I. **Branching Process Introduction**

A. General

1. Branching processes (BP) are particularly relevant for biological processes.
2. BP models populations and tracks the size of the population over time.
3. DTBP is a type of DTMC with infinite state space (Markov matrix is less convenient to work with).
4. In a DTBP model, particles in a population die and reproduce at each time-point to produce the next generation. The offspring of each particle are IID.

B. **Definition:** discrete-time branching process
Let $P(\xi = k) = p_k, k = 0, 1, \dots$ be the probability mass function for the number of offspring generated by each particle. If the particles behave independently, then the size of each generation $\{X_n, n \geq 0\}$ follows a DTBP.

C. To use DTBP, the following are requirements:

1. All particles behave independently
2. A reasonable description of the *offspring distribution* (p_k) exists
3. Initial population size distribution $\Leftrightarrow$ initial state distribution

D. Death followed by reproduction
Branching processes model reproduction as occurring simultaneous with death. If you wish to model a particle that gives birth multiple times before dying, the solution is to have the particle reborn at each death. Then, after the first generation, there is one new particle and the original particle, now reborn. When the particle eventually dies, no particle appears to replace it in the next generation.

E. Fundamental branching process equation
$\{X_n\}$ is a Markov process. To see that $P(X_n = j \mid X_0, X_1, \dots, X_{n-1} = i) = P(X_n = j \mid X_{n-1} = i)$, consider that if there are i independent particles in the $(n-1)$th generation, then

$$X_n = \xi_1 + \xi_2 + \cdots + \xi_i,$$

where ξ_k are the iid number of offspring for all parent particles k. This is the *fundamental branching process equation.*

F. Examples

1. Surname or families

2. Survival of mutant genes or organisms
   Assume that one rare mutant initiates process. Assume all mutants reproduce in the same fashion (their offspring are iid r.v.). But, what is a reasonable offspring distribution?
   The Poisson distribution is frequently used, so

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!},$$

   where λ is the expected number of offspring. The Poisson distribution is justified as follows. Suppose that a mother producess a lot of eggs, say N. Suppose the probability that an egg is successfully fertilized is p. Suppose that the probability that the fertilized egg successfully grows to adulthood is q. Then, the number of offspring of this mother is

$$Y \sim \text{Binomial}(N, pq)$$

   follows a Binomial distribution with success probability pq. However, if the population size M is not expanding or shrinking, then she produces $2 = \lambda = Npq$ offspring on average to replace herself and the sperm-donor. Now, let $N \to \infty$ and $pq \to 0$. The law of rare events insures that Y is approximately Poisson distributed with mean λ.

3. How many grandchildren produced out of this room?
   We shall assume we and our children will behave independently and identically at least with respect to the number of children we and they will produce.

Number of offspring	$P(\xi = i)$
0	$\frac{1}{12}$
1	$\frac{1}{12}$
2	$\frac{5}{12}$
3	$\frac{3}{12}$
4	$\frac{2}{12}$
$E(\xi)$	$\frac{28}{12} = \frac{7}{3}$

   Let X_i be the number of children of the ith person in the room.
   Let X_{ij} be the number of children of the jth child of the ith person in this room.
   Then, the total number of grandchildren is

$$X = X_{11} + X_{12} + \cdots + X_{1X_1} + X_{21} + X_{22} + \cdots X_{2X_2} + \cdots + X_{12X_{12}}.$$

Take expectations,

$$\begin{aligned} E(X) &= E\left(X_{11} + X_{12} + \cdots + X_{1X_1}\right) + E\left(X_{21} + X_{22} + \cdots + X_{2X_2}\right) + \\ &\quad \cdots + E\left(X_{12,1} + \cdots X_{12X_{12}}\right), \end{aligned}$$

but

$$E(X_{j1} + \cdots + X_{jX_j}) = E(X_j)E(\xi) = [E(\xi)]^2$$

for all people j and where ξ generically represents the number of offsprings of these independent and reproductively identical particles. Thus, the total number of grandchildren is

$$E(X) = 12\left(\frac{7}{3}\right)^2.$$

We expect some 65 grandchildren to come out of this class. From 12 (actually 24 when you include the second parent) to 65. We're growing!

II. Means and Variances

A. Preliminaries

Let $E(\xi) = \mu$. Let $\text{Var}(\xi) = \sigma^2$.
Let $M(n) = E(X_n)$. Let $V(m) = \text{Var}(X)$.
Assume $X_0 = 1$.

B. Recursion for means

Claim: $M(n+1) = \mu M(n)$
Proof:

$$\begin{aligned} M(n+1) &= E(X_{n+1}) \\ &= E(\xi_1 + \xi_2 + \cdots + \xi_{X_n}) \\ &= E(X_n)E(\xi) \\ &= \mu M(n) \end{aligned}$$

Claim: The solution of the above recursion equation is $M(n) = \mu^n$.
Proof: The proof is by induction.
Anchor: Show $M(0) = \mu^0 = 1$. $X_0 = 1$ by assumption, so this is true.
Assume true for n: $M(n) = \mu^n$
Prove true for $n+1$: $M(n+1) = \mu M(n)$ by recursion, but the assumption shows $M(n+1) = \mu\mu^n = \mu^{n+1}$.

C. Recursion for variances

Claim: $V(n+1) = M(n)\sigma^2 + V(n)\mu^2$

Proof:

The proof uses the variance formula for sums or independent random variables.

$$\begin{aligned} V(n+1) &= \mathrm{Var}(X_{n+1}) \\ &= E(X_n)\mathrm{Var}(\xi) + \mathrm{Var}(X_n)\left[E(\xi)\right]^2 \\ &= M(n)\sigma^2 + V(n)\mu^2. \end{aligned}$$

Claim:

$$V(n) = \sigma^2\mu^{n-1}\begin{cases} n & \mu = 1 \\ \frac{1-\mu^n}{1-\mu} & \mu \neq 1 \end{cases}$$

Proof:

Anchor: Show formula true for $n = 0$, i.e. $V(0) = 0$. But $X_0 = 1$, so it is true.

Assume true for n.

Prove true for $n+1$:

$$\begin{aligned} V(n+1) &= M(n)\sigma^2 + V(n)\mu^2 \\ &= \mu^n\sigma^2 + \sigma^2\mu^{n+1}\begin{cases} n & \mu = 1 \\ \frac{1-\mu^n}{1-\mu} & \mu \neq 1 \end{cases} \\ &= \begin{cases} \sigma^2(n+1) & \mu = 1 \\ \sigma^2\mu^n\left[1 + \frac{\mu(1-\mu^n)}{1-\mu}\right] & \mu \neq 1 \end{cases} \\ &= \begin{cases} \sigma^2(n+1) & \mu = 1 \\ \sigma^2\mu^n\left[\frac{1-\mu+\mu-\mu^{n+1}}{1-\mu}\right] & \mu \neq 1 \end{cases} \\ &= \begin{cases} \sigma^2(n+1) & \mu = 1 \\ \sigma^2\mu^n\left[\frac{1-\mu^{n+1}}{1-\mu}\right] & \mu \neq 1 \end{cases} \end{aligned}$$

D. Implications

Condition on μ	Outcome
$\mu = 1$	Population mean unchanging, but variance increases linearly.
$\mu > 1$	Population mean and variance grow geometrically.
$\mu < 1$	Population mean and variance shrink geometrically.

III. Extinction

A. Preliminaries

When $X_n = 0$, then $X_{n+i} = 0$ for all $i \geq 0$ since particles cannot be generated from nothing. In other words, 0 is an absorbing state.

Let T be the random time of extinction, i.e.

$$T = \min_{n \geq 0}\{X_n = 0\}.$$

B. Solution by First-Step Analysis

Let $U_n = P(N \leq n \mid X_0 = 1) = P(X_n = 0 \mid X_0 = 1)$. Then,

$$\begin{aligned}
U_n &= p_0 + p_1 P(\text{the single offspring's family dies out in } n-1 \text{ remaining generations}) \\
&\quad + p_2 P(\text{the two offsprings' families die out in } n-1 \text{ remaining generations}) \\
&\quad + \cdots \\
&= p_0 + p_1 U_{n-1} + p_2 \left(U_{n-1}\right)^2 + \cdots \\
&= \sum_{k=0}^{\infty} p_k \left(U_{n-1}\right)^k
\end{aligned}$$

The preceding recursion equation can be solved to find U_n.