Probability Generating Function

A. Definition: probability generating function(pgf)

The nonnegative integer-valued r.v. X with probability mass function(pmf) $p_k = P(X = k)$ has probability generating function

$$\phi(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k, \tag{0.1}$$

where $0 \leq s \leq 1$ is a dummy variable.

B. Knowing the pgf is equivalent to knowing the pmf(pgf $\equiv$ pmf)

1. Obviously knowing the pmf allows us to apply equation (1) to get the pgf.
2. Retrieving the pmf from pgf involves derivatives

$$\begin{aligned}
p_0 &= \phi(0) \\
p_1 &= \frac{1}{1!}\frac{d}{ds}\phi(0) = \frac{1}{1!}\frac{d}{ds}\sum_{k=0}^{\infty} p_k s^k|_{s=0} = \sum_{k=1}^{\infty} k p_k s^{k-1}|_{s=0} = 1.p_1 = p_1 \\
p_2 &= \frac{1}{2!}\frac{d^2}{ds^2}\phi(0) = \frac{1}{2!}\frac{d}{ds}\sum_{k=0}^{\infty} p_k s^k|_{s=0} = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-2}|_{s=0} = \frac{1}{2}.2p_2 = p_2 \\
\vdots &= \vdots \\
p_n &= \frac{1}{n!}\frac{d^n}{ds^n}\phi(0) = \frac{1}{n!}\frac{d^n}{ds^n}\sum_{k=0}^{\infty} p_k s^k|_{s=0} = \ldots = p_n
\end{aligned}$$

C. Product formula for generating functions

Suppose $Y_n = X_1 + X_2 + \ldots + X_n$ be the sume of n independent random variables where X_i has pgf $\phi_i(s)$. Then,

$$\phi_{Y_n}(s) = \phi_1(s)\ldots\phi_n(s).$$

Proof. Proof by induction.

Note: Clearly $\phi_{Y_1}(s) = \phi_1(s)$ when $n = 1$. Assume: $\phi_{Y_{n-1}}(s) = \phi_1(s)\ldots\phi_{n-1}(s)$. Show the result: Let $Y_{n-1} = X_1 + X_2 + \ldots + X_{n-1}$. By assumption, its pgf is

$$\phi_{Y_{n-1}}(s) = \phi_1(s)\ldots\phi_n(s).$$

And X_n has pgf $\phi_n(s)$. Now

$$Y_n = Y_{n-1} + X_n.$$

Consider its pgf

$$\begin{aligned}
\phi_{Y_n}(s) &= \mathrm{E}(s^{Y_{n-1}+X_n}) = \mathrm{E}(s^{Y_{n-1}} s^{X_n}) \\
&= \mathrm{E}(s^{Y_{n-1}})\mathrm{E}(s^{X_n})(\text{by independence of } Y_{n-1} \text{ and } X_n) \\
&= \phi_{Y_{n-1}}(s)\phi_n(s) = \phi_1(s)\ldots\phi_{n-1}(s)\phi_n(s).
\end{aligned}$$

D. Retrieving mean and variance

$$\frac{d\phi(s)}{ds}|_{s=1} = \sum_{k=1}^{\infty} k p_k s^{k-1}|_{s=1} = \sum_{k=1}^{\infty} k p_k = \mathrm{E}(X).$$

$$\frac{d^2\phi(s)}{ds^2}|_{s=1} = \sum_{k=1}^{\infty} k(k-1)p_k|_{s=1} = \sum_{k=2}^{\infty} k(k-1)p_k = \mathrm{E}(X(X-1)) = \mathrm{E}(X^2) - \mathrm{E}(X).$$

Recall

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\
&= \frac{d^2\phi(s)}{ds^2}|_{s=1} + \frac{d\phi(s)}{ds}|_{s=1} - \left[\frac{d\phi(s)}{ds}|_{s=1}\right]^2.
\end{aligned}$$

E. Examples

1. Poisson random variable: The pmf is $p_k = \dfrac{e^{-\lambda}\lambda^k}{k!}$. gives us the pgf

$$\phi(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} \frac{(s\lambda)^k e^{-\lambda-\lambda s+\lambda s}}{k!} = e^{\lambda s-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k e^{-\lambda s}}{k!} = e^{\lambda(s-1)}.$$

To obtain the mean and variance, take derivatives

$$\begin{aligned}
\frac{d\phi(s)}{ds}|_{s=1} &= \lambda e^{\lambda(s-1)}|_{s=1} = \lambda \\
\frac{d^2\phi(s)}{ds^2}|_{s=1} &= \lambda^2 e^{\lambda(s-1)}|_{s=1} = \lambda^2 \\
\mathrm{Var}(X) &= \lambda^2 + \lambda - \lambda^2 = \lambda.
\end{aligned}$$

2. Geometric The pmf

$$p_k = p(1-p)^k, \quad k = 0, \dots$$

gives the pgf

$$\phi(s) = \sum_{k=0}^{\infty} s^k (1-p)^k p = \sum_{k=0}^{\infty} ps(1-p)^k = \frac{p}{1-s(1-p)}.$$

To obtain the mean, we take the derivative

$$\frac{d\phi(s)}{ds}|_{s=1} = \frac{[1-s(1-p)].0 + p(1-p)}{[1-s(1-p)]^2} = \frac{p(1-p)}{[1-(1-p)]^2} = \frac{(1-p)}{p}.$$

The variance can obtained similarly.

F. Random sums

Suppose $Y = X_1 + X_2 + \dots + X_N$, where N is a random variable with pgf $\phi_N(s)$ and X_i are i.i.d. with pgf $\phi_X(s) = \mathrm{E}(s^X)$.

Claim: $\phi_Y(s) = \phi_N[\phi_X(s)]$.

Proof:

$$\begin{aligned}
\phi_Y(s) &= \sum_{k=0}^{\infty} P(Y=k)s^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} P[Y=k|N=n)P(N=n) \right] s^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} P(X_1 + X_2 + \dots + X_n = k|N=n)P(N=n) \right] s^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} P(Y=k)P(N=n) \right] s^k \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} P(Y=k)s^k \right] P(N=n) \\
&= \sum_{n=0}^{\infty} [\phi_X(s)]^n P(N=n) \\
&= \phi_N[\phi_X(s)].
\end{aligned}$$

II. pgf's and branching processes

A. Generating function of a branching process(BP)

1. Definition: Offspring generating function

Let the offspring probability function be $\phi_X(s)$. We call this the *offspring generating function.*

2. **Claim:** probability generating function of BP at generation n is nth functional iteration of offspring generating function. Recall $Y_n = X_1 + X_2 + \ldots X_{Y_{n-1}}$, so the nth generation is a random sum of length Y_{n-1}. Then assuming $Y_0 = 1$,

$$\begin{aligned}
\phi_{Y_1}(s) &= \phi_X(s) \\
\phi_{Y_2}(s) &= \phi_X[\phi_X(s)] \\
\vdots &= \vdots \\
\phi_{Y_{n(s)}} &= \phi_Y(\phi_Y(\ldots\phi_Y(s))).
\end{aligned}$$

So the generating function of the nth generation is obtained by functional iteration of the function $\phi_X(s)$ n times.

B. Generating function for extinction probability.

Recall that $u_n = P(Y_n = 0)$ is the probability that extinction (absorption) has happened by time n. It can be shown that

$$u_n = \sum_{k=0}^{\infty} p_k(u_{n-1})^k = \phi_X(u_{n-1}),$$

so the extinction probabilities can be obtained through functional recursion on the offspring generating function.

III. Example: birth and death process

Suppose that the particles modeled in a branching process can either die at the end of time period, leaving no offspring, or can give birth to one offspring while continuing to live to the next time point. In other words, each particle either produces 0 or 2 offspring after each interval in time. So, the offspring generating function is

$$\phi_X(s) = p_0 + p_2 s^2,$$

where $p_2 = 1 - p_0$ is the probability of birth and p_0 is the probability of death. If we assume the process starts with one particle ($Y_0 = 1$).

$$
\begin{aligned}
\phi_0(s) &= s \\
\phi_1(s) &= p_0 + p_1 s^2 \\
\phi_2(s) &= p_0 + p_1(p_0 + p_1 s^2)^2 \\
\phi_3(s) &= p_0 + p_1(p_0 + p_1(p_0 + p_1 s^2))^2 \\
\vdots &= \vdots \\
u_0 &= 0 \\
u_1 &= p_0 \\
u_2 &= p_0 + p_2 p_0^2 \\
u_3 &= p_0 + p_2 p_0^2 (1 + p_0 p_2)^2 \\
\vdots &= \vdots
\end{aligned}
$$