STAT 305

Solutions

1. Page 226 numbers 2 and 3.

S has eight states

$$\{SSS, SSR, SRS, SRR, RSS, RSR, RRS, RRR\}$$

Notice that the first two letters in state $n+1$ must match the last two letters in state n because they refer to the same days.

For the states in the order above:

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

2. Page 226 number 5.

Notice that $\{Y_k = 1\}$ is the same as $\{X_k = 0\}$ and that any time $Y_k=0$ we must have $Y_{k+1}=1$. Notice also that if two consective Y s are equal to 1 then both corresponding X values must be 1, not 2. Compute

$$\begin{aligned} P(Y_4 = 1 | Y_3 = 1, Y_2 = 1, Y_1 = 1, Y_0 = 0) \\ &= P(X_4 = 1 | X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 0) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} P(Y_4 = 1 | Y_3 = 1, Y_2 = 0, Y_1 = 1, Y_0 = 0) \\ &= P(X_4 \in 1,2 | X_3 \in \{1,2\}, X_2 = 0, X_1 = 1, X_0 = 0) \\ &= \frac{1}{4} \end{aligned}$$

If Y were a Markov chain these two probabilities would have had to be equal so Y is **not** a Markov Chain.

I saw a lot of answers that said $P(Y_2=1|Y_1=1)$ depends on X_1. This is not true. If A and B define events then $P(A|B)$ never ``depends'' on some other event. However, in this problem

$$P(Y_2=1|Y_1=1)$$

does depend on the initial distribution of the X_n chain.

3. Page 228 number 14.

If X_n is the coin you flip on the n th toss (labelling the first toss with 0) then this is a two state Markov chain with

$$\mathbf{P} = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}$$

Setting $\alpha \mathbf{P} = \alpha$ and $\alpha_1 + \alpha_2 = 1$ gives

$\alpha_1 = .6\alpha_1 + .5\alpha_2$

$\alpha_2 = 1 - \alpha_1$

which gives $.9\alpha_1 = .5$ or $\alpha_1 = 5/9$ and $\alpha_2 = 4/9$. Coin 1 is used 5/9 of the time.

Compute $\mathbf{P}^4$ and take out the 1,1 element to get 1111/2500=0.4444. I also accepted, though I think the English is fairly clear, use of $\mathbf{P}^5$ to get
11111/25000 which is approximately 0.44444.

4. Page 152 number 45.

We are told

$$\mathrm{E}(X_n | X_{n-1} = x) = 0$$

and

$$\mathrm{Var}(X_n | X_{n-1} = x) = \mathrm{E}(X_n^2 X_{n-1} = x) = \beta x^2$$

Thus:

$$\mathrm{E}(X_n | X_{n-1}) = 0$$

and

$$\mathrm{E}(X_n^2 | X_{n-1}) = \beta X_{n-1}^2$$

Now

$$\begin{aligned} \mathrm{E}(X_n) &= \mathrm{E}\left\{\mathrm{E}(X_n | X_{n-1})\right\} \\ &= \mathrm{E}(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(X_n) &= \mathrm{E}(X_n^2) \\ &= \mathrm{E}\left\{\mathrm{E}(X_n^2 | X_{n-1})\right\} \\ &= \beta \mathrm{E}(X_{n-1}^2) \\ &= \beta \mathrm{Var}(X_{n-1}) \end{aligned}$$

Thus

$$\mathrm{Var}(X_n) = \beta^{n-1} \mathrm{Var}(X_1)$$

Now starting from $X_0=x_0$ we know

$$\mathrm{Var}(X_1) = \beta x_0^2$$

so

$$\mathrm{Var}(X_n) = \beta^n x_0^2$$

5. Page 152 number 43.

First

$$\begin{aligned}\mathrm{E}\left\{\mathrm{Var}(X|Y)\right\} &= \mathrm{E}\left(\mathrm{E}\left[\{X - E(X|Y)\}^2 \,|Y\right]\right)\\ &= \mathrm{E}\left[\{X - E(X|Y)\}^2\right]\\ &= \mathrm{E}(X^2) - 2\mathrm{E}\left\{X\mathrm{E}(X|Y)\right\} + \mathrm{E}\left[\{\mathrm{E}(X|Y)\}^2\right]\end{aligned}$$

Also

$$\begin{aligned}\mathrm{Var}\{\mathrm{E}(X|Y)\} &= \mathrm{E}\left[\{\mathrm{E}(X|Y)\}^2\right] - [\mathrm{E}\left\{\mathrm{E}(X|Y)\right\}]^2\\ &= \mathrm{E}\left[\{\mathrm{E}(X|Y)\}^2\right] - \{\mathrm{E}(X)\}^2\end{aligned}$$

Now the tricky bit. Remember that $\mathrm{E}(X|Y)$ is a function of Y and that

$$\mathrm{E}(A(Y)X|Y) = A(Y)\mathrm{E}(X|Y)$$

for **any** function $A(Y)$. Then

$$\{\mathrm{E}(X|Y)\}^2 = \mathrm{E}\left\{X\mathrm{E}(X|Y)|Y\right\}$$

so

$$\mathrm{E}\left[\{\mathrm{E}(X|Y)\}^2\right] = \mathrm{E}\left\{X\mathrm{E}(X|Y)\right\}$$

Then $\mathrm{E}\left\{\mathrm{Var}(X|Y)\right\} = \mathrm{E}(X^2) - \mathrm{E}\left\{X\mathrm{E}(X|Y)\right\}$

and

$$\mathrm{Var}\{\mathrm{E}(X|Y)\} = \mathrm{E}\left\{X\mathrm{E}(X|Y)\right\} - \{\mathrm{E}(X)\}^2$$

Add these together to get

$$\mathrm{E}\left\{\mathrm{Var}(X|Y)\right\} + \mathrm{Var}\{\mathrm{E}(X|Y)\} = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2 = \mathrm{Var}(X)$$

6. Page 156 number 65.

The probability that player 1 wins the tournament is just the probability s/he wins n consecutive games. S/he has probability p of winning a given game so the probability of winning n consecutive is p^n.

There are two ways for player 2 to win the tournament. One way is for player 1 to be knocked out before the round where 1 and 2 would meet. Given that this happens player 2 must win n games all against opponents numbered more than 2. Thus the conditional probability that 2 wins the tournament given s/he never has to play 1 is p^n. The other way for 2 to win is to beat play 1 when they meet and win n -1 other games against opponents numbered more than 2. Given that player 1 survives to the round where 1 and 2 are to play the probability that 2 wins is thus $p^{n-1}(1-p)$.

In round 1 there are 2^n-1 possible opponents for player 1. There is 1 chance in 2^n-1 that player 2 is the chosen opponent so the chance that players 1 and 2 are scheduled to meet in round 1 is $1/(2^n-1)$. Player 2 is scheduled to meet player 1 in round 2 if player 2 is assigned to either of the slots in the pair whose winner plays the winner of player 1s first round pair. Thus the probability they are scheduled to meet in round 2 is $2/(2^n-1)$. In general the probability they are scheduled to meet in round k is $2^{k-1}/(2^n-1)$. Given that they are scheduled to meet in round k the chance that player 1 survives to that round is p^{k-1}. This gives a total probability of

$$\sum_{k=1}^{n} \frac{2^{k-1}}{2^n-1}\left\{p^{k-1}p^{n-1}(1-p)+(1-p^{k-1})p^n\right\}$$

This simplifies to

$$p^n + p^{(n-1)}(1-(2p)^n)/(2^n-1)$$

(If $p>1/2$ the second term is actually negative.)

7. Page 150 number 31.

Condition on Z the number of sixes on the first roll. Then

$$\mathrm{E}(N)=\mathrm{E}(\mathrm{E}(N|Z))$$

But

$$\mathrm{E}(N|Z=z)=1+m_{n-z}$$

where $m_0=0$ by convention. Now Z has a Binomial(n,1/6) distribution so

$$\mathrm{E}(N)=1+\sum_{0}^{n} m_{n-z}\binom{n}{z}\left(\frac{1}{6}\right)^z\left(\frac{5}{6}\right)^{n-z}$$

For $n=1$ N has a Geometric(1/6) distribution so $m_1=6$. Then

$$m_2=1+2\frac{5}{36}m_1+\frac{25}{36}m_2$$

giving

$$m_2=\frac{96}{11}$$

Next

$$m_3=1+\frac{125}{216}m_3+3\frac{25}{216}m_2+3\frac{5}{216}m_1$$

which makes

$$m_3=\frac{10566}{1001}$$

Similarly

$$m_4 = \frac{728256}{61061}$$

and

$$m_5 = \frac{3698650986}{283994711}$$

In part b let Y_i be the number of throws made using die number i. Then Y_i is geometric, $\mathrm{E}(Y_i) = 6$ and

$$\sum_1^n Y_i = \sum_{i=1}^N X_i$$

so the desired expectation is $6n$.

8. Page 227 number 10.

For $\mathbf{P}_1$ there is only the class $\{1, 2, 3\}$ which is recurrent.

For $\mathbf{P}_2$ there is only the class $\{1, 2, 3, 4\}$ which is recurrent.

For $\mathbf{P}_3$ the classes are $\{1, 3\}$, $\{2\}$ and $\{4, 5\}$. The first and last are recurrent; $\{2\}$ is transient.

For $\mathbf{P}_4$ the classes are $\{1, 2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. Classes $\{1, 2\}$ and $\{3\}$ are recurrent; the other two are transient.

9. Page 228 number 18.

The Markov Chain X_n is the remainder after Y_n is divided by 13. The transition matrix is

$$\begin{bmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

This matrix has columns which add to 1 so according to number 16 the stationary transition probabilities are all 1/13 which is the desired answer.

10. Page 228 number 19.

The state space for this chain is $\{SS, SF, FS < FF\}$ where *SF* means, for instance that the current trial was a failure and the one previous a success. Labelling the states in this order

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

The equations $\pi\mathbf{P} = \pi$ and $\sum \pi_i = 1$ become

$$\pi_1 = 0.8\pi_1 + 0.5\pi_3$$

$$\pi_2 = 0.2\pi_1 + 0.5\pi_3$$

$$\pi_3 = 0.5\pi_2 + 0.5\pi_4$$

$$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4$$

From the first equation

$$0.2\pi_1 = 0.5\pi_3$$

Substitute in the second equation to get

$$\pi_2 = \pi_3$$

Use this in the third equation to get

$$\pi_3 = \pi_4$$

All these in the final equation give

$$1 = \pi_3(5/2 + 1 + 1 + 1)$$

so that the solution is

$$(\frac{5}{11}, \frac{2}{11}, \frac{2}{11} \frac{2}{11})$$

Whenever the chain is in state *SS* or *FS* the current trial is a success. This happens

$$\frac{5}{11} + \frac{2}{11} = \frac{7}{11}$$

of the time.