STAT305, Winter 2004, Assignment 3 Due: Feb 25th

1. A commuter has two possible routes to work, A and B. There is construction activity on route A about 1 day in 20, and on route B about 1 day in 10. If the commuter takes route A and finds construction she switches to route B for the next day otherwise she uses A again. If the commuter takes route B and finds construction she switches to route A for the next day; otherwise she uses B again.

   In the long run on what fraction of days does she commute via route A and on what fraction of her trips does she find construction? Your answer must clearly state what assumptions you are making to answer the question. Answers with inadequate explanations will get low marks.

2. In a sequence of Bernoulli trials we say that at time n the state E_1 is observed if the trials number $n-1$ and n resulted in SS. Similarly E_2, E_3, E_4 stand for SF, Find the matrix P and all its powers.

3. A Markov Chain has state space $\{1, 2, 3, 4\}$ and transition matrix

$$\mathbf{P}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   a) Identify all the communicating classes and say whether or not each state is transient.

   b) let $q_k = \mathrm{P}(\text{there is an n such that } X_n = 4 | X_0 = k\ )$, Derive the equations $q_2 = (q_2 + q_3)/3$ and $q_3 = (q_2 + 2q_3 + 1)/4$.

4. **Branching with immigration** Each generation of a branching process (with a single progenitor) is augmented by a random number of immigrants who are indistinguishable from the other members of the population. Suppose that the numbers of immigrants in different generations are independent of each other and of the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function G_n of the size of the nth generation satisfies $G_{n+1}(s) = G_n(G(s))H(s)$, where G is the probability generating function of a typical family of offspring.

5. A bank assesses the credit-worthness of various firms every quarter; the ratings are, in order of decreasing merit, A, B, C, D (default). A researcher proposes a Markov chain model for the credit rating of a typical firm with the following transition matrix:

$$\mathbf{P}_2 = \begin{pmatrix} 1\text{-}\alpha - \alpha^2 & \alpha & \alpha^2 & 0 \\ \alpha & 1\text{-}2\alpha - \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & 1\text{-}2\alpha - \alpha^2 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some parameter α.

(a) Draw the transition graph of the chain and determine the range of values of α for which the matrix P is a valid transition matrix.

(b) State, with reasons, whether the chain is irreducible and aperiodic.

(c) Derive the stationary probability distribution for the chain.

(d) For the value $\alpha = 0.1$, calculate the probability that the company's rating in the third quarter, X_3, is in the default state D:

(i) given that the company's rating in the first quarter, X_1, is A

(ii) given that the company's rating in the first quarter, X_1, is B

(iii) given that the company's rating in the first quarter, X_1, is C

(iv) given that the company's rating in the first quarter, X_1, is D

6. The members of a disability insurance scheme are classified as "active"(A),"temporarily disabled" (T), "permanetly disabled" (P) or "dead" (D). Members are entitled to benifits when they are in state T or P. Using the following discrete time (in years) Markov chain model with transtion matrix

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.1 & 0.05 & 0.1 \\ 0.5 & 0.3 & 0.1 & 0.1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Draw the transition graph of the chain and find its stationary probability distribution.

(ii) Calculate the mean duration of permanent disability benifit.

(iii) Caculate the probability that a member, initially active, is either temporarily or permanently disabled three years after the start of the scheme.

(iv) Calculate the probability that a member, initially active, will never draw any benifit from the scheme.

7. Consider the following strategy for comparing two medical treatments, say treatment A and treatment B. Patients are treated one at a time and the result of each treatment is recorded as a Success or Failure. Every time a treatment succeeds the next patient is treated with the same treatment which was just successful. When a treatment fails, the next patient is treated with the other treatment. Suppose that the probability that treatment A succeeds is P_A while the probability that treatment B succeeds is P_B. In the long run what fraction of patients are treated with treatment B?

8. (a) Exercise 6, on page 253 of the text.

(b) Exercise 14, page 254 of the text.

(c) Exercise 25, page 256 of the text.