)) \\ = & \mathrm{Var}(X_2 - 2(X_5-X_2)) = \mathrm{Var}(X_2) + (-2)^2\mathrm{Var}(X_5-X_2) \\ = & (2)(4)p(1-p) + (-2)^2(3)(4)p(1-p) = 56p(1-p). \end{aligned}$$

(iii)

$$\begin{aligned} & \mathrm{Var}(3X_4-2X_5+4X_{10}) = \mathrm{Var}((3-2+4)X_4 + (-2+4)(X_5-X_4) + 4(X_{10}-X_5)) \\ = & \mathrm{Var}(5X_4 + 2(X_5-X_4) + 4(X_{10}-X_5)) \\ = & \mathrm{Var}(5X_4) + \mathrm{Var}(2(X_5-X_4) + \mathrm{Var}(4(X_{10}-X_5)) \\ = & (5)^2\mathrm{Var}(X_4) + (2)^2\mathrm{Var}(X_5-X_4) + (4)^2\mathrm{Var}(X_{10}-X_5) \\ = & (5)^2(4)(4)p(1-p) + (2)^2(1)(4)p(1-p) + (4)^2(5)(4)p(1-p) = 736p(1-p) \end{aligned}$$

Exercise 4.1. *Let $\{S_n : n \geq 0\}$ be a random walk with $S_0 = 0$ and $P[S_{n+1} = i+1 \mid S_n = i] = p$ and $P[S_{n+1} = i-1 \mid S_n = i] = 1-p$. Find:*
(i) $P[S_{10} = 4]$.
(ii) $P[S_4 = -2, S_{10} = 2]$.
(iii) $P[S_3 = -1, S_9 = 3, S_{14} = 6]$.
(iv) $\mathrm{Var}(S_7)$.
(v) $\mathrm{Var}(5 - S_3 - 2S_{10} + 3S_{20})$.
(vi) $\mathrm{Cov}(-2S_3 + 5S_{10}, 5 - S_4 + S_7)$.

Given $k > 0$, let T_k be the first time that $S_n = k$, i.e.

$$T_k = \inf\{n \geq 0 : S_n = k\}.$$

Then, T_k has negative binomial distribution,

$$P(T_k = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}, n \geq k.$$

We have that $E[T_k] = \frac{k}{p}$ and $\mathrm{Var}(T_k) = \frac{kq}{p}$.

Exercise 4.2. *The probability that a given driver stops to pick up a hitchhiker is $p = 0.004$; different drivers, of course, make their decisions to stop or not independently of each other. Given that our hitchhiker counted 30 cars passing her without stopping, what is the probability that she will be picked up by the 37-th car or before?*

Solution: You need to find the probability that any of the seven cars stops. This is the complementary that none of the cars stops. So, it is $1 - (1 - 0.004)^7 = 0.02766623$.

Gambler's ruin problem. Imagine a game played by two players. Player A starts with \$$k$ and player B starts with \$$(N-k)$. They play successive games until one of them ruins. In every game, they bet \$1, the probability that A wins is p and the probability that A losses is $1-p$. Assume that the outcomes in different games are independent. Let X_n be player A's money after n games. After, one of the player ruins, no more wagers are done. If $X_n = N$, then $X_m = N$, for each $m \geq n$. If $X_n = 0$, then $X_m = 0$, for each $m \geq n$. For $1 \leq k \leq N-1$,

$$P(X_{n+1} = k+1 | X_n = k) = p, \; P(X_{n+1} = k-1 | X_n = k) = 1-p.$$

Let P_k be the probability that player A wins (player B gets broke). We will prove that

$$P_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{k}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

P_k is the probability that a random walk with $X_0 = k$, reaches N before reaching 0. P_k is the probability that a random walk goes up $N-k$ before it goes down k. If $N \to \infty$,

$$P_k = \begin{cases} 1 - \left(\frac{q}{p}\right)^k & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}$$

Now, P_k is the probability that a random walk with $X_0 = k$, where $k > 0$, never reaches 0.

Exercise 4.3. *Two gamblers, A and B make a series of \$1 wagers where B has .55 chance of winning and A has a .45 chance of winning on each wager. What is the probability that B wins \$10 before A wins \$5?*

Solution: Here, $p = 0.55$, $k = 5$, $N - k = 10$. So, the probability that B wins \$10 before A wins \$5 is

$$\frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} = \frac{1-\left(\frac{0.45}{0.55}\right)^5}{1-\left(\frac{0.45}{0.55}\right)^{15}}$$

Exercise 4.4. *Suppose that on each play of a game a gambler either wins \$1 with probability p or losses \$1 with probability $1-p$, where $0 < p < 1$. The gamble continuous betting until she or he is either winning n or losing m. What is the probability that the gambler quits a winner assuming that she/he start with \$i?*

Solution: We have to find the probability that a random walk goes up n before it goes down m. So, $N - k = n$ and $k = m$, and the probability is

$$\frac{1-\left(\frac{q}{p}\right)^m}{1-\left(\frac{q}{p}\right)^{m+n}}, \text{if } p \neq \frac{1}{2}$$
$$\frac{m}{m+n}, \text{if } p = \frac{1}{2}$$

Problem 4.1. *(# 7, November 2002). For an allosaur with 10,000 calories stored at the start of a day:*
(i) The allosaur uses calories uniformly at a rate of 5,000 per day. If his stored calories reach 0, he dies.
(ii) Each day, the allosaur eats 1 scientist (10,000 calories) with probability 0.45 and no

scientist with probability 0.55.
(iii) The allosaur eats only scientists.
(iv) The allosaur can store calories without limit until needed.
Calculate the probability that the allosaur ever has 15,000 or more calories stored.
(A) 0.54 (B) 0.57 (C) 0.60 (D) 0.63 (E) 0.66

Solution: This is the gamblers' ruin problem with $p = 0.45$, $q = 0.55$ and $X_0 = 2$. We need to find

$$\mathrm{P}(X_k \text{ is up 1 before going down 2}) = \frac{1 - \left(\frac{0.55}{0.45}\right)^2}{1 - \left(\frac{0.55}{0.45}\right)^3} = 0.598.$$

4.3 Markov Chains

Definition 4.3.1. *A Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ is a stochastic process with countably many states such that for each states $i_0, i_1, \dots, i_n, j$,*

$$\begin{aligned}&\mathrm{P}(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n)\\&= \mathrm{P}(X_{n+1} = j | X_n = i_n)\end{aligned}$$

In words, for a Markov chain the conditional distribution of any future state X_{n+1} given the past states $X_0, X_1, \dots, X_{n-1}$ and the present state X_n is independent of the past values and depends only on the present state. Having observed the process until time n, the distribution of the process from time $n + 1$ on depends only on the value of the process at time n.

Let E the set consisting the possible states of the Markov chain. Usually, $E = \{0, 1, 2, \dots\}$ or $E = \{1, 2, \dots, m\}$. We will assume that $E = \{0, 1, 2, \dots\}$.

We define $P_{ij} = P(X_{n+1} = j | X_n = i_n)$. Note that $P(X_{n+1} = j | X_n = i_n)$ is independent of n. P_{ij} is called the one-step transition probability from state i into state j. We define $P_{ij}^{(k)} = P(X_{n+k} = j | X_n = i_n)$. $P_{ij}^{(k)}$ is called the k-step transition probability from state i into state j.

We denote $P = (P_{ij})_{i,j}$ to the matrix consisting by the one-step transition probabilities, i.e.

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

We denote $P^{(k)} = (P_{ij}^{(k)})_{i,j}$ to the matrix consisting by the k-step transition probabilities. We have that $P^{(1)} = P$. Every row of the matrix $P^{(k)}$ is a conditional probability mass function. So, the sum of the elements in each row of the matrix $P^{(k)}$ adds 1, i.e. for each i, $\sum_{j=0}^{\infty} P_{ij}^{(k)} = 1$.

Kolmogorov-Chapman equations. Let $\alpha_i = \mathrm{P}(X_0 = i)$, then

$$\mathrm{P}(X_0 = i_0, X_1 = i_1, X_{n-1} = i_{n-1}, X_n = i_n) = \alpha_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$$
$$P_{ij}^{(2)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}$$
$$P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik}^{(n-1)} P_{kj}$$
$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$
$$P^{(n+m)} = P^n P^{(m)}$$
$$P^{(n)} = P^n$$
$$\mathrm{P}(X_n = j) = \sum_{i=0}^{\infty} \alpha_i P_{ij}^{(n)}$$

Proof.

$$\begin{aligned}
&\mathrm{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n) \\
&= \mathrm{P}(X_0 = i_0) P(X_1 = i_1 | X_0 = i_0) P(X_2 = i_2 | X_0 = i_0, X_1 = i_1) \\
&\cdots P(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots X_{n-1} = i_{n-1}) \\
&= \mathrm{P}(X_0 = i_0) P(X_1 = i_1 | X_0 = i_0) P(X_2 = i_2 | X_1 = i_1) P(X_n = i_n | X_{n-1} = i_{n-1}) \\
&= \alpha(i_0) P_{i_0,i_1} P_{i_1,i_2} \cdots P_{i_{n-1},i_n}.
\end{aligned}$$

For $0 < n < m$,

$$\begin{aligned}
&\mathrm{P}(X_0 = i, X_n = j, X_m = k) \\
&= \mathrm{P}(X_0 = i) P(X_n = j | X_0 = i) P(X_m = k | X_0 = i, X_n = k) \\
&= \mathrm{P}(X_0 = i) P(X_n = j | X_0 = i) P(X_m = k | X_n = k) = \alpha(i) P_{i,j}^{(n)} P_{j,k}^{(m-n)}.
\end{aligned}$$

$$\begin{aligned}
&P_{ij}^{(n+m)} = \mathrm{P}(X_{n+m} = j | X_0 = i) = \frac{\mathrm{P}(X_0 = i, X_{n+m} = j)}{\mathrm{P}(X_0 = i)} = \sum_{k=0}^{\infty} \frac{\mathrm{P}(X_0 = i, X_n = k, X_{n+m} = j)}{\mathrm{P}(X_0 = i)} \\
&= \sum_{k=0}^{\infty} \frac{\alpha(i) P_{i,k}^{(n)} P_{k,j}^{(m)}}{\alpha(i)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}.
\end{aligned}$$

Finally,

$$\mathrm{P}(X_n = j) = \sum_{i=0}^{\infty} \mathrm{P}(X_0 = i) \mathrm{P}(X_n = j | X_0 = i) = \sum_{i=0}^{\infty} \alpha_i P_{ij}^{(n)}.$$

Q.E.D.

Exercise 4.5. *Let* $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and $\alpha = (1/2, 1/3, 1/6)$. *Find:*
(i) P^2,
(ii) P^3,
(iii) $P(X_2 = 2)$,
(iv) $P(X_0 = 1, X_3 = 3)$, *(v)* $P(X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1)$,
(vi) $P(X_2 = 3 | X_1 = 3)$,
(vii) $P(X_{12} = 1 | X_5 = 3, X_{10} = 1)$,
(viii) $P(X_3 = 3, X_5 = 1 | X_0 = 1)$,
(ix) $P(X_3 = 3 | X_0 = 1)$.

Solution: (i)

$$P^2 = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$$

(ii)

$$\begin{aligned} P^3 &= \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix} \\ &= \begin{pmatrix} 0.3425926 & 0.3796296 & 0.2777778 \\ 0.3888889 & 0.2569444 & 0.3541667 \\ 0.3506944 & 0.3159722 & 0.3333333 \end{pmatrix} \end{aligned}$$

(iii)

$$\alpha P^2 = (1/2, 1/3, 1/6) \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix} = (0.375, 0.3125, 0.3125).$$

So, $P(X_2 = 2) = 0.3125$.

(iv)

$$\begin{aligned} & P(X_0 = 1, X_3 = 3) = P(X_0 = 1)P(X_3 = 3|X_0 = 1) \\ = \;& P(X_0 = 1)P^{(2)}_{1,3} = (0.5)(0.33333) = 0.16667. \end{aligned}$$

(v)

$$P(X_1 = 2, X_2 = 3, X_3 = 1|X_0 = 1) = \quad P(X_0 = 1)P_{1,2}P_{2,3}P_{3,1} = (1/2)(2/3)(1/2)(1/4) = 1/24.$$

(vi)

$$P(X_2 = 3|X_1 = 3) = P_{3,3} = 1/2.$$

(vii)

$$P(X_{12} = 1|X_5 = 3, X_{10} = 1) = P(X_{12} = 1|X_{10} = 1) = P^{(2)}_{1,1} = 0.444444.$$

(viii)

$$P(X_3 = 3, X_5 = 1|X_0 = 1) = P(X_3 = 3|X_0 = 1)P(X_5 = 1|X_3 = 3) == P^{(2)}_{1,3}P^{(2)}_{3,1} = (0.3333)(0.3333) = 0.1$$

(ix)

$$P(X_3 = 3|X_0 = 1) = P^{(2)}_{1,3} = 0.3333.$$

Exercise 4.6. *Let P be a three state Markov Chain with transition matrix*

$$P = \begin{pmatrix} 0.25 & 0.50 & 0.25 \\ 0.10 & 0.20 & 0.70 \\ 0.80 & 0.10 & 0.10 \end{pmatrix}$$

Suppose that chain starts at time 0 in state 2.
(a) Find the probability that at time 3 the chain is in state 1.
(b) Find the probability that the first time the chain is in state 1 is time 3.
(c) Find the probability that the during times 1,2,3 the chain is ever in state 2.

Exercise 4.7. *Suppose that 3 white balls and 5 black balls are distributed in two urns in such a way that each contains 4 balls. We say that the system is in state i is the first urn contains i white balls, $i = 0, 1, 2, 3$. At each stage 1 ball is drawn from each urn at random add the ball drawn from the first urn is placed in the second, and conversely the ball of the second urn is placed in the first urn. Let X_n denote the state of the system after the n–th stage. Prove $\{X_n\}_{n=1}^{\infty}$ is a Markov chain. Find the matrix of transition probabilities.*

Exercise 4.8. *Let $E = \{1, 2\}$, transition matrix*

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

Prove by induction that:

$$P^n = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{pmatrix}.$$

Problem 4.2. *(# 38, May, 2000). For Shoestring Swim Club, with three possible financial states at the end of each year:*
(i) State 0 means cash of 1500. If in state 0, aggregate member charges for the next year are set equal to operating expenses.
(ii) State 1 means cash of 500. If in state 1, aggregate member charges for the next year are set equal to operating expenses plus 1000, hoping to return the club to state 0.
(iii) State 2 means cash less than 0. If in state 2, the club is bankrupt and remains in state 2.
(iv) The club is subject to four risks each year. These risks are independent. Each of the four risks occurs at most once per year, but may recur in a subsequent year.
(v) Three of the four risks each have a cost of 1000 and a probability of occurrence 0.25 per year.
(vi) The fourth risk has a cost of 2000 and a probability of occurrence 0.10 per year.
(vii) Aggregate member charges are received at the beginning of the year.
(viii) $i = 0$
Calculate the probability that the club is in state 2 at the end of three years, given that it is in state 0 at time 0.
(A) 0.24 (B) 0.27 (C) 0.30 (D) 0.37 (E) 0.56

Solution: We may joint states 0 and 1 in a unique state. Then,

$$\begin{aligned} &P_{11} = P(\text{no big risk and at most one small risk}) \\ &= (0.9)(.75)^3 + (0.9)(3)(.25)(.75)^2 = 0.759375 \end{aligned}$$

So,

$$P_{11}^{(3)} = P_{11}^3 = (0.7594)^3 = 0.4379$$
$$P_{12}^{(3)} = 1 - P_{11}^3 = 0.5621$$

Problem 4.3. *(#26, Sample Test) You are given:*

- *The Driveco Insurance company classifies all of its auto customers into two classes: preferred with annual expected losses of 400 and standard with annual expected losses of 900.*
- *There will be no change in the expected losses for either class over the next three years.*
- *The one year transition matrix between driver classes is given by:*

$$T = \begin{pmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{pmatrix}$$

  with states $E = \{\text{preferred}, \text{standard}\}$.
- $i = 0.05$.
- *Losses are paid at the end of each year.*
- *There are no expenses.*
- *All drivers insured with Driveco at the start of the period will remain insured for three years.*

Calculate the 3-year term insurance single benefit premium for a standard driver.

Solution: Let T be the transition matrix. Then:

$$T^2 = \begin{pmatrix} 0.8125 & 0.1875 \\ 0.75 & 0.25 \end{pmatrix}$$

Hence, the 3-year insurance single benefit premium for a standard driver is

$$\begin{aligned} & \mathrm{P}(X_1 = 1|X_0 = 2)(900)(1.05)^{-1} + \mathrm{P}(X_1 = 2|X_0 = 2)(400)(1.05)^{-1} \\ + \quad & \mathrm{P}(X_2 = 1|X_0 = 2)(900)(1.05)^{-2} + \mathrm{P}(X_2 = 2|X_0 = 2)(400)(1.05)^{-2} \\ + \quad & \mathrm{P}(X_3 = 1|X_0 = 2)(900)(1.05)^{-3} + \mathrm{P}(X_3 = 2|X_0 = 2)(400)(1.05)^{-3} \\ = \quad & (1)(900)(1.05)^{-1} + (0)(400)(1.05)^{-1} + (0.85)(900)(1.05)^{-2} + (0.15)(400)(1.05)^{-2} \\ + \quad & (0.8125)(900)(1.05)^{-3} + (0.1875)(400)(1.05)^{-3} = 1854.875 \end{aligned}$$

Problem 4.4. *(# 22, May 2001). The Simple Insurance Company starts at time* $t = 0$ *with a surplus of* $S = 3$. *At the beginning of every year, it collects a premium of* $P = 2$. *Every year, it pays a random claim amount:*

Claim Amount	*Probability of Claim Amount*
0	*0.15*
1	*0.25*
2	*0.50*
4	*0.10*

Claim amounts are mutually independent. If, at the end of the year, Simple's surplus is more than 3, it pays a dividend equal to the amount of surplus in excess of 3. If Simple is unable to pay its claims, or if its surplus drops to 0, it goes out of business. Simple has no administrative expenses and its interest income is 0.
Calculate the expected dividend at the end of the third year.
(A) 0.115 (B) 0.350 (C) 0.414 (D) 0.458 (E) 0.550

Solution: The Markov chain has states $E = \{0, 1, 2, 3\}$ and transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.10 & 0.50 & 0.25 & 0.15 \\ 0.10 & 0 & 0.50 & 0.40 \\ 0 & 0.10 & 0 & 0.90 \end{pmatrix}$$

We have that $\alpha = (0, 0, 0, 1)$, $\alpha P = (0, 0.10, 0, 0.90)$ and
$\alpha P^2 = (0.01, 0.14, 0.025, 0.825)$. The expected dividend at the end of the third year is

$$(0.025)(1)(0.15) + (0.825)(2)(0.15) + (0.825)(1)(0.25) = 0.4575.$$

Problem 4.5. *(# 23, November 2001). The Simple Insurance Company starts at time 0 with a surplus of 3. At the beginning of every year, it collects a premium of 2. Every year, it pays a random claim amount as shown:*

Claim Amount	*Probability of Claim Amount*
0	*0.15*
1	*0.25*
2	*0.40*
4	*0.20*

If, at the end of the year, Simple's surplus is more than 3, it pays a dividend equal to the amount of surplus in excess of 3. If Simple is unable to pay its claims, or if its surplus drops to 0, it goes out of business. Simple has no administration expenses and interest is equal to 0. Calculate the probability that Simple will be in business at the end of three years.
(A) 0.00 (B) 0.49 (C) 0.59 (D) 0.90 (E) 1.00

Solution: Let states 0, 1, 2, 3 correspond to surplus of 0, 1, 2, 3. The probability that Simple will be in business at the end of three years is $P[X_3 \neq 0 | X_0 = 3]$. The one year transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.20 & 0.40 & 0.25 & 0.15 \\ 0.20 & 0.00 & 0.40 & 0.40 \\ 0.00 & 0.20 & 0.00 & 0.80 \end{pmatrix}$$

We have that $\alpha = (0, 0, 0, 1)$, $\alpha P = (0, 0.2, 0, 0.8)$, $\alpha P^2 = (0.04, 0.24, 0.05, 0.67)$ and $\alpha P^3 = (0.098, 0.230, 0.080, 0.592)$. So, $P[X_3 \neq 0 | X_0 = 3] = 1 - 0.098 = 0.902$.

Problem 4.6. *(# 21, November 2002). An insurance company is established on January 1.*
(i) The initial surplus is 2.

(ii) At the 5th of every month a premium of 1 is collected.
(iii) At the middle of every month the company pays a random claim amount with distribution as follows

x	$p(x)$
1	0.90
2	0.09
3	0.01

(iv) The company goes out of business if its surplus is 0 or less at any time.
(v) $i = 0$
Calculate the largest number m such that the probability that the company will be out of business at the end of m complete months is less than 5%.
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: Let states 0, 1, 2, correspond to surplus of 0, 1, 2. The one year transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0.01 & 0.09 & 0.9 \end{pmatrix}$$

We have that $\alpha = (0, 0, 1)$, $\alpha P = (0.01, 0.09, 0.90)$, $\alpha P^2 = (0.028, 0.162, 0.81)$ and. $\alpha P^3 = (0.0523, 0.2187, 0.729)$. So, the answer is 2.

Problem 4.7. *(# 29, November 2001). A machine is in one of four states (F, G, H, I) and migrates annually among them according to a Markov process with transition matrix:*

	F	G	H	I
F	0.20	0.80	0.00	0.00
G	0.50	0.00	0.50	0.00
H	0.75	0.00	0.00	0.25
I	1.00	0.00	0.00	0.00

At time 0, the machine is in State F. A salvage company will pay 500 at the end of 3 years if the machine is in State F. Assuming $\nu = 0.90$, calculate the actuarial present value at time 0 of this payment.
(A) 150 (B) 155 (C) 160 (D) 165 (E) 170

Solution: We need to find $P^{(3)}_{1,1}$. We have that $\alpha = (1, 0, 0, 0)$ and

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.75 & 0 & 0 & 0.25 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

So, $\alpha P = (0.2, 0.8, 0, 0)$, $\alpha P^2 = (044, 0.16, 0.4, 0)$, and $\alpha P^3 = (0.469, 0.352, 0.08, 0.1)$. Hence, $P^{(3)}_{1,1} = 0.468$. The actuarial present value at time 0 of this payment is

$$(0.468)(500)(0.9)^3 = 170.58.$$

Random Walk: A random walk Markov chain with states $E = \{0, \pm 1, \pm 2, \dots\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$.

Absorbing Random Walk with one barrier: The absorbing random walk with one barrier is the Markov chain with states $E = \{0, 1, 2, \dots\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $i \geq 1$, and $P_{0,0} = 1$.

Absorbing Random Walk with two barriers: The absorbing random walk with two barriers is the Markov chain with states $E = \{0, 1, 2, \dots, N\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $1 \leq i \leq N - 1$, and $P_{0,0} = P_{N,N} = 1$.

Reflecting Random Walk with one barrier: The reflecting Random Walk with one barrier is the Markov chain with states $E = \{0, 1, 2, \dots\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $i \geq 1$, and $P_{0,1} = 1$.

4.4 Classification of States

4.4.1 Accessibility, communication

State j is said to be accessible form state i if $P_{ij}^{(n)} > 0$, for some $n \geq 0$. This is equivalent to $\mathrm{P}((X_n)_{n\geq 0} \text{ ever enters } j | X_0 = i) > 0$. If this happens, we say that $i \to j$. We say that $i \leftrightarrow j$ if $i \to j$ and $j \to i$. Since $P_{ii}^{(0)} = 1$, $i \leftrightarrow i$.

We can put the information about accessibility in a graph. We put an arrow from i to j if $P_{ij} > 0$. Usually, we do not write the arrows from an element to itself.

Exercise 4.9. *Let $E = \{1, 2, 3\}$ and let*

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Represent using graphs the one-step accessibility of states.

Exercise 4.10. *Let $E = \{1, 2, 3\}$ and let the transition matrix be*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/2 \end{pmatrix}.$$

Represent using graphs the one-step accessibility of states.

The relation $i \leftrightarrow j$ is an equivalence relation, i.e. it satisfies
(i) For each i, $i \leftrightarrow i$.
(ii) For each i, j, $i \leftrightarrow j$ if and only if $j \leftrightarrow i$.
(iii) For each i, j, k, if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

So, we may decompose E into disjoint equivalent classes. A class consists of elements which communicate between themselves and no element from outside. A Markov chain is said to be irreducible if there is only one class, that is, if all state communicate with each other.

Exercise 4.11. *Prove that if the number of states in a Markov chain is M, where M is a positive integer, and if state j can be reached from state i, where $i \neq j$, then it can be reached in $M-1$ steps or less.*

Exercise 4.12. *Consider the Markov chain with states $E = \{1,2,3,4,5\}$ and transition matrix*

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

Represent using graphs the one-step accessibility of states. Identity the communicating classes.

Exercise 4.13. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 & 0 & 0 \\ .2 & .4 & 0 & 0 & .1 & 0 & .1 & .2 \\ 0 & 0 & .3 & 0 & 0 & .4 & 0 & .3 \end{pmatrix}.$$

and states $E = \{1,2,3,4,5,6,7,8\}$. Represent using graphs the one-step accessibility of states. Find the communicating classes.

4.4.2 Recurrent and transient states

Let f_i denote the probability that, starting in state i, the process will ever reenters state i, i.e.

$$f_i = \mathrm{P}(X_n = i, \quad \text{for some } n \geq 1 | X_0 = i).$$

State i is said to be recurrent if $f_i = 1$. State i is said to be transient if $f_i < 1$. A state i is absorbing if $P_{ii} = 1$. An absorbing state is recurrent.

If state i is recurrent,

$$\mathrm{P}(X_n = i, \quad \text{infinitely often } n \geq 1 | X_0 = i) = 1.$$

If state i is transient,

$$\begin{aligned} &\mathrm{P}((X_n)_{n\geq 1} \quad \text{is never } i | X_0 = i) = (1 - f_i) \\ &\mathrm{P}((X_n)_{n\geq 1} \quad \text{enters } i \quad \text{exactly } k \text{ times } n \geq 1 | X_0 = i) = f_i^k (1 - f_i), k \geq 0 \\ &\mathrm{P}((X_n)_{n\geq 0} \quad \text{enters } i \quad \text{exactly often } k \text{ times } | X_0 = i) = f_i^{k-1} (1 - f_i), k \geq 0. \end{aligned}$$

Let N_i be the number of times, which the Markov chain stays at state i, i.e. $N_i = \sum_{n=0}^{\infty} I(X_n = i)$. Then,

(i) If state i is recurrent,

$$P(N_i = \infty | X_0 = i) = 1, E[N_i | X_0 = i] = \infty.$$

(ii) If state i is transient,

$$\begin{array}{l} P(N_i = k | X_0 = i) = f_i^{k-1}(1 - f_i), k \geq 0, \\ E[N_i | X_0 = i] = \frac{1}{1-f_i} \end{array}$$

We have that

$$E[N_i | X_0 = i] = E[\sum_{n=0}^{\infty} I(X_n = i) | X_0 = i)] = \sum_{n=0}^{\infty} P_{ii}^{(n)}.$$

Theorem 4.1. *State i is recurrent, if $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$.*
State i is transient, if $\sum_{n=0}^{\infty} P_{ii}^{(n)} < \infty$.
If state i is transient, the averate number of times spent at i is $\frac{1}{1-f_i} = \sum_{n=0}^{\infty} P_{ii}^{(n)}$.

Corollary 4.1. *If state i is recurrent and states i and j communicate, then state j is recurrent.*

Proof. If states i and j communicate, then there are integers k and m such that $P_{ij}^{(k)} > 0$ and $P_{ji}^{(m)} > 0$. For each integer n,

$$P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} P_{ii}^{(n)} P_{ji}^{(k)}.$$

So,

$$\sum_{n=0}^{\infty} P_{jj}^{(m+n+k)} \geq \sum_{n=0}^{\infty} P_{ji}^{(m)} P_{ii}^{(n)} P_{ji}^{(k)} = \infty.$$

Q.E.D.

All states in the same communicating class have the same type, either all are recurrent or all are transient.

Theorem 4.2. *A finite state Markov chain has at least one recurrent class.*

Theorem 4.3. *A communicating class C is closed if it cannot leave C, i.e. for each $i \in C$, each $j \notin C^c$ and each $n \geq 0$, $P_{ij}^{(n)} = 0$.*

Theorem 4.4. *A recurrent class is closed.*

We may have elements i of a transient class such that $i \to j$, for some $j \notin C$. We may have elements i of a recurrent class such that $j \to i$, for some $j \notin C$.

Theorem 4.5. *A closed finite class is recurrent.*

So, if E has finite many states, a class C is recurrent if and only if it is closed.

Exercise 4.14. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

and states $E = \{0, 1, 2, 3, 4\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.15. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

and states $E = \{1, 2, 3\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.16. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.17. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.3 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5, 6, 7\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.18. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0.5 \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.19. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 & 0 & 0 \\ .2 & .4 & 0 & 0 & .1 & 0 & .1 & .2 \\ 0 & 0 & .3 & 0 & 0 & .4 & 0 & .3 \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find the communicating classes. Classify the states as transient or recurrent.

Exercise 4.20. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 & 0 \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5, 6, 7\}$. Find the communicating classes. Classify the states as transient or recurrent.

Problem 4.8. *(# 21, May 2001). The Simple Insurance Company starts at time $t = 0$ with a surplus of $S = 3$. At the beginning of every year, it collects a premium of $P = 2$. Every year, it pays a random claim amount:*

Claim Amount	*Probability of Claim Amount*
0	*0.15*
1	*0.25*
2	*0.50*
4	*0.10*

Claim amounts are mutually independent. If, at the end of the year, Simple's surplus is more than 3, it pays a dividend equal to the amount of surplus in excess of 3. If Simple is unable to pay its claims, or if its surplus drops to 0, it goes out of business. Simple has no administrative expenses and its interest income is 0.
Determine the probability that Simple will ultimately go out of business.
(A) 0.00 (B) 0.01 (C) 0.44 (D) 0.56 (E) 1.00

Solution: The Markov chain has states $E = \{0, 1, 2, 3\}$ and transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.10 & 0.50 & 0.25 & 0.15 \\ 0.10 & 0 & 0.50 & 0.40 \\ 0 & 0.10 & 0 & 0.90 \end{pmatrix}$$

State 0 is recurrent. So, the probability that Simple will ultimately go out of business is 1.

Random walk: The random walk is the Markov chain with state space $\{0, \pm 1, \pm 2, \dots\}$ and transition probabilities $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$. The Markov chain is irreducible. By the law of the large numbers, given $X_0 = i$, with probability one, $\lim_{n\to\infty} \frac{X_n}{n} = 2p - 1$. So, if $p > 1/2$, with probability one, $\lim_{\to\infty} X_n = \infty$. If $p < 1/2$, with probability one, $\lim_{\to\infty} X_n = -\infty$. So, if $p \neq 1/2$, the Markov chain is transient.

We have that $P_{00}^{(2n)} = \binom{2n}{n} p^n (1-p)^n$. By the Stirling formula,

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}}.$$

So,

$$\lim_{n\to\infty} \frac{P_{00}^{2n}}{\frac{(4p(1-p))^n}{\sqrt{\pi n}}}.$$

Hence, for $p = 1/2$,

$$\lim_{n\to\infty} \frac{P_{00}^{2n}}{\frac{1}{\sqrt{\pi n}}}.$$

and $\sum_{n=0}^{\infty} P_{00}^n = \infty$. The random walk is recurrent for $p = 1/2$.

4.4.3 Periodicity

State i has period d if $P_{ii}^{(n)} = 0$ whenever n is not divisible by d and d s the largest integer with this property. Starting in i, the Markov chain can only return at i at the times $d, 2d, 3, d \dots$ and d is the largest integer satisfying this property. A state with period 1 is said to be aperidodic.

Theorem 4.6. *If i has period d and $i \leftrightarrow j$, then j has period d.*

Example 4.4. *Consider the Markov chain with state space $E = \{1, 2, 3\}$ and transition matrix*

$$P = \begin{pmatrix} 0 & 0.4 & 0.6 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the communicating classes. Classify the communicating classes as transient or recurrent. Find the period d of each communicating class.

Example 4.5. *Consider a random walk on $E = \{0, 1, 2, 3, 4, 5, 6\}$ with transition matrix*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 & .5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the communicating classes. Classify the communicating classes as transient or recurrent. Find the period d of each communicating class.

Example 4.6. *Consider a random walk on $E = \{0, 1, 2, 3, 4, 5, 6\}$ with transition matrix*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Find the communicating classes. Classify the communicating classes as transient or recurrent. Find the period d of each communicating class.

Example 4.7. *Consider the Markov chain with state space $E = \{1, 2, 3, 4\}$ and transition matrix*

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the communicating classes. Classify the communicating classes as transient or recurrent. Find the period d of each communicating class.

4.5 Limiting Probabilities

A recurrent state i is said to be positive recurrent is starting at i, the expected time until the process returns to state i is finite. A recurrent state i is said to be null recurrent is starting at i, the expected time until the process returns to state i is infinite. A positive recurrent aperiodic state is called ergodic.

Theorem 4.7. *In a finite state Markov chain all recurrent states are positive recurrent.*

Theorem 4.8. *For an irreducible ergodic Markov chain, $\lim_{n\to\infty} P_{ij}^{(n)}$ exists and is independent of i. Furthermore, letting $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$, $j \geq 0$, then π_j is the unique nonnegative solution of*

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \ j \geq 0$$
$$1 = \sum_{i=0}^{\infty} \pi_i.$$

The equation above can be written as

$$\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ .\\ .\end{pmatrix} = \begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \cdots\end{pmatrix}\begin{pmatrix}P_{00} & P_{01} & P_{02} & \cdots\\ P_{10} & P_{11} & P_{22} & \cdots\\ P_{20} & P_{21} & P_{22} & \cdots\\ . & . & . & \cdots\\ . & . & . & \cdots\end{pmatrix}$$

π_i is called the long run proportion of time spent at state i. Let $a_j(N) = \sum_{n=1}^N I(X_n = j)$ be the amount of time the Markov chain spends in state j during the periods $1, \dots, N$. Then, with probability one,

$$\lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} I(X_n = j) = \pi_j.$$

Let $T_i = \inf\{X_n : n \geq 1, X_n = i\}$. Given $X_0 = i$, T_i is the time to return to state i. We have that $f_i = \mathrm{P}(T_i < \infty | X_0 = i)$. A state is recurrent if $f_i = 1$. A state is positive recurrent if $E[T_i|X_0 = i] < \infty$. A state is null recurrent if $E[T_i|X_0 = i] = \infty$. We have that $\pi_i = \frac{1}{E[T_i|X_0=i]}$.

$\frac{1}{\pi_i}$ is the average time to return to state i.

Definition 4.5.1. *A distribution $(\alpha_0, \alpha_1, \alpha_2, \dots)$ of initial probabilities is called stationary distribution. If given that X_0 has this distribution, then $P(X_N = j) = \alpha_j$, for each $j \geq 0$.*

$(\alpha_0, \alpha_1, \alpha_2, \dots)$ has a stationary distribution if

$$(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_0, \alpha_1, \alpha_2, \dots)\begin{pmatrix}P_{00} & P_{01} & P_{02} & \cdots\\ P_{10} & P_{11} & P_{12} & \cdots\\ P_{20} & P_{21} & P_{22} & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{pmatrix}$$

The long run probabilities $(\pi_0, \pi_1, \pi_2, \dots)$ determines a stationary distribution.

Theorem 4.9. *Let $\{X_n : n \geq 0\}$ be an irreducible Markov chain with long run probabilities $(\pi_0, \pi_1, \pi_2, \dots)$, and let r be a bounded function on the state space. Then, with proability one,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} r(X_n) = \sum_{j=0}^{\infty} r(j)\pi_j.$$

Exercise 4.21. *Consider the Markov chain with state space $E = \{1, 2, 3, 4, 5, 6\}$ and transition matrix*

$$P = \begin{pmatrix}0.5 & 0.5 & 0 & 0 & 0 & 0\\ 0.8 & 0.2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.7 & 0.3 & 0 & 0\\ 0 & 0 & 0.4 & 0.6 & 0 & 0\\ 0 & 0 & 0.3 & 0 & 0.2 & 0.5\\ 0.5 & 0 & 0 & 0 & 0.2 & 0.3\end{pmatrix}.$$

Compute the matrix F. Compute $\lim_{n\to\infty} P^n$. Compute

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=1}^{n} E[r(X_m)|X_0 = i]$$

where $r = (1, -2, 3, -4, 5, 6)$.

Exercise 4.22. *Let $E = \{1, 2\}$ and let*

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Show that if $0 < \alpha, \beta < 1$, the Markov chain is irreducible and ergodic. Find the limiting probabilities.

Exercise 4.23. *Find the following limit*

$$\lim_{n\to\infty} \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}^n,$$

where $0 < a, b < 1$.

Exercise 4.24. *Consider the Markov chain with state space $E = \{1, 2, 3\}$ and let*

$$P = \begin{pmatrix} 0 & 0.4 & 0.6 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}.$$

Show that the Markov chain is irreducible and aperiodic. Find the limiting probabilities.

Exercise 4.25. *A particle moves on a circle through points which have been marked 0,1,2,3, and 4 (in a clockwise order). At the each step it has probability p of moving to the right (clockwise) and $1-p$ to the left (counter wise). Let X_n denote its location on the circle after the n–th step. The process $\{X_n : n \geq 0\}$ is a Markov chain.*
(a) Find the transition probability matrix.
(b) Calculate the limiting probabilities.

Exercise 4.26. *Automobile buyers of brands A, B and C, stay with or change brands according to the following matrix:*

$$P = \begin{pmatrix} \frac{4}{5} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{4} \end{pmatrix}$$

After several years, what share of the market does brand C have?

Exercise 4.27. *Harry, the semipro. Our hero, Happy Harry, used to play semipro basketball where he was a defensive specialist. His scoring productivity per game fluctuated between three state states: 1 (scored 0 or 1 points), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inevitably, if Harry scored a lot of points in one game, his jealous teammates* *refused to pass him the ball in the next game, so his productivity in the next game was nil. The team statistician, Mrs. Doc, upon observing the transition between states concluded these transitions could be modeled by a Markov chain with transition matrix*

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{pmatrix}.$$

What is the long run proportion of games that our hero had high scoring games? Check that the Markov chain is irreducible and ergodic. Find the long run proportion vector (π_1, π_2, π_3)*.*

Exercise 4.28. *Harry, the semipro. Our hero, Happy Harry, used to play semipro basketball where he was a defensive specialist. His scoring productivity per game fluctuated between three state states: 1 (scored 0 or 1 ponts), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inveyitaby, if Harry scored a lot of points in one game, his jealous teammates refused to pass him the ball in the next game, so his productivity in the next game was nill. The team statistician, Mrs. Doc, upon observing the transition between states concluded these transitions could be modelled by a Markov chain with transition matrix*

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{pmatrix}.$$

What is the long run proportion of games that our hero had high scoring games? The salary structure in the the semipro leagues includes incentives for scoring. Harry was paid $40/game for a high scoring performance, $30/game when he scored between 2 and 5 points and only $20/game when he scored one or less. What was the long run earning rate of our hero?

Exercise 4.29. *Suppose that whether or not it rains tomorrow depends on past weather conditions only through the last two days. Specifically, suppose that if it has rained yesterday and today then it will rain tomorrow with probability .8; if it rained yesterday but not today then it will rain tomorrow with probability .3; if it rained today but not yesterday then it will rain tomorrow with probability .4; and if it has not rained whether yesterday or today then it will rain tomorrow with probability .2. What proportion of days does it rain?*

Exercise 4.30. *Harry, the semipro. Our hero, Happy Harry, used to play semipro basketball where he was a defensive specialist. His scoring productivity per game fluctuated between three state states: 1 (scored 0 or 1 ponts), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inveyitaby, if Harry scored a lot of points in one game, his jealous teammates refused to pass him the ball in the next game, so his productivity in the next game was nill. The team statistician, Mrs. Doc, upon observing the transition between states concluded these transitions could be modelled by a Markov chain with transition matrix*

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{pmatrix}.$$

What is the long run proportion of games that our hero had high scoring games? The salary structure in the the semipro leagues includes incentives for scoring. Harry was paid $40/game for a high scoring performance, $30/game when he scored between 2 and 5 points and only $20/game when he scored one or less. What was the long run earning rate of our hero?

Exercise 4.31. *At the start of a business day an individual worker's computer is one of the three states: (1) operating, (2) down for the day, (3) down for the second consecutive day. Changes in status take place at the end of a day. If the computer is operating at the beginning of a day there is a 2.5 % chance it will be down at the start of the next day. If the computer is down for the first day at the start of day, there is a 90 % chance that it sill be operable for the start of the next day. If the computer is down for two consecutive days, it is automatically replaced by a new machine for the following day. Calculate the limiting probability that a worker will have an operable computer at the start of a day. Assume the state transitions satisfy the Markov chain assumptions.*

Problem 4.9. *(# 33, May, 2000). In the state of Elbonia all adults are drivers. It is illegal to drive drunk. If you are caught, your driver's license is suspended for the following year. Driver's licenses are suspended only for drunk driving. If you are caught driving with a suspended license, your license is revoked and you are imprisoned for one year. Licenses are reinstated upon release from prison. Every year, 5% of adults with an active license have their license suspended for drunk driving. Every year, 40% of drivers with suspended licenses are caught driving. Assume that all changes in driving status take place on January 1, all drivers act independently, and the adult population does not change. Calculate the limiting probability of an Elbonian adult having a suspended license.*
(A) 0.019 (B) 0.020 (C) 0.028 (D) 0.036 (E) 0.047

Solution: We have a Markov chain with states $E = \{1, 2, 3\}$, where $1 = \{\text{ok license}\}$, $2 = \{\text{suspended license}\}$, and $3 = \{\text{prison for one year}\}$. The transition matrix is

$$\begin{pmatrix} 0.95 & 0.05 & 0 \\ 0.6 & 0 & 0.4 \\ 1 & 0 & 0 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, π_3, we solve

$$\begin{aligned} \pi_1 &= 0.95\pi_1 + 0.6\pi_2 + \pi_3 \\ \pi_2 &= 0.05\pi_1 \\ 1 &= \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

We get $\pi_1 = \frac{100}{107}, \pi_2 = \frac{5}{107}, \pi_3 = \frac{2}{107}$. So, the answer is $\pi_2 = \frac{5}{107} = 0.0467$.

Problem 4.10. *(# 40, May, 2000). Rain is modeled as a Markov process with two states:*
(i) If it rains today, the probability that it rains tomorrow is 0.50.
(ii) If it does not rain today, the probability that it rains tomorrow is 0.30.
Calculate the limiting probability that it rains on two consecutive days.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.19 (E) 0.22

Solution: We have a Markov chain with states $E = \{1, 2\}$, where $1 = \{\text{rain}\}$ and $2 = \{\text{not rain}\}$. The transition matrix is

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, we solve

$$\begin{aligned} \pi_1 &= 0.5\pi_1 + 0.3\pi_2 \\ 1 &= \pi_1 + \pi_2. \end{aligned}$$

We get $\pi_1 = \frac{3}{8}, \pi_2 = \frac{5}{8}$. So, the answer is

$$\lim_{n\to\infty} P[X_n = 1, X_{n+1} = 1] = \lim_{n\to\infty} P[X_n = 1]P_{11} = \frac{3}{8}\frac{1}{2} = \frac{3}{16} = 0.1875.$$

Problem 4.11. *(# 34, November 2000).The Town Council purchases weather insurance for their annual July 14 picnic.*
(i) For each of the next three years, the insurance pays 1000 on July 14 if it rains on that day.
(ii) Weather is modeled by a Markov chain, with two states, as follows:

- *The probability that it rains on a day is 0.50 if it rained on the prior day.*
- *The probability that it rains on a day is 0.20 if it did not rain on the prior day.*

(iii) $i = 0.10$.
Calculate the single benefit premium for this insurance purchased one year before the first scheduled picnic.
(A) 340 (B) 420 (C) 540 (D) 710 (E) 760

Solution: We have a Markov chain with states $E = \{1, 2\}$, where $1 = \{\text{rain}\}$ and $2 = \{\text{not rain}\}$. The transition matrix is

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, we solve

$$\begin{aligned} \pi_1 &= 0.5\pi_1 + 0.2\pi_2 \\ 1 &= \pi_1 + \pi_2. \end{aligned}$$

We get $\pi_1 = \frac{2}{7}, \pi_2 = \frac{5}{7}$. So, the benefit premium is

$$\tfrac{2}{7}(1000)(1.1)^{-1} + \tfrac{2}{7}(1000)(1.1)^{-2} + \tfrac{2}{7}(1000)(1.1)^{-3} = \tfrac{2}{7}(1000)(1.1)^{-3}\left(\tfrac{(1.1)^3-1}{1.1-1}\right) = 710.53.$$

Problem 4.12. *(# 7, May 2001). A coach can give two types of training, " light" or "heavy," to his sports team before a game. If the team wins the prior game, the next training is equally*

likely to be light or heavy. But, if the team loses the prior game, the next training is always heavy. The probability that the team will win the game is 0.4 after light training and 0.8 after heavy training. Calculate the long run proportion of time that the coach will give heavy training to the team.
(A) 0.61 (B) 0.64 (C) 0.67 (D) 0.70 (E) 0.73

Solution: We have the following options

Option	Probability
light $\mapsto$ win $\mapsto$ light	$(0.4)(.5) = 0.2$
light $\mapsto$ win $\mapsto$ heavy	$(0.4)(.5) = 0.2$
light $\mapsto$ lose $\mapsto$ heavy	$(0.6)(1) = 0.6$
heavy $\mapsto$ win $\mapsto$ light	$(0.8)(.5) = 0.4$
heavy $\mapsto$ win $\mapsto$ heavy	$(0.8)(.5) = 0.4$
heavy $\mapsto$ lose $\mapsto$ heavy	$(0.2)(.5) = 0.2$

We have a Markov chain with states $E = \{1, 2\}$, where $1 = \{\text{light}\}$ and $2 = \{\text{heavy}\}$. The transition matrix is

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, we solve

$$\pi_1 = 0.2\pi_1 + 0.4\pi_2$$
$$\pi_1 + \pi_2 = 1$$

We get $\pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}$.

Problem 4.13. *(# 22, November 2001). A taxi driver provides service in city R and city S only. If the taxi driver is in city R, the probability that he has to drive passengers to city S is 0.8. If he is in city S, the probability that he has to drive passengers to city R is 0.3. The expected profit for each trip is as follows:*
a trip within city R: 1.00
a trip within city S: 1.20
a trip between city R and city S: 2.00
Calculate the long-run expected profit per trip for this taxi driver.
(A) 1.44 (B) 1.54 (C) 1.58 (D) 1.70 (E) 1.80

Solution: We have a Markov chain with states $E = \{R, S, \}$, where R means the taxi is in city R and S means the taxi is in city S. The transition matrix is

$$\begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, we solve

$$\pi_1 + \pi_2 = 1$$
$$\pi_1 = 0.2\pi_1 + 0.3\pi_2$$
$$\pi_2 = 0.8\pi_1 + 0.7\pi_2$$

We get $\pi_1 = \frac{3}{11}, \pi_2 = \frac{8}{11}$. The long expected profit is

$$\frac{3}{11}(0.2)(1) + \frac{3}{11}(0.8)(2) + \frac{8}{11}(0.3)(2) + \frac{8}{11}(0.7)(1.2) = 1.54.$$

Problem 4.14. *(# 29, November 2002). Homerecker Insurance Company classifies its insureds based on each insured's credit rating, as one of Preferred, Standard or Poor. Individual transition between classes is modeled as a discrete Markov process with a transition matrix as follows:*

	Preferred	*Standard*	*Poor*
Preferred	*0.95*	*0.04*	*0.01*
Standard	*0.15*	*0.80*	*0.05*
Poor	*0.00*	*0.25*	*0.75*

Calculate the percentage of insureds in the Preferred class in the long run.
(A) 33% (B) 50% (C) 69% (D) 75% (E) 92

Solution: We have a Markov chain with states $E = \{\text{Preferred}, \text{Standard}, \text{Poor}\}$, The transition matrix is

$$\begin{pmatrix} 0.95 & 0.04 & 0.01 \\ 0.15 & 0.80 & 0.05 \\ 0.00 & 0.25 & 0.75 \end{pmatrix}$$

To find the limiting probabilities π_1, π_2, π_3, we solve

$$\begin{array}{l} \pi_1 + \pi_2 + \pi_3 = 1 \\ \pi_1 = \frac{19}{20}\pi_1 + \frac{3}{20}\ \pi_2 \\ \pi_2 = \frac{1}{25}\pi_1 + \frac{4}{5}\pi_3 + \frac{1}{4}\pi_3 \end{array}$$

We get

$$\pi_1 = \frac{75}{108}, \pi_2 = \frac{25}{55}, \pi_3 = \frac{8}{108}.$$

The percentage of insureds in the Preferred class in the long run is $\pi_1 = \frac{75}{108} = 69.44\%$.

Example 4.8. *Let X be an irreducible aperiodic Markov chain with $m < \infty$ states, and suppose its transition matrix P is doubly Markov. Show that then $\pi(i) = \frac{1}{m}$, $i \in E$ is the limiting distribution.*

4.6 Hitting Probabilities

Given a set $A \subset E$, let

$$h_i = \mathrm{P}((X_n)_{n\geq 0} \text{ is ever in } A | X_0 = i).$$

h_i is the probability that the process hits A given that it starts at i.

Theorem 4.10. $(h_0, h_1, \dots)$ *is the minimal nonnegative solution to the system of linear equations,*

$$\begin{array}{l} h_i = 1, \text{ for } i \in A, \\ h_i = \sum_{j=0}^{\infty} P_{ij} h_j, \text{ for } i \notin A. \end{array}$$

(Minimality means that if $(x_0, x_1, \dots)$ is another solution with $x_i \geq 0$, for each i, then $x_i \geq h_i$ for each i.)

Given a set $A \subset E$, let

$$H = \inf\{n \geq 0 : X_n \in A\}.$$

H is the hitting time, i.e. it is the time the Markov chain hits A. We have that $h_i = \mathrm{P}(H < \infty | X_0 = i)$. Let

$$k_i = E[H|X_0 = i] = \sum_{n=1}^{\infty} n\mathrm{P}(H = n|X_0 = i) + \infty\mathrm{P}(H = \infty|X_0 = i).$$

k_i is the mean hitting time.

Theorem 4.11. $(k_0, k_1, \dots)$ *is the minimal nonnegative solution to the system of linear equations,*

$$\begin{aligned} &k_i = 0, \text{ for } i \in A, \\ &k_i = 1 + \textstyle\sum_{j \notin A} P_{ij} k_j, \text{ for } i \notin A. \end{aligned}$$

We may also consider hitting times, requiring that a hits only occurs for times bigger than 1. Given a set $A \subset E$, let

$$\tilde{H} = \inf\{n \geq 1 : X_n \in A\}.$$

Let

$$\tilde{k}_i = E[\tilde{H}|X_0 = i] = \sum_{n=1}^{\infty} n\mathrm{P}(\tilde{H} = n|X_0 = i) + \infty\mathrm{P}(\tilde{H} = \infty|X_0 = i).$$

We can find $\tilde{k}_i$, using the formula

$$\tilde{k}_i = \sum_{j \in A} P_{i,j} + \sum_{j \notin A} P_{i,j}(1 + k_j),$$

where $k_i = E[H|X_0 = i]$ and $H = \inf\{n \geq 0 : X_n \in A\}$.

If E is an irreducible ergodic Markov chain and $A\{i\}$, then,

$$\tilde{k}_i = \frac{1}{\pi_i}.$$

Theorem 4.12. $(k_0, k_1, \dots)$ *is the minimal nonnegative solution to the system of linear equations,*

$$\begin{aligned} &k_i = 0, \text{ for } i \in A, \\ &k_i = 1 + \textstyle\sum_{j \notin A} P_{ij} k_j, \text{ for } i \notin A. \end{aligned}$$

Exercise 4.32. *For the random walk, i.e. the Markov chain with states $E = \{0, \pm 1, \pm 2, \cdots,\}$ the transition probabilities are $P_{i,i+1} = p$, and $P_{i,i-1} = 1 - p$. The Markov chain is irreducible. If $p \neq \frac{1}{2}$, the chain is transient. If $p = \frac{1}{2}$, the chain is null recurrent. Using the previous theorem, it is possible to prove that the mean hitting time to $\{0\}$ is ∞. Note that there is no solution to*

$$\begin{aligned} &k_0 = 0, \\ &k_i = 1 + \textstyle\sum_{j \neq 0} P_{ij} k_j, \text{ for } i \neq 0. \end{aligned}$$

Exercise 4.33. *For the random walk with a reflecting barrier, i.e. the Markov chain with states $E = \{0, 1, 2, \dots\}$ and transition probabilities $P_{i,i+1} = p$, and $P_{i,i-1} = 1 - p$ for $i \geq 1$, and $P_{01} = 1$. The Markov chain is irreducible. For $A = \{0\}$,*

$$h_i = \begin{cases} \left(\frac{q}{p}\right)^i & \text{if } p > \frac{1}{2} \\ 1 & \text{if } p \leq \frac{1}{2}, \end{cases}$$

if $i > 0$. If follows from this that for a regular Markov chain,

$$f(0,0) = \mathrm{P}((X_n)_{n\geq 1} \text{ ever transits to } 0 | X_0 = 0) = \begin{cases} 2(1-p) & \text{if } p > \frac{1}{2} \\ 2p & \text{if } p \leq \frac{1}{2}. \end{cases}$$

Exercise 4.34. *A mouse is in a 2×2 maze, form a room, the mouse can go any available room which is either up, or down or to the right or to the left at random. There is a cat in room 4. If the mouse gets to either room 4, stays there forever. Find the average life of the mouse assuming that it starts at room i.*

1	2
3	4 CAT

Solution: The states are $E = \{1, 2, 3, 4\}$. The transition matrix is

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $A = \{1\}$, we need to find k_i. We setup the equations

$$\begin{aligned} k_4 &= 0 \\ k_1 &= 1 + (0.5)k_2 + (0.5)k_3 \\ k_2 &= 1 + (0.5)k_1 \\ k_3 &= 1 + (0.5)k_1 \end{aligned}$$

The solutions are $k_2 = k_3 = 3$ and $k_1 = 4$.

Exercise 4.35. *A mouse is in a 2×3 maze, form a room, the mouse can go any available room which is either up, or down or to the right or to the left at random. Room 6 is full of cheese and there is a cat in room 3. If the mouse gets to either room 3 or 6, stays there forever. Find the probability that the mouse end up in room 6 assuming that it starts at room i.*

1	2	3 CAT
4	5	6 CHEESE

Solution: The states are $E = \{1, 2, 3, 4, 5, 6\}$. The transition matrix is

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For $A = \{1\}$, we need to find h_i. We setup the equations

$$\begin{aligned} &h_6 = 1, h_3 = 0 \\ &h_1 = \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4 \\ &h_2 = \tfrac{1}{3}h_1 + \tfrac{1}{3}h_5 \\ &h_4 = \tfrac{1}{2}h_1 + \tfrac{1}{2}h_5 \\ &h_5 = \tfrac{1}{3}h_4 + \tfrac{1}{3}h_2 + \tfrac{1}{3} \end{aligned}$$

The solutions are $h_1 = \frac{5}{11}$, $h_2 = \frac{4}{11}$, $h_4 = \frac{6}{11}$, $h_5 = \frac{7}{11}$.

Exercise 4.36. *A mouse is in a 3×3 maze, form a room, the mouse can go any available room which is either up, or down or to the right or to the left at random. Room 9 is full of cheese and there is a cat in room 3. If the mouse gets to either room 3 or 9, stays there forever. Find the probability that the mouse end up in room 3 assuming that it starts at room i. Hint: By symmetry $h_4 = h_5 = h_6 = \frac{1}{2}$.*

1	2	3 CAT
4	5	6
7	8	9 CHEESE

Exercise 4.37. *A student goes to an Atlantic City casino during a spring break to play a roulette table. The student begins with \$50 and wants to simply win \$150 (or go broke) and go home. The student bets on red each play. In this case, the probability of winning her bet (and her doubling her ante) at each play is 18/38. To achieve her goal of reaching a total of \$200 of wealth quickly, the student will bet her entire fortune until she gets \$200.*

(a) What is the probability the student will go home a winner?

(b) How many plays will be needed, on average, to reach a closure at the roulette table?

Solution: $E = \{0, 50, 100, 200\}$ and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 10/19 & 0 & 9/19 & 0 \\ 0 & 10/19 & 0 & 9/19 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) With $A = \{200\}$, we need to find h_{50}. We have that $h_0 = 0$ and $h_{200} = 1$,

$$h_{50} = (9/19)h_{100}, h_{100} = h_{200}.$$

So, $h_{50} = (9/19)^2 = 0.2241$.

(i) With $A = \{0, 200\}$, we need to find k_{50}, the equations to solve are $k_0 = 0$ and $k_{200} = 0$,

$$k_{50} = 1 + (9/19)k_{100}, k_{100} = 1.$$

So, $k_{50} = (28/19) = 1.473684$.

Exercise 4.38. *In a certain state, a voter is allowed to change his or her party affiliation (for primary elections) only by abstaining from the next primary election. Experience shows that, in the next primary, a former Democrat will abstain 1/3 of the time, whereas a Republican will abstain 1/5 of the time. A voter who abstains in a primary election is equally likely to vote for either party in the next election (regardless of their past party affiliation).*
(i) Model this process as a Markov chain by writing and labeling the transition probability matrix.
(ii) Find the percentage of voters who vote Democrat in the primary elections. What percentage vote Republican? Abstain?
For the remaining questions, suppose Eileen Tudor-Wright is a Republican.
(iii) What is the probability that, three elections from now, Eileen votes Republican?
(iv) How many primaries, on average, does it take before Eileen will again vote Republican? Until she abstains?

Solution: (i) We have a Markov chain with state space $E = \{1, 2, 3\}$, where 1,2,3 correspond to Democrat, Republican and Independent, respectively. The transition matrix is

$$P = \begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 4/5 & 1/5 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(ii) We find the long run probabilities, solving

$$\begin{aligned} \pi_1 &= \tfrac{2}{3}\pi_1 + \tfrac{1}{2}\pi_3 \\ \pi_2 &= \tfrac{4}{5}\pi_2 + \tfrac{1}{2}\pi_3 \\ \pi_3 &= \tfrac{1}{3}\pi_1 + \tfrac{1}{5}\pi_2 \\ 1 &= \pi_1 + \pi_2 + \pi_3 \end{aligned}$$

We get $\pi_1 = \frac{3}{10}$, $\pi_2 = \frac{1}{2}$, $\pi_3 = \frac{2}{10}$

(iii) We need to find $P^{(3)}_{2,2}$. We find $(0, 1, 0)P^3$. We have that

$$(0,1,0)P = (0,1,0)\begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 4/5 & 1/5 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (0, \tfrac{4}{5}, \tfrac{1}{5})$$

$$(0,1,0)P^2 = (0, \tfrac{4}{5}, \tfrac{1}{5})P = (0, \tfrac{4}{5}, \tfrac{1}{5})\begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 4/5 & 1/5 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (\tfrac{1}{10}, \tfrac{37}{50}, \tfrac{4}{25})$$

$$(0,1,0)P^3 = (\tfrac{1}{10}, \tfrac{37}{50}, \tfrac{4}{25})P = (\tfrac{1}{10}, \tfrac{37}{50}, \tfrac{4}{25})\begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 4/5 & 1/5 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (\tfrac{11}{75}, \tfrac{84}{125}, \tfrac{68}{375})$$

The probability that, three elections from now, Eileen votes Republican is $\frac{84}{125}$.

(iv) We take $A = \{2\}$, we need to find $\tilde{k}_2$ and k_3. First, we find k_1, k_2, k_3. We have that $k_2 = 0$,

$$\begin{aligned} k_1 &= 1 + \tfrac{2}{3}k_1 + \tfrac{1}{3}k_3, \\ k_3 &= 1 + \tfrac{1}{2}k_1, \end{aligned}$$

We get that $k_1 = 8$ and $k_3 = 5$. Hence, the average time until she votes Republican again is $\tilde{k}_2 = (4/5)(1) + (1/5)(1 + 5) = 2$. The average time until she votes abstain is $k_3 = 5$.

Using Theorem 4.10, it is possible to prove the following theorem:

Theorem 4.13. *(Gambler's ruin probability) Consider the Markov chain with states $E = \{0, 1, 2, \dots, N\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $1 \le i \le N - 1$, and $P_{0,0} = P_{N,N} = 1$. Let h_i be the probability that the Markov chain hits 0. Then,*

$$h_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

Theorem 4.14. *(Gambler's ruin probability against an adversary with infinite amount of money) Consider the Markov chain with states $E = \{0, 1, 2, \dots\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $1 \le i$, and $P_{0,0} = 1$. Let h_i be the probability that the Markov chain hits N. Then,*

$$P_k = \begin{cases} 1 - \left(\frac{q}{p}\right)^k & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \le \frac{1}{2} \end{cases}$$

Exercise 4.39. *A gambler has \$2 and he needs \$10. He play successive plays until he is broke or he gets \$10. In every play, he stakes \$1. The probability that the wins is $\frac{1}{2}$ and the probability that the losses is $\frac{1}{2}$. Find the probability that the gambler gets \$10. Find the average amount of time that the game last.*

Solution: The states are $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For $A = \{0, 10\}$, we need to find k_i. We setup the equations

$$\begin{aligned} k_0 &= k_{10} = 0 \\ k_9 &= \tfrac{1}{2}k_{10} + \tfrac{1}{2}k_8 \\ k_8 &= \tfrac{1}{2}k_9 + \tfrac{1}{2}k_7 \\ k_7 &= \tfrac{1}{2}k_8 + \tfrac{1}{2}k_6 \\ k_6 &= \tfrac{1}{2}k_7 + \tfrac{1}{2}k_5 \\ k_5 &= \tfrac{1}{2}k_6 + \tfrac{1}{2}k_4 \\ k_4 &= \tfrac{1}{2}k_5 + \tfrac{1}{2}k_3 \\ k_3 &= \tfrac{1}{2}k_4 + \tfrac{1}{2}k_2 \\ k_2 &= \tfrac{1}{2}k_3 + \tfrac{1}{2}k_1 \\ k_1 &= \tfrac{1}{2}k_2 + \tfrac{1}{2}k_0. \end{aligned}$$

which are equivalent to

$$\begin{aligned}
k_2 - k_1 &= k_1 - k_0 - 2 \\
k_3 - k_2 &= k_2 - k_1 - 2 \\
k_4 - k_3 &= k_3 - k_2 - 2 \\
k_5 - k_4 &= k_4 - k_3 - 2 \\
k_6 - k_5 &= k_5 - k_4 - 2 \\
k_7 - k_6 &= k_6 - k_5 - 2 \\
k_8 - k_7 &= k_7 - k_6 - 2 \\
k_9 - k_8 &= k_8 - k_7 - 2 \\
k_{10} - k_9 &= k_9 - k_8 - 2
\end{aligned}$$

So,

$$\begin{aligned}
k_1 - k_0 &= k_1 - k_0 \\
k_2 - k_1 &= k_1 - k_0 - 2(1) \\
k_3 - k_2 &= k_1 - k_0 - 2(2) \\
k_4 - k_3 &= k_1 - k_0 - 2(3) \\
k_5 - k_4 &= k_1 - k_0 - 2(4) \\
k_6 - k_5 &= k_1 - k_0 - 2(5) \\
k_7 - k_6 &= k_1 - k_0 - 2(6) \\
k_8 - k_7 &= k_1 - k_0 - 2(7) \\
k_9 - k_8 &= k_1 - k_0 - 2(8) \\
k_{10} - k_9 &= k_1 - k_0 - 2(9)
\end{aligned}$$

Adding all the last equations, we get that $0 = (10)k_1 - 2(1+2+\cdots+9) = 10k_1 - 90$. Hence, $k_1 = 9$ and $k_2 = 16$.

Exercise 4.40. *A gambler has \$2 and he needs \$10 in a hurry. He play successive plays until he is broke or he gets \$10. If has \$5 or less, he bets all his money. If he has more than \$5, he bets just enough to get \$10. The probability that the wins is $\frac{1}{2}$ and the probability that the losses is $\frac{1}{2}$. Find the probability that the gambler gets \$10. Find the average amount of time that the game last.*

Theorem 4.15. *Consider the Markov chain with states $E = \{0, 1, 2, \dots, N\}$ and one-step transition probabilities, $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$ for $1 \le i \le N-1$, and $P_{0,0} = P_{N,N} = 1$. Let k_i be the mean hitting time of $\{0, N\}$. Then,*

$$k_i = \begin{cases} \frac{1}{p-q}\left(\frac{N\left(1-\left(\frac{q}{p}\right)^i\right)}{1-\left(\frac{q}{p}\right)^N} - i\right) & \text{if } p \neq \frac{1}{2} \\ i(N-i) & \text{if } p = \frac{1}{2} \end{cases}$$

Let

$$f(i,j) = \mathrm{P}((X_n)_{n\ge1} \text{ ever transits to } j | X_0 = i)$$

If $f(j,j) = 1$, state j is recurrent. If $f(j,j) < 1$, state j is transient. If $i \neq j$, then $f(i,j) > 0$ if and only if $i \to j$.

Theorem 4.16. *Let N_j be the number of visits of $(X_n)_{n\ge 0}$ to state j. Let $s_{i,j} = E[N_j|X_0 = i]$. Then,*

(a) If j is recurrent, then $P(N_j = \infty|X_0 = j) = 1$ and $s_{j,j} = \infty$.

(b) If j is transient, then $P(N_j = m|X_0 = j) = (f(j,j))^{m-1}(1 - f(j,j))$, for $m = 1, 2, \dots$; and $s_{j,j} = \frac{1}{1-f(j,j)}$.

(c) If $i \ne j$ and $f(i,j) = 0$, then $P(N_j = 0|X_0 = i) = 1$, and $s_{i,j} = 0$.

(d) If $i \ne j$, $f(i,j) > 0$ and j is recurrent, then $P(N_j = 0|X_0 = i) = 1 - f(i,j)$ and $P(N_j = \infty|X_0 = j) = f(i,j)$; and $s_{i,j} = \infty$.

(e) If $i \ne j$, $f(i,j) > 0$ and j is transient, then

$$P(N_j = m|X_0 = i) = \begin{cases} 1 - f(i,j) & \text{if } m = 0 \\ f(i,j)(f(j,,j))^{m-1}(1 - f(j,j)) & \text{if } m = 1, 2, \dots \end{cases}$$

and $s_{i,j} = \frac{f(i,j)}{1-f(j,j)}$.

Theorem 4.17.

$$s_{ij} = \begin{cases} \frac{f(i,j)}{1-f(j,j)} & \text{if } i \ne j, f(j,j) < 1 \text{ and } f(i,j) > 0 \\ \frac{1}{1-f(j,j)} & \text{if } i = j, f(j,j) < 1 \text{ and } f(i,j) > 0 \\ \infty & \text{if } f(j,j) = 1 \text{ and } f(i,j) > 0 \\ 0 & \text{if } f(i,j) = 0 \end{cases}$$

Hence, for $f(j,j) < 1$ and $f(i,j) > 0$,

$$f(i,j) = \frac{s_{ij} - \delta_{ij}}{s_{jj}} \tag{4.1}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

To find $s_{i,j}$, when i and j are transient states, we can use the following method. Suppose that the states are numbered so that $T = \{1, 2, \dots, t\}$ are the transient states. Let i and let j be two transient states. Then, for any recurrent state k, $s_{k,j} = 0$. So, for $i \ne j$, $s_{ij} = \sum_{k=1}^{t} P_{ik}s_{kj}$ and $s_{jj} = 1 + \sum_{k=1}^{t} P_{ik}s_{kj}$. In order works,

$$s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik}s_{kj}, \text{ for } k = 1, \dots, t. \tag{4.2}$$

Let

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdot & \cdot & s_{1t} \\ s_{21} & s_{22} & s_{23} & \cdot & \cdot & s_{2t} \\ s_{31} & s_{32} & s_{33} & \cdot & \cdot & s_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{t1} & s_{t2} & 0 & \cdot & \cdot & s_{tt} \end{pmatrix}$$

let

$$P_T = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdot & \cdot & P_{1t} \\ P_{21} & P_{22} & P_{23} & \cdot & \cdot & P_{2t} \\ P_{31} & P_{32} & P_{33} & \cdot & \cdot & P_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{t1} & P_{t2} & 0 & \cdot & \cdot & P_{tt} \end{pmatrix}$$

and let

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 \end{pmatrix}.$$

In matrix notation, Equation (3.2) is $S = I + P_T S$. So, $S = (I - P_T)^{-1}$.

To find the inverse of a 2×2 matrix, we may use the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To find $f(i, j)$, when i is transient and j is recurrent, we may need to use Theorem 4.15. If $A = \{j\}$ and j is recurrent, then for each i, $h_i = f_{i,j}$. If $A = \{j\}$, then for each $i \neq j$, $h_i = f(i, j)$. If $A = \{j\}$ and j is transient, then $h_j = 1 > f(j, j)$.

Exercise 4.41. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.7 & 0 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}.$$

and states $E = \{1, 2, 3\}$. Find the communicating classes. Classify the states as transient or positive recurrent. Compute the matrix $(s_{i,j}))_{1 \leq i,j \leq 3}$ and the matrix $(f(i, j))_{1 \leq i,j \leq 3}$.

Solution: $\{1, 2\}$ is a recurrent communicating class. $\{3\}$ is a transient class. $P_T = \begin{pmatrix} 0.4 \end{pmatrix}$, $I - P_T = \begin{pmatrix} 0.6 \end{pmatrix}$ and $(I - P_T)^{-1} = \begin{pmatrix} 5/3 \end{pmatrix}$. Hence,

$$S = \begin{pmatrix} \infty & \infty & 0 \\ \infty & \infty & 0 \\ \infty & \infty & 5/3 \end{pmatrix}.$$

and

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0.4 \end{pmatrix}.$$

Exercise 4.42. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.2 & 0.4 & 0.4 \\ 0 & 0 & 1 \end{pmatrix}.$$

and states $E = \{1, 2, 3\}$. Find the communicating classes. Classify the states as transient or positive recurrent. Compute the matrix $(s_{i,j}))_{1\le i,j\le 3}$ and the matrix $(f(i,j))_{1\le i,j\le 3}$.

Solution: $\{1, 2\}$ is a transient communicating class. $\{3\}$ is a recurrent class. $P_T = \begin{pmatrix} 0.3 & 0.7 \\ 0.2 & 0.4 \end{pmatrix}$, $I - P_T = \begin{pmatrix} 0.7 & -0.7 \\ -0.2 & 0.6 \end{pmatrix}$ and

$$(I - P_T)^{-1} = \frac{1}{(0.7)(0.60) - (-0.7)(-0.2)} \begin{pmatrix} 0.7 & -0.7 \\ -0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} \frac{15}{7} & \frac{5}{2} \\ \frac{5}{7} & \frac{5}{2} \end{pmatrix}$$

Hence,

$$S = \begin{pmatrix} \frac{15}{7} & \frac{5}{2} & \infty \\ \frac{15}{7} & \frac{5}{2} & \infty \\ 0 & 0 & \infty \end{pmatrix}.$$

We also have that

$$\begin{aligned} f(1,1) &= \tfrac{s(1,1)-1}{s(1,1)} = \tfrac{\frac{15}{7}-1}{\frac{15}{7}} = \tfrac{8}{15} \\ f(1,2) &= \tfrac{s(1,2)}{s(2,2)} = \tfrac{\frac{5}{2}}{\frac{5}{2}} = 1 \\ f(2,1) &= \tfrac{s(2,1)}{s(1,1)} = \tfrac{\frac{5}{7}}{\frac{15}{7}} = \tfrac{1}{3} \\ f(2,2) &= \tfrac{s(2,2)-1}{s(2,2)} = \tfrac{\frac{5}{2}-1}{\frac{5}{2}} = \tfrac{3}{5} \end{aligned}$$

and

$$F = \begin{pmatrix} \frac{8}{15} & 1 & 1 \\ \frac{1}{3} & \frac{3}{5} & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4.43. *Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix*

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0.5 \end{pmatrix}.$$

and states $E = \{1, 2, 3, 4, 5\}$. Find the communicating classes. Classify the states as transient or positive recurrent. Compute the matrix $(s_{i,j}))_{1\le i,j\le 3}$ and the matrix $(f(i,j))_{1\le i,j\le 3}$.

Exercise 4.44. *Compute the matrix* $(s_{i,j})_{1,j=1,\dots,4}$ *and the matrix* $(f_{i,j})_{1,j=1,\dots,4}$ *for the transition matrix*

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

Exercise 4.45. *An HMO plans to assess resource requirements for its elderly members by determining the distribution of numbers of members whose health is classified into one of three states: state 1: healthy; state 2: severely impaired; state 3: dead. Changes in health are to be modeled using a 3-state, discrete-time, Markov chain. Transitions may occur at the end of each year and the matrix of one-step (one-year) transition probabilities is given by*

$$P = \begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.10 & 0.65 & 0.25 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) If an individual is currently healthy, what is the probability that she will ever become severely impaired?

(b) If an individual is severely impaired what is the probability that she will ever return to be healthy?

(c) For an insured individual who is currently healthy what is the expected number of years until death?

(d) For an insured individual who is currently severely impaired what is the expected number of years until death?

Solution: 1 and 2 are transient states. Hence,

$$P_T = \begin{pmatrix} \frac{3}{4} & \frac{3}{20} \\ \frac{1}{10} & \frac{13}{20} \end{pmatrix},$$

$$I - P_T = \begin{pmatrix} \frac{1}{4} & \frac{-3}{20} \\ \frac{-1}{10} & \frac{7}{20} \end{pmatrix}.$$

and

$$(I - P_T)^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{-3}{20} \\ \frac{-1}{10} & \frac{7}{20} \end{pmatrix}^{-1} = \frac{1}{\frac{1}{4}\frac{7}{20} - \frac{-3}{20}\frac{-1}{10}} \begin{pmatrix} \frac{1}{4} & \frac{-3}{20} \\ \frac{-1}{10} & \frac{7}{20} \end{pmatrix} = \begin{pmatrix} \frac{35}{16} & \frac{15}{16} \\ \frac{10}{16} & \frac{25}{16} \end{pmatrix}.$$

(a) $f(1,2) = \frac{s(1,2)}{s(2,2)} = \frac{\frac{15}{16}}{\frac{26}{16}} = \frac{3}{5}$.
(b) $f(2,1) = \frac{s(2,1)}{s(1,1)} = \frac{\frac{10}{16}}{\frac{35}{16}} = \frac{2}{7}$.
(c) $s(1,1) + s(1,2) = \frac{50}{16}$.
(d) $s(2,1) + s(2,2) = \frac{35}{16}$.

Problem 4.15. *(12, November 2003) A new disease has the following characteristics:*
(i) Once an individual contracts the disease, each year they are in only one of the following states with annual treatment costs as shown:

State	*Annual Treatment Costs*
Acutely ill	*10*
In remission	*1*
Cured or dead	*0*

Annual treatment costs are assumed not to change in the future.
(ii) Changes in state occur only at the end of the year.
(iii) 30% of those who are acutely ill in a given year are in remission in the following year and 10% are cured or dead.
(iv) 20% of those who are in remission in a given year become acutely ill in the following year and 30% are cured or dead.
(v) Those who are cured do not become acutely ill or in remission again.
Recall: To find the inverse of a 2×2 *matrix, we may use the formula*

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} a_{2,2}/d & -a_{1,2}/d \\ -a_{2,1}/d & a_{1,1}/d \end{pmatrix} \text{ where } d = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Calculate the expected total treatment costs, for the current and future years, for an individual currently acutely ill.
(A) 18 (B) 23 (C) 28 (D) 33 (E) 38

Solution: Let states be 1,2,3 for acutely ill, in remission, cured/dead. The transition matrix is

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}.$$

States 1 and 2 are transient. State 3 is recurrent. So,

$$P_T = \begin{pmatrix} 0.6 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}, I - P_T = \begin{pmatrix} 0.4 & -0.3 \\ -0.2 & 0.5 \end{pmatrix}, \text{ and } (I - P_T)^{-1} = \begin{pmatrix} \frac{25}{7} & \frac{15}{7} \\ \frac{10}{17} & \frac{20}{7} \end{pmatrix}$$

The expected total treatment costs, for the current and future years, for an individual currently acutely ill is

$$10s_{1,1} + s_{1,2} = 10\frac{25}{7} + \frac{15}{7} = \frac{265}{7} = 37.85714.$$

Exercise 4.46. *A restaurant business fluctuates in successive yeas between three states: 1 (solvency), 2 (serious debt) and 3 (bankruptcy). The transition probability between the states is*

$$P = \begin{pmatrix} 0.8 & 0.05 & 0.15 \\ 0.05 & 0.75 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the average life of new restaurant business. Find the average number of years which a new restaurant spends in serious debt before bankruptcy. If a restaurant is in serious debt, what is the average number of years it will last before bankruptcy? If a restaurant is in debt, what is the probability that it will get to the state of solvency.

Exercise 4.47. *In a certain company, a pc is one of the following states:*

State 1, working;

State 2, down for the day;

State 3, recycling room.

Changes in state take place at the end of a day. If the computer is operating at the beginning of a day there is a 2.5 % chance it will be down at the start of the next day. If the computer is down for the first day at the start of day, there is a 90 % chance that it still be operable for the start of the next day. If the computer is down for two consecutive days, it is automatically replaced by a new machine for the following day. Once that the pc gets to the recycling room is dismantled.

Calculate the average number of days a pc is being repaired before it is destroyed. Calculate the average number of days a pc is being used before it is destroyed.

Exercise 4.48. *On any given day Buffy is either cheerful (c), so–so (s) or gloomy (g). If she is cheerful today then she will be (c), (s), or (g) tomorrow with respective probabilities .7, .2 and .1. If she is so–so today then she will be (c), (s), or (g) tomorrow with respective probabilities .4, .3 and .3. If she is gloomy today then she will be (c), (s), or (g) tomorrow with respective probabilities .2, .4 and .4. What long run proportion of time is Buffy cheerful? If she is gloomy today, what is the average number of days that it will take her to get cheerful?*

Exercise 4.49. *(Skater boy problem) A boy and a girl move into a two–bar town on the same day. Each night the boy visits one of the bars, starting in bar A, according to a Markov chain with transition matrix*

$$P_{\text{boy}} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Likewise, the girl visits one of the other two bars according to a Markov chain transition matrix

$$P_{\text{girl}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

but starting at bar B. Assume that the two Markov chains are independent. Naturally the game ends when boy meets girl, i.e., when they go to the same bar. Argue that the progress of the game can described by a theree–state Markov chain:

State 1 is when the boy is at bar A and the girl at bar B.

State 2 is when the boy is at bar B and the girl at bar A.

State 3 is when the boy and the girl go to the same bar.

Find the transition matrix of this Markov chain. Let N denote the number of the night on which the boy meets the girl. Find $E[N]$.

4.7 Branching Processes

Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual will, by the end of its lifetime, have produced j new off springs

with probability P_j, $j \geq 0$, independent of the number produced by any other individuals. Let X_n be the size of the n–th generation. $(X_n)_{n\geq 0}$ is a Markov chain with state space $E = \{0, 1, 2, \dots\}$.
0 is an absorbing state: $P_{00} = 1$.
If $P_0 > 0$, the states $\{1, 2, \dots\}$ are transient.
If $P_0 = 0$ and $P_1 = 1$, all states are recurrent.
If $P_0 = 0$ and $P_1 < 1$, the states $\{1, 2, \dots\}$ are transient.

If $P_0 = 0$ and $P_1 < 1$, with positive probability $\lim_{n\to\infty} X_n = \infty$). This happens because any finite set of transient states $\{1, 2, \dots, N\}$ will be visited only finitely often. So, the population will either die out or its size will converge to infinity.

From now, let us assume that $P_j < 1$, for each $j \geq 0$.

Let $\mu = \sum_{j=0}^{\infty} jP_j$ be the mean number of the offspring of a single individual. Let $\sigma^2 = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$ be the variance of the number of offspring produced by a single individual. Let $Z_1, \dots, Z_{X_{n-1}}$ be the offspring of each of the X_{n-1} individuals, then $X_n = \sum_{i=1}^{X_{n-1}} Z_i$. We have that

$$E[X_n|X_{n-1}] = E[\textstyle\sum_{i=1}^{X_{n-1}} Z_i|X_{n-1}] = X_{n-1}\mu,$$
$$\mathrm{Var}(X_n|X_{n-1}) = \mathrm{Var}(\textstyle\sum_{i=1}^{X_{n-1}} Z_i|X_{n-1}) = X_{n-1}\sigma^2$$

So,

$$\begin{aligned} & E[X_n] = E[E[X_n|X_{n-1}]] = E[X_{n-1}]\mu, \\ & \mathrm{Var}(X_n) = E[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(E[X_n|X_{n-1}]) \\ = \ & E[X_{n-1}]\sigma^2 + \mathrm{Var}(X_{n-1})\mu^2 \end{aligned}$$

If $X_0 = 1$, by induction,

$$E[X_n] = \mu^n,$$
$$\mathrm{Var}(X_n) = \begin{cases} \sigma^2\mu^{n-1}\frac{\mu^n-1}{\mu-1} & \text{if } \mu = 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

Assuming that $P_0 > 0$ and $X_0 = 1$, let

$$h_i = \mathrm{P}((X_n)_{n\geq 0} \text{ hits } 0|X_0 = i).$$

h_i is the probability that the population dies out. Then,

$$h_i = \begin{cases} 1 & \text{if } \mu \geq 1 \\ h_1^i & \text{if } \mu < 1, \end{cases}$$

where h_1 is the smallest positive solution of the equation

$$h_1 = \sum_{j=0}^{\infty} P_j h_1^j = 1.$$

Observe that the function $f(x) = -x + \sum_{j=0}^{\infty} P_j$ is a convex function with $f(0) = 1$, $f(1) = 0$ and $f'(1) = \mu - 1$.

Exercise 4.50. *For a branching process, calculate h_1, when:*
(i) $P_0 = \frac{1}{2}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{4}$.
(ii) $P_0 = P_1 = P_2 = P_3 = \frac{1}{4}$.
(iii) $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$.

Exercise 4.51. *Consider a branching process X_n with $p_0 = \frac{1}{6}$ and $p_k = \frac{5}{2}\left(\frac{1}{4}\right)^k$ for $k \geq 1$. Find*

$$P\{\lim_{n\to\infty} X_n = \infty | X_0 = i\}.$$

Exercise 4.52. *Consider a branching process X_n whose family–size distribution $\{N(i) : 1 \leq i\}$ has mean μ and variance σ^2. Find*

$$E[X_j X_k | X_0 = i],$$

where $1 \leq j \leq k$. Hint:

$$E[X_n^2 | X_0 = 1] = \sigma^2 \mu^{n-1} \sum_{l=0}^{n-1} \mu^l + \mu^{2n}.$$

Exercise 4.53. *Consider a branching process X_n whose family–size distribution $\{N(i) : 1 \leq i\}$ has a binomial distribution with parameters 2 and p, $0 < p < 1$. Find the probability that X_n will become eventually extinct, if $X_0 = 3$. Hint:* $\Pr\{N(1) = 0\} = (1-p)^2$, $\Pr\{N(1) = 1\} = 2p(1-p)$ *and* $\Pr\{N(1) = 2\} = p^2$.

Exercise 4.54.

Let μ and σ^2 be the mean and variance of the family-size distribution of a branching process X_n. Find $\mathrm{Var}(X_n | X_0 = i)$. *Conclude that*

$$\mathrm{Var}(X_n | X_0 = i) \to \infty, \quad \text{if} \quad m \geq 1, i \geq 1$$

and

$$\mathrm{Var}(X_n | X_0 = i) \to 0, \quad \text{if} \quad m < 1, i \geq 1.$$

Exercise 4.55. *Consider a branching process X_n whose family–size distribution is* $\Pr\{N(1) = k\} = (1-r)r^k$, *for each $k \geq 0$. Find*

$$P\{\lim_{n\to\infty} X_n = \infty | X_0 = i\}.$$

Exercise 4.56. *Consider a braching process X_n whose family–size distribution is $P\{N(1) = 0\} = p$ and $P\{N(1) = 1\} = 1 - p$, where $0 < p < 1$. Find the probability that the process becomes extinct at or before the n–th generation, given that $X_0 = i$, $i \geq 1$. Hint: First, do the case $i = 1$. In the general case, let T be the time of extinction and let $T^{(k)}$ be the time of the extinction of the descendents of the k-th initial individual, $1 \leq k \leq i$. Observe that*

$$P\{T \leq n | X_0 = i\} = P\{T^{(k)} \leq n, \text{ for } \ 1 \leq k \leq i | X_0 = i\} = \prod_{1\leq k\leq i} P\{T^{(k)} \leq n | X_0 = i\}.$$

Exercise 4.57. *Let μ and σ^2 be the mean and variance of the family–size distribution of a branching process X_n. Show that*

$$E[X_n^2|X_{n-1}] = \sigma^2 X_{n-1} + \mu^2 X_{n-1}^2,$$

and

$$E[X_n^2|X_0 = i] = i\sigma^2\mu^{n-1} + \mu^2 E[X_{n-1}^2|X_0 = i].$$

Use this to show, by induction, that

$$E[X_n^2|X_0 = 1] = \sigma^2 m^{n-1} \sum_{j=0}^{n-1} \mu^j + \mu^{2n}.$$

Problems

1. Let X be a Markov chain with state space $E = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0.5 \end{pmatrix}.$$

   Find the communicating classes. Classify the states as transient or positive recurrent. Find the period of each communicating class.

2. Consider the Markov chain with state space $E = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0.25 & 0 & 0.75 & 0 \\ 0 & .6 & 0 & 0.4 \end{pmatrix}.$$

   Find the communicating classes. Classify the states as transient or positive recurrent. Find the period of each communicating class.

3. Let $\{X_n\}$ be a Markov chain with one–step transition probability matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 & 0 \end{pmatrix}.$$

   and states $E = \{1, 2, 3, 4, 5, 6, 7\}$. Find the communicating classes. Classify the states as transient or positive recurrent. Find the period of each communicating class.

4. **Random walk with an absorbing barrier.** Consider a Markov process on $E = \{0, 1, 2, \dots\}$ with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ q & 0 & p & 0 & . & . & . \\ 0 & q & 0 & p & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix},$$

where $0 < p < 1$. Find the communicating classes. Find the period. Classify the states as transient, positive recurrent or null recurrent. Find $h(i) = P\{(X_n)_{n=0}^{\infty} \text{ hits } 0 \mid X_0 = i\}$ and $k_i = E[H \mid X_0 = i]$, $i = 0, 1, \dots$, where H is the time the chain hits 0.

5. Compute the matrix $(s_{i,j})_{1,j=1,\dots,4}$ and the matrix $(f_{i,j})_{1,j=1,\dots,4}$ for the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

6. On any given day Buffy is either cheerful (c), so–so (s) or gloomy (g). If she is cheerful today then she will be (c), (s), or (g) tomorrow with respective probabilities .7, .2 and .1. If she is so–so today then she will be (c), (s), or (g) tomorrow with respective probabilities .4, .3 and .3. If she is gloomy today then she will be (c), (s), or (g) tomorrow with respective probabilities .2, .4 and .4. What long run proportion of time is Buffy cheerful? If she is gloomy today, what is the average number of days that it will take her to get cheerful?

7. Compute the potential matrix S and the matrix F for the transition matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0.5 \end{pmatrix}.$$

8. Consider a random walk on $E = \{0, 1, 2, \dots,\}$ with an reflecting barrier at 0. This is Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ q & 0 & p & 0 & \dots \\ 0 & q & 0 & p & \dots \\ 0 & 0 & q & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where $0 < p < 1$ and $p + q = 1$. Find a strictly positive solution to $\nu = \nu P$. Observe that you can find a solution even in the case when the Markov chain is transient.