# STAT305, Fall 2001

1. Suppose that in an experiment, a fair die is rolled twice. Let $A$=\{the first outcome is even\}, $B$=\{the total score is 4\}, $C$= the total score, $D$=the absolute difference between two scores.

   (a) Which of $A$, $B$, $C$, $D$ are events? Which of them are random variables?

   (b) Which of the following make sense? Which of them do not?

   (i) $A \cup B$, (ii) $P(C)$, (iii) $E(A)$, (iv) Var(D).

2. Let $S$ be the sample space of an particular experiment, $A$ and $B$ be events, and $P$ be a probability measure. Which of the followings are Axioms, definitions and formulas?

   (i) $P(A \cup B) = P(A) + P(B) - P(AB)$.

   (ii) $P(S) = 1$.

   (iii) $P(A|B) = P(AB)/P(B)$ when $P(B) \neq 0$.

3. If $X$ and $Y$ are two random variables, what do we mean by

   (i) $F(x)$ is the cumulative distribution function of $X$?

   (ii) $X \leq 4$ is independent of $Y \geq 2$?

4. Using only the axioms of probability, show that

   1) $P(A \cup B) = P(A) + P(B) - P(AB)$

   2) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$.

5. a) Prove that $P(ABC) = P(A|BC)P(B|C)P(C)$.

   b) Prove that if $A$ and $B$ are independent, then so are $A^c$ and $B^c$.

6. Let $A$ and $B$ be two events.

(a) Show that in general, if $A$ and $B$ are mutually exclusive, then they are not necessarily independent. (you only need to give an example, say based on the experiment of tossing a coin once)

(b) Find a particular pair of events $A$ and $B$ such that they are both mutually exclusive and independent. (if you do not have any idea, try all possible pairs of events in the experiment of tossing a coin once)

7. Prove Boole's inequalities:

   (a) $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$, (b) $P(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n P(A_i^c)$.

8. Let $A_1 \supset A_2 \supset \cdots$ be a sequence of events. If $\bigcap_{i=1}^\infty A_i = \phi$(empty), show that

$$\lim_{n\to\infty} P(A_n) = 0.$$

9. A random number $N$ of fair dice is thrown. $P(N = n) = 2^{-n}, n \geq 1$. Let $S$ be the sum of the scores. Find the probability that

   a) $N = 2$ given $S = 4$

   b) $S = 4$ given $N = 2$.

   c) $S = 4$ given $N$ is even

   d) the largest number shown by any die is $r$.

10. A biased coin ($p = P(head)$) is tossed repeatedly. Let $P_n$ be the probability that an even number of heads has occurred after $n$ tosses. Note that $P_o = 1$ and show that for $n \geq 1$,

$$P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1}.$$

   Solve this difference equation to obtain an expression for $P_n$.

11. A coupon is selected at random from a series of $k$ coupons and placed in each box of cereal. A house-husband has bought $N$ boxes of cereal. What is the probability that all $k$ coupons are obtained? (Hint: Consider the event that the $i$th coupon is not obtained. The answer is in nice summation format.)

12. If birthdates are equally likely to fall in each of the twelve months of the year, find the probability that all twelve months are represented among the birthdates of 20 people selected at random.

    (Hint: let $A_i$ be the event that the $i$th month is not included and consider $A_1 \cup A_2 \cdots \cup A_{12}$)

13. Let $X$ be a random variable and $g(\cdot)$ be a real valued function.

    (a) What do we mean by $X$ is discrete?

    (b) If $X$ is a discrete random variable, argue that $g(X)$ is also a random variable and discrete.

    (c) If $X$ is a continuous random variable, is $g(X)$ necessarily a continuous random variable? Why?

14. Let $X$ and $Y$ be independent Poisson variables with means $\lambda$ and $\mu$. Show that

    a) $T = X + Y$ is Poisson with mean $\lambda + \mu$.

    b) the conditional distribution of $X$ given $T = n$ is binomial and find its parameters.

    c) Find the conditional distribution of $T$ given $X = k$.

15. An integer $N$ is chosen from the geometric distribution with probability function

    $$f_N(n) = \theta(1-\theta)^{n-1}, n = 1, 2, \ldots$$

    Given $N = n$, $X$ has the uniform distribution on $1, 2, \ldots, n$.

    a) Find the joint p.f. of $X$ and $N$.

    b) Find the conditional p.f. of $N$ given $X = x$.

16. The number of fish that Elise catches in a day is a Poisson random variable with mean 30. However, on the average, Elise tosses back two out of every three fish she catches. What is the probability that, on a given day, Elise takes home$n$ fish. What is the mean and variance

of (a) the number of fish she catches, (b) the number of fish she takes home? (What independence assumptions have you made?)

17. Let $X_1, X_2, X_3$ be independent random variables taking values in the positive integers and having probability function given by $P(X_i = x) = (1 - p_i)p_i^{x-1}$ for $x = 1, 2, \ldots,$ and $i = 1, 2, 3$.

    (a) Show that

    $$P(X_1 < X_2 < X_3) = \frac{(1 - p_1)(1 - p_2)p_2 p_3^2}{(1 - p_2 p_3)(1 - p_1 p_2 p_3)}.$$

    (b) Find $P(X_1 \leq X_2 \leq X_3)$.

18. Let $a$ and $b$ be independent random variables uniformly distributed in (0, 1). What is the probability that $x^2 + ax + b = 0$ has no real roots?

19. Express the distribution functions of

    $$X^+ = \max\{0, X\}, \quad X^- = -min\{0, X\}, \quad |X| = X^+ + X^-, \quad -X$$

    in terms of the distribution function $F$ of the random variable $X$.

20. Is it generally true that $E(1/X) = 1/E(X)$? Is it ever true that $E(1/X) = 1/E(X)$?

21. Suppose that 13 cards are selected at random from a regular deck of 52 playing cards. (a) If it is known that at least one ace has been selected, what is the probability that at least two aces have been selected? (b) If it is known that the ace of heart has been selected, what is the probability that at least two aces have been selected?

22. The number of children $N$ in a randomly chosen family has mean $\mu$ and variance $\sigma^2$. Each child is a male with probability $p$ independently and $X$ represents the number of male children in a randomly chosen family. Find the mean and variance of $X$.

23. Suppose we have ten coins which are such that if the $i$th one is flipped then heads will appear with probability $i/10$, $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows head. What is the conditional probability that it was the fifth coin?

24. Find the mean and variance of $X$ when

(a) X has Poisson distribution with $p(x) = \frac{\mu^x}{x!} e^{-\mu}$, $x = 0, 1, \ldots$.

(b) X has exponential distribution with $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

25. (a) If $X$ and $Y$ are exponentially distributed with rate $\lambda = 1$ and independent of each other, find the density function of $X + Y$.

(b) If $X$ and $Y$ are geometrically distributed with parameter $p$ and independent of each other, find the probability mass function of $X + Y$.

(c) Find a typical discrete distribution and a typical continuous distribution (not discussed in class) to repeat question (a) and (b).

26. Suppose that given $N = n$, $X$ has binomial distribution with parameters $n$ and $p$. Suppose also $N$ has Poisson distribution with parameter $\mu$. Use the technique of generating functions to find

[5](a) the marginal distribution function of $X$.

[5](b) the distribution of $N - X$.

27. A coin is tossed repeatedly, heads appearing with probability $p = 2/3$ on each toss.

(a) Let $X$ be the number of tosses until the first occasion by which two heads have appeared successively. Write down a difference equation for $f(k) = P(X = k)$.

(b) Show the generating function of $f(k)$ is given by

$$F(s) = \frac{4}{27} s^2 [\frac{2}{1 - \frac{2}{3}s} + \frac{1}{1 + \frac{1}{3}s}].$$

(c) Find an explicit expression for $f(k)$ and calculate $E(X)$.

28. Let $X$ and $Y$ be independent random variables with negative binomial distribution and probability function

$$p_i = \binom{-k}{i} p^k (p-1)^i, \qquad i = 0, 1, \ldots.$$

(a) Show that the probability generating function of $X$ is given by $G(s) = \frac{p^k}{(1+(p-1)s)^k}$.

(b) Find the probability function of $X + Y$.

(c) Calculate $E(e^X)$ and $Var(e^X)$ (Bonus marks if you find that a condition on the size of $p$ is needed).

29. The random variables $X$ and $Y$ are said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}]\}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, where $\sigma_x, \sigma_y, \mu_x, \mu_y$, and $\rho$ are constants such that $-1 < \rho < 1$, $\sigma_x > 0$, $\sigma_y > 0$.

(a) Show that $X$ is normally distributed with mean $\mu_x$ and variance $\sigma_x^2$ and $Y$ is normally distributed with mean $\mu_y$ and variance $\sigma_y^2$.

(b) Show that the conditional density of $X$ given that $Y = y$ is normal with mean $\mu_x + (\rho\sigma_x/\sigma_y)(y - \mu_y)$ and variance $\sigma_x^2(1 - \rho^2)$.

30. Suppose that 10 cards, of which 5 are red and 5 are green, are put at random into 10 envelops, of which 7 are red and 3 are green, so that each envelop will contain a card. Determine the probability that exactly $k$ envelopes will contain a card with a matching color(k=0, 1, ..., 10).

31. Give the sequences generated by the following:

1) $A(s) = (1 - s)^{-1.5}$;

2) $B(s) = (s^2 - s - 12)^{-1}$;

3) $C(s) = s\log(1 - \theta s^2)/\log(1-\theta)$;

4) $D(s) = s/(5 + 3s)$;

5) $E(s) = (3 + 2s)/(s^2 - 3s - 4)$;

6) $F(s) = (p + qs)^n$.

32. Turn the following equation systems into equations in generating functions.

    1) $b_0 = 1$; $b_j = b_{j-1} + 2 * a_j$, $j = 1, 2, \ldots$; $a_0 = 0$.

    2) $b_0 = 0$, $b_1 = p$, $b_n = q\sum_{r=1}^{n-1} b_r b_{n-1-r}$, $n = 2, 3, \ldots$.

33. 1) Find the generating function of the sequence $a_j = j(j+1)$, $j = 0, 1, 2, \ldots$.

    2) Find the generating function of the sequence $a_j = j/(j+1)$, $j = 0, 1, 2, \ldots$.

    3) Let $X$ be a non-negative integer valued random variable and define $r_j = P(X \le j)$. Find the generating function of $\{r_j\}$ in terms of the probability generating function of $X$.

34. 1) Negative Binomial

$$p_j = \binom{-k}{j}(-p)^j(1-p)^k, j = 0, 1, \ldots$$

    where $k > 0$ and $0 < p < 1$.

    2) Let $r_o = 0$, $r_j = c/j(j+2)$, $j = 1, 2, \ldots$ (find the constant c by yourselves).

    Find the means and the variances of the above distributions whichever exists.

35. Find the probability generating function of the following distributions:

    1. Discrete uniform on $0, 1, \ldots, N$.

2. Geometric.

3. Binomial.

4. Poisson.

36. Let $\{a_n\}$ be a sequence with generating function $A(s)$, $|s| < R$, $R > 0$. Find the generating functions of

1) $\{c + a_n\}$ where $c$ is a real number.

2) $\{ca_n\}$ where $c$ is a real number.

3) $\{a_n + a_{n+2}\}$.

4) $\{(n+1)a_n\}$.

5) $\{a_{2n}\} = \{a_0, 0, a_2, 0, a_4, \ldots\}$.

6) $\{a_{3n}\} = \{a_0, 0, 0, a_3, 0, 0, a_6, \ldots\}$.

37. For a branching process with family size distribution given by

$$P_0 = 1/6, P_2 = 1/3, P_3 = 1/2;$$

calculate the probability generating function of $Z_2$ given $Z_o = 1$, where $Z_2$ is the population of the second generation. Find also, the mean and variance of $Z_2$ and the probability of extinction. Repeat the same calculation when $Z_o = 3$ and

$$P_0 = 1/6, P_1 = 1/2, P_3 = 1/3.$$

38. Let the probability $p_n$ that a family has exactly $n$ children be $\alpha p^n$ when $n \geq 1$, and $p_0 = 1 - \alpha p(1 + p + p^2 + \cdots)$. Assume that all $2^n$ sex sequences in a family of $n$ children have probability $2^{-n}$. Show that for $k \geq 1$, the probability that a family has exactly $k$ boys is $2\alpha p^k/(2-p)^{k+1}$. Given that a family includes at least one boy, what is the probability that there are two or more boys?

39. Let $X_i$, $i \geq 1$, be independent uniform (0, 1) random variables, and define $N$ by

$$N = \min\{n : X_n < X_{n+1}\}$$

where $X_0 = x$. Let $f(x) = E(N)$.

(a) Derive an integral equation for $f(x)$ by conditioning on $X_1$.

(b) Differentiate both sides of the equation derived in (a).

(c) Solve the resulting equation obtained in (b).

40. Consider a sequence defined by $r_0 = 0$, $r_1 = 1$ and $r_j = r_{j-1} + 2r_{j-2}$, $j \geq 2$. Find the generating function R(s) of $\{r_j\}$, determine $r_{25}$. For what region of $s$ values does the series for $R(s)$ converge?

41. Let $X_1, X_2, \cdots$ be independent random variables with common p.g.f. $G(s) = E(s^{X_i})$. Let $N$ be a random variable with p.g.f. $H(s)$ independent of the $X_i$'s. Let $T$ be defined as 0 if $N = 0$ and $\sum_{i=1}^{N} X_i$ if $N > 0$.s $H(G(s))$. Hence find $E(T)$ and Var(T) in terms of $E(X)$, $Var(X)$, $E(N)$ and $Var(N)$. Find the p.g.f. of $T$ where the $X_i$'s are $B * 1, p)$, and $N$ is $P_\theta$. What is the distribution of $T$?

42. Consider a branching process in which the family size distribution is Poisson with mean $\lambda$.

(a) Under what condition will the probability of extinction of the process be less than 1?

(b) Find the extinction probability when $\lambda = 2.5$ numerically.

(c) When $\lambda = 2.5$ find the expected size of the 10th generation, and the probability of extinction by the 5th generation. Comment on the relationship between this second number and the ultimate extinction probability obtained in (b).

43. Consider a branching process in which the family size distribution is geometric with parameter $p$. (The geometric distribution has p.m.f $p_j = p(1-p)^j, j = 0, 1, \ldots$).

(a) Under what condition will the probability of extinction of the process be less than 1?

(b) Find the probability of extinction when $p = 1/3$.

(c) When $p = 1/3$, find the expectation and variance of the size of the 10th generation and the probability of extinction by the 5th generation.

44. Let $\{Z_n\}_{n=0}^{\infty}$ be an usual branching process with $Z_0 = 1$. It is known $P_0 = 1 - p, P_2 = p$ with $0 \leq p \leq 1$.

    (a) Find a condition on the size of $p$ such that the probability of extinction is 0.

    (b) Find the range of $p$ such that the probability of extinction is smaller than 1. Calculate the probability of extinction as a function of $p$.

    (c) Calculate the mean and the variance of $Z_n$ when $p = \frac{2}{3}$.

45. Let $X_1, X_2, \ldots$ be independent random variables with common p.g.f. $G(s) = E(s^{X_i})$. Let $N$ be a random variable with p.g.f. $H(s)$. Show that

$$T = \begin{cases} \sum_{i=1}^{N} X_i & N \geq 1 \\ 0 & N = 0 \end{cases}$$

    has p.g.f. $H(G(s))$. Hence, find the mean and variance of $T$ in terms of the means and variances of $X_i$ and $N$. Remark: Can you see the relevance between this problem and the usual branching process?

46. Let $T$ denote the total progeny of a branching process that begins with a single root ancestor (T does not include the ancestor). Show that $K(s)$, the p.g.f. of T, satisfies $K(s) = G(sK(s))$ where $G(s)$ is the p.g.f. of the family size distribution.

47. **Branching with immigration** Each generation of a branching process (with a single progenitor) is augmented by a random number of immigrants who are indistinguishable from the other members of the

population. Suppose that the numbers of immigrants in different generations are independent of each other and of the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function $G_n$ of the size of the $n$th generation satisfies $G_{n+1}(s) = G_n(G(s))H(s)$, where $G$ is the probability generating function of a typical family of offspring.

48. Consider the random walk $X_0 = 0$, $X_n = X_{n-1} + Z_n$ where $P(Z_n = +1) = p$, $P(Z_n = -1) = q$, $n = 1, 2, \ldots$ independently ($p + q = 1$). Find the probability that the event $X_n = r$ will ever occur where $r$ is a fixed positive integer. If $p > q$, find the expected time until its first occurrence.

49. Consider the random walk $X_0 = 0$, $X_n = X_{n-1} + Z_n$ where $P(Z_n = +1) = p$, $P(Z_n = -2) = q$, $n = 1, 2, \ldots$ independently ($p + q = 1$). Let $\Lambda^{(r)}(s)$ and $\lambda_n$ be defined as in the class. Show that $\Lambda^{(r)}(s) = [\Lambda(s)]^r$ and derive the relationship

$$\lambda_n = q\lambda^{(3)}_{n-1}, \quad n = 2, 3, \ldots.$$

Hence, show that

$$qs[\Lambda(s)]^3 - \Lambda(s) + ps = 0.$$

50. Consider the random walk $X_o = 0$, $X_n = X_{n-1} + Z_n$ where $Z_1, Z_2, \ldots$ are independent, but $P(Z_n = 1) = p$, $P(Z_n = -1) = q$ and $P(Z_n = 0) = r$, $n = 1, 2, \ldots$ ($p + q + r = 1$). Let

$$f_n = P(X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0),$$

$$\lambda_n = P(X_1 < 1, \ldots, X_{n-1} < 1, X_n \geq 1).$$

Find their generating functions $\Lambda(s)$ and $F(s)$. Hence, obtain the probability the 0 ever recurs and the probability that the walk ever passes through 1.

51. If an unbiased coin is tossed repeatedly, show that the probability that the number of heads ever exceeds twice the number of tails is $(\sqrt{5}-1)/2$.

52. Let $P_{r,k}$ be the probability that the simple random walk visits state $r$ $(r > 0)$ exactly $k$ times.

    a) If $p = q = 0.5$, show that $p_{r,k} = 0,\ k = 0, 1, 2, \ldots$.

    b) If $p > q$, show that

$$p_{r,k} = \begin{cases} 0, & k = 0; \\ (1-\theta)^{k-1}\theta, & k = 1, 2, \ldots \end{cases}$$

    where $\theta = |p - q|$.

    c) If $p < q$, show that

$$p_{r,k} = \begin{cases} 1-\lambda, & k = 0; \\ \lambda(1-\theta)^{k-1}\theta, & k = 1, 2, \ldots \end{cases}$$

    where $\lambda = (p/q)^r$.

53. Consider a gambler who at each play of the game has probability $p$ of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with $i$ unit, the gambler's fortune will reach $N$ before reaching 0?

    Hint: Let $P_i, i = 0, \ldots, N$ denote that probability that, starting with $i$, the gambler's fortune will eventually reach $N$. Derive a relationship between $P_i$'s.

54. Using the fact that $u_{2n+1} = 0$ and $u_{2n} = \binom{2n}{n} p^n q^n$ to show

$$U(s) = (1 - 4pqs^2)^{-1/2}.$$

55. Let $\lambda$ be the recurrent event FS in a sequence of Bernoulli trials.

    a) Determine $\{u_n\}$, $U(s)$ and $F(s)$.

b) For the case $p = q = 1/2$, find $\{f_n\}$.

c) Let the $K_n$ represent the number of occurrences of $\lambda$ in the first $n$ trials. Find the mean and variance of $K_n$.

56. Use probabilistic arguments to verify that

$$u_n = \sum_{k=0}^{n} f_k u_{n-k}, \quad n = 1, 2, \ldots$$

and so show that $U(s) = 1 + F(s)U(s)$.

57. In a simple random walk exmine if the following are renewal events.

(a) $\epsilon$ is said to occur at trial $n$ if at trial $n$, a return to origin from the positive side takes place.

(b) $\epsilon$ is said to occur at trial $n$ if at trial $n$, the walk is to the right side of the origin.

58. Lifetime distribution of a fuse is given by $f_n = \theta^{n-1}(1-\theta)$, $n = 1, 2, \ldots$.

(a) Show that $P(X = m + n | X > m) = f_n, n = 1, 2, \ldots$.

(b) Suppose that a new fuse is place in service on day 0 and immediately upon its failure is replaced with an identical fuse. Also assume that the lifetimes are i.i.d. random variables with distribution given above. The event $\lambda$ is said to occur at trial $n$ if a new fuse is put at trial $n$. We note that $\lambda$ is a recurrent event. Obatin $F(s)$ and $U(s)$, and hence determin $u_n$ for this event.

(c) Let $T$ be the survival time of the fuse in service at time $n$. (if a failure occurs at time $n$, the fuse in service is the replacement). Write $T$ as a sum of indicator variables $Y_0, Y_1, \ldots$, where $Y_i = 1$ if the fuse in service at $n$ is also in service at time $i$, and 0 otherwise. Show that

$$E(T) = \frac{1 + \theta - \theta^{n+1}}{1 - \theta}.$$

Note that as $n \to \infty$, $E(T) \to (1+\theta)/(1-\theta)$ which is strictly greater than the mean interrecurrence time. Can you explain this fact on intuitive grounds?

59. In a symmetric random walk in two dimensions, a particle begins at the origin and then moves 1 unit to the N, S, E, or W each with probability 1/4. Let $\lambda$ designate return to the origin.

    (a) Whoat that $u_{2n+1} = 0$ and

    $$u_{2n} = \binom{2n}{n} 4^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2.$$

    (b) Show that the particle returns to the origin with probability 1. Argue from this result that the particle must pass through every point in the integer lattice.

60. Consider a renewal event with the $\{f_n\}$ sequence having generating function $F(s)$. Let $N_k$ denote the number of occurrences in the first $k$ trials and let $q_{k,n} = P(N_k = n)$.

    Show that $Q_n(s) = \sum_k q_{k,n} s^k$ is given by

    $$Q_n(s) = \frac{\{1 - F(s)\} F^n(s)}{1 - s}.$$

    Hint: $q_{k,n} = \sum_{\text{combination of } \alpha} F(T_1 = \alpha_1) P(T_2 = \alpha_2) \cdots P(T_n = \alpha_n) P(T_{n+1} > \beta)$ where $T_i$ is the number of trials between the $(i-1)$th and the $i$-th occurence and $\beta = k - \alpha_1 - \cdots - \alpha_n$.

61. (Self-organizing data retrieval system). consider a shelf continaing two books, $B_1$ and $B_2$ (among others). These books have two possible orders on the shelf, namely $B_1 B_2$ or $B_2 B_1$. Assume that at epoches $n = 0, 1, 2, \ldots$, a book is required by a library user, and that at any epoch the probability that $B_j$ is needed is $p_j$, $j = 1, 2$, independently of what happens in other apochs. Assume $p_1 > 0, p_2 > 0$, $p_1 + p_2 < 1$. To obtain the required book, the librarian always searches the book-shelf

from left to right, so that average search time for the requested book is minimized if the book with higher $pj$ value is on the left.

However, the librarian does not know which book is more popular, and therefore cannot decide whether $b_1B_2$ or $B_2B_1$ is the better arrangement. To increase the chance of having the requested book nearer the left much of the time, the foolowing algorithm has been devised. Whenever any book is requested, it is placed to the end of the shelf when it is returned. Thus if $B_2$ is demanded and the shelf order is $B_1B_2$, the new arrangement will be$B_2B_1$ once $B_2$ is returned.

(a) Let $\lambda$ be the renewal event "shelf order of $B_1$ and $B_2$ is $B_1B_2$". Let $f_n$ be the lifetime sequence for $\lambda$, having generating function $F(s)$. Show that

$$F(s) = (1 - p_2)s + p_1p_2s^2/(1 - (1 - p_1)s).$$

(b) Show that $\lambda$ is aperiodic and recurrent. Hence determine $\lim u_n = P(\lambda \text{at epoch} n)$. Determine also the long run probability that the self order is $B_2B_1$.

62. Let $\{X_n\}_{n=0}^{\infty}$ be a stochastic process.

[3](a) If for each fixed $n$, $X_n$ has density function

$$f(x) = 1 \qquad \text{when } x \in [n, n+1],$$

write down the state space of this process.

[3](b) If for each fixed $n$, $P(X_n = n) = P(X_n = -1) = 0.5$, write down the state space of this process.

[3](c) Which of the above state spaces, the one in (a) or in (b), is countable?

63. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may

be analyzed by using a Markov chain. How many states are needed? Define the stochastic process, list the state space.

Suppose also that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine the the transition matrix.

64. Let the transition probability matrix of a two-state Markov chain be given by

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{pmatrix} 0.5 + 0.5(2p-1)^n & 0.5 - 0.5(2p-1)^n \\ 0.5 - 0.5(2p-1)^n & 0.5 + 0.5(2p-1)^n \end{pmatrix}.$$

65. Let the one step transition matrix of an MC be

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 < a, \quad b < 1.$$

Show that the $n$-step transition matrix

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

Use matrix multiplication directly to obtain $P^3$ when $a = b = 0.25$. Verify the result by using the formula you just obtained.

66. Specify the classes of the following Markov chains and determine whether they are transient or recurrent, whether they are periodic or aperiodic.

For recurrent states, find their mean recurrence time.

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P}_3 = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \mathbf{P}_4 = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{P}_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \quad \mathbf{P}_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix}.$$

$$\mathbf{P}_7 = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \quad \mathbf{P}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 \\ 1/3 & 0 & 1/6 & 1/6 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

67. Prove that if the number of states in a Markov chain is $M$, and if state $j$ can be reached from state $i$, it can be reached in $M$ steps or less.

68. A transition matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is

$$\sum_i p_{ij} = 1, \text{ for all } j.$$

If such a chain is irreducible and aperiodic and consists of $M+1$ states, 0, 1, ..., M, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \qquad j = 0, 1, ..., M$$

69. Let $\{X_n\}_{n=0}^{\infty}$ be a Markov Chain with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1) Classify the state space into classes.

2) Which of them are recurrent, or transient?

3) Find the period of state 2. (assume the state space is { 0, 1, 2, 3}).

4) Find the expected inter-recurrent times for all recurrent states. (The answers to some states should be obvious; Limiting probabilities)

70. Consier the transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0\ \frac{2}{3} & 0 & \end{bmatrix}.$$

(a) Show that $S$ consists of 2 closed classes and 2 open classes. What are these classes?

(b) Determine the period of each of the closed classes.

Note it is impossibel to return to either of the transient states 2 and 4 in this chain. In this case, we set the period of the state to be infinity, to indicate that we cannot return.

(c) Find the unique steady state corresponding to each of the closed classes.

(d) Write down the general form of all steady states for $P$.

(2) If $X_0 = 2$, what is the probability of absorption in to the class $\{0, 1\}$? If $X_0 = 4$, what is the probability of absorption in to the class $\{0, 1\}$?

71. Consier the transition matrix

$$P = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

(a) Show that $S$ consists of two closed classes and one open class.

(b) Find the period of each of the three classes.

(c) Find the unique steady state corresponding to each closed class, and write down the general form of all steady states for $P$.

(d) Find the probability of absorption into $\{1, 3\}$ from state 0 and the probability of absorption into $\{1, 3\}$ from state 5. What can you say about the probabilities of absorption in $\{2, 4\}$ from states 0 and 5 respectively?

72. Consier the transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Check that $P$ is irreducible and find the period of $P$.

(b) Solve for the unique steady state of $P$.

(c) Use the periodic form of the Main Convergence Theorem to find the mean recurrence time of each of the states.

73. Consider a chain with states 0, 1, 2, ..., $a$ with

$$p_{0,1} = 1, \quad P_{a,a-1} = 1$$

and

$$P_{ij} = \{ \begin{array}{ll} i^2/a^2 & j = i-1 \\ (a-i)^2/a^2 & j = i+1 \\ 2i(a-i)/a^2 & j = i, (i \neq 0, a) \end{array}$$

Show that the chain is ergodic and obtain the stationary distribution.

74. One form of a random walk with two reflecting barriers has transition matrix given by

$$P_{00} = 1 - p, \quad P_{01} = p;$$

$$P_{j,j-1} = 1, \quad P_{j,j} = 4, P_{j,j+1} = p;$$

$$P_{a,a-1} = q, \quad P_{a,a} = 1 - q.$$

where $p + q + r = 1$ Whoat that the chain is irreducible and aperiodic. Determine the stationary distribution for this chain. (Can you spot the minor missing assumption we really need?)

75. Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process with the family size distribution given by $P(x = 0) = 1/3$, $P(X = 2) = 2/3$.

1) State the definition of the Markov chain.

2) Verify that $\{Z_n\}_{n=0}^{\infty}$ is a Markov chain. Calculate the transition probabilities $p_{ij}$. (Think about situations such as $i = 0$; $j = 0$; $j$ is odd etc).

3) Classify the state space. Indicate whether they are recurrent or transient. Give a one line explanation.

4) Can you find a stationary distribution?

76. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of $k$ pairs of running shoes, what proportion of the time does he run barefooted?

77. The proof copy of a book is read by an infinite sequence of editors checking for mistakes. Each mistake is detected with probability $p$ at each reading; between readings the printer corrects the detected mistakes but introduces a random number of new errors (errors may be introduced even if no mistakes were detected). Assume as much independence as usual and that the numbers of new errors after different readings are independent and have Poisson distribution. Find the stationary distribution of the number $X_n$ of errors after the $n$th editor-printer cycle.

78. For a series of dependent trials the probability of success on any trial is $(k+1)/(k+2)$ where $k$ is equal to the number of successes on the previous two trials. Compute $\lim_{n\to\infty} P(\text{success on the } n\text{th trial})$.

79. It is known that for a Markov Chain, the limit probabilities exist if it is ergodic and aperiodic. Find a simple example of Markov chain such that it does not satisfy all the conditions but the limit probabilities still exist.

80. Suppose there are 5 white and 5 black balls in an urn. On each day, a ball is selected randomly and replaced by a ball with other color. Let $X_n = 0$ if a while ball is selected and $X_n = 1$ otherwise. Also, let $Y_n$ be the number of white balls in the urn after $n$th selection. Are $\{X_n\}$ and $\{Y_n\}$ Markov chain? If not, explain why. If yes, list the state space and obtain the transition matrix.

81. Suppose 4 balls are placed into two urns A and B. On each day, One ball is selected such that each of the four balls is equally likely to be selected and the ball is then placed into the other urn.

Let $X_n$ be the number of balls in urn A on the $n$th day. Let $Y_n$ be the number of balls in urn A on the $2n$th day.

a) Are $\{X_n\}_{n=0}^{\infty}$ and $\{Y_n\}_{n=0}^{\infty}$ Markov chains. If any of them are, write down their state spaces and transition matrices and do the usual classification.

b) Given $X_0 = 1$, find the probability function of $X_2$.

c) In a long run, what proportion of times when at least one urn is empty?

d) Given $X_0 = k$, calculate the probability that number of balls in urn A reaches 0 before the number of balls in urn B reaches 0 for $k = 0, 1, 2, 3$ and 4.

82. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially (on the 0th day) flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1?

83. A particle moves on a circle through points which have been marked 0, 1, 2, 3, 4 (in clockwise order). At each step it has a probability $p$ of moving to the right (clockwise) and $1-p$ to the left (counterclockwise). Let $X_n$ denote its location on the circle after the $n$th step. Show that the process $\{X_n, n \geq 0\}$ is a Markov chain.

(a) Find the transition probability matrix.

(b) If we know $X_1 = 2$, what is the probability of $X_3 = 4$?

(c) If $X_0$ is equally likely to be 0, 1, 2, 3, 4, what is the probability of $X_3 = 4$?

84. Consider a process $\{X_n, n = 1, 2, \ldots\}$ which takes on the values 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_o = i_o\} = \begin{cases} P_{ij}^{I}, & \text{when n is even} \\ P_{ij}^{II}, & \text{when n is odd} \end{cases}$$

where $\sum_{j=0}^{2} P_{ij}^{I} = \sum_{j=0}^{2} P_{ij}^{II} = 1$, $i = 0, 1, 2$. Is $\{X_n, n \geq 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

85. Show that if state $i$ is recurrent and state $j$ does not communicate with state $j$, then $P_{ij} = 0$. This implies that once a process enters a recurrect class of states it can never leave that class. For this reason, a recurrent class if often referred to as a closed class.

86. For a Poisson process show, for $s < t$, that

$$P\{N(s) = k | N(t) = n\} = \binom{n}{k} (\frac{s}{t})^k (1 - \frac{s}{t})^{n-k}, \quad k = 0, 1, \ldots, n.$$

87. Cars pass a point on the highway at a Poisson rate of one per minute. If five percent of the cars on the road are Dodges, then

(a) what is the probability that at least one Dodge passes by during a hour?

(b) given that ten Dodges have passed by in an hour, what is the expected number of cars to have passed by in that time?

(c) if 50 cars have passed by in an hour, what is the probability that five of them were Dodges?

88. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$. Let $S_n$ denote the time of the $n$th event. Find

(a) $\mathrm{E}(S_4)$,

(b) $\mathrm{E}[S_4|N(1) = 2]$,

(c) $\mathrm{E}[N(4) - N(2)|N(1) = 3]$.

89. Two individuals, $A$ and $B$, both require kidney transplants. If she does not receive a new kidney, then $A$ will die after an exponential time with rate $\mu_A$, and $B$ after an exponential time with rate $\mu_B$. New kidneys arrive in accordance with a Poisson process having rate $\lambda$. It has been decided that the first kidney will go to $A$ (or to $B$ if $B$ is alive and $A$ is not at that time) and the next one to $B$ (if still living).

    (a) What is the probability $A$ obtains a new kidney?

    (b) What is the probability $B$ obtains a new kidney?

90. Telephone calls arrive at a switchboard in a Poisson process at the rate of 2 per minute. A random one-tenth of the calls are long distance.

    (a) What is the probability that no call arrives between 9:00-9:05am?

    (b) What is the probability that at least 2 calls arrive between 10:00-10:02am?

    (c) What is the probability of at least one long distance call in a ten minute period?

    (d) Given that there have been 8 long distance calls in an hour, what is the expected number of calls to have arrived in the same period?

    (e) Given that there were 90 calls in an hour, what is the probability that 10 were long distance?

91. Three customers A, B and C enter a bank. A and B to deposit money and C to buy a money order. Suppose that the time it takes to deposit money is exponentially distributed with mean 2 minutes, and that the time it takes to buy a money order is exponentially distributed with mean 4 minutes. If all three customers are served immediately, what is the probability that C is finished first? That A is finished last?

92. Suppose that a one-celled organism can be in one of two states– either A or B. An individual in state A will change to state B at an exponential rate $\alpha$; an individual in state B divides into two new individuals of type A at an exponential rate $\beta$. Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate parameters for this model.

93. Potential customers arrive at a single-server station in accordance with a Poisson process with rate $\lambda$. However, if the arrival finds $n$ customers already in the station, then he will enter the system with probability $\alpha_n$. Assuming an exponential service rate $\mu$, set this up as a birth and death process and determine the birth and death rates.

94. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.

    (a) Determine the expected time to go from state 0 to state 2.

    (b) Determine the expected time to go from state 2 to state 3.

    (c) Determine the variances in parts (a) and (b).

95. A job shop consists of three machines and two repairmen. The amount of time a machines works before breaking down is exponentially distributioned with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then

    (a) what is the average number of machines not in use?

    (b) what proportion of the time are both repairmen busy?

96. Each individual in a biological population is assumed to give birth at an exponential rate $\lambda$, and to die at an exponential rate $\mu$. In addition, there is an exponential rate of increase $\theta$ due to immigration. However, immigration nor birth are allowed when the population size reaches $N$.

    (a) Set this up as a birth and death model.

(b) If $N = 3$, $\theta = \lambda = 1$, $\mu = 2$, determine the proportion of time that immigration is restricted.

97. Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

    (a) What fraction of the attendant's time will be spent servicing cars?

    (b) What fraction of potential customers are lost?

98. A parking lot has $N$ spaces. The incoming traffic is of Poisson type at a rate of $\lambda$ cars per hour whereas the occupancy times are exponentially distributed with a mean of $\beta$ hours.

    (1) Find the appropriate differential equations for the probabilities, $P_n(t)$, of finding exactly $n$ spaces occupied at time $t$.

    (2) When $N = 5$, $\lambda = 2$ and $\beta = 1$, obtain the variance of the number of spaces occupied if the process has been operating for a very long time.

99. A small appropriate, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean 1/4 hour. What is

    (a) the average number of customers in the shop?

    (b) the proportion of potential customers that enter the shop?

    (c) If the barber could work twice as fast, how much more business would he do?

100. Consider two machines, both of which have an exponential lifetime with mean $1/\lambda$. There is a single repairman that can service machines at an

exponential rate $\mu$. Set up the Kolmogorov backward equations; you need not solve them. If you can solve this equation, what questions will you be able to answer?

101. Consider two machines. Machine $i$ operates for an exponential time with rate $\lambda_i$ and then fails; its repair time is exponential with rate $\mu_i$, $i = 1, 2$. The machines act independently of each other. Define a four-state continuous-time Markov chain which jointly describes the condition of the two machines. Use the assumed independence to compute the transition probabilities for this chain and then verify that these transition probabilities satisfy the forward and backward equations.

102. Let $\{X(t)\}$ be a typical birth and death process with birth rates $\lambda_n$ and death rates $\mu_n$, $n = 0, 1, \ldots$, and $\mu_0 = 0$. (You are responsible to know any other assumptions made in a general birth and death process).

    (a) In this set up, let $B_n$ be the waiting time until a birth when $X(t) = n$ and $D_n$ be the waiting time until a death when $X(t) = n$. What are the distributions of $B_n$ and $D_n$ and their related parameters?

    (b) Let $T_n = \min\{B_n, D_n\}$. Calculate $P(T_n > t)$ for $t > 0$. What is the distribution of $T_n$?

    (c) Calculate $P(B_n < D_n)$ for any non-negative integer $n$.

    (d) Verify that $\{X(t)\}$ is a continuous time Markov chain. Identifying corresponding parameters: the exponential rate $v_n$ and conditional transition probability $p_{ij}$. (given that $X(t)$ is leaving $i$, the chance it enters $j$).

103. A computer can handle $N$ tasks simultanuously. The tasks are submitted to the computer as a Poisson process with a rate of $\lambda$ per second and the amount of time it takes to complete a task is independent of other tasks and has exponential distribution with a mean of $\beta$ seconds. The tasks submitted while the computer is at full load will be lost without any warnings.

(a) Set up a birth and death process to model this process. This includes: define $\{X(t), t \geq 0\}$; write down its state space and its birth and death rates.

(b) Write down its infinitesimal generator $G$.

(c) Assume $N = 3$, $\lambda = 4$ and $\beta = 1$,

(i) obtain the limiting probabilities of this process.

(ii) obtain the mean number of tasks the computer handles at any moment if the computer has been operating for a very long time.

(iii) what proportion of the jobs you submitted will get lost in a long run?

104. Suppose that the interarrival distribution for a renewal process is Poisson distributed with mean $\mu$. That is, suppose

$$P(X_n = k) = \frac{\mu^{k-1}}{(k-1)!} \exp(-\mu), \quad k = 1, 2, \ldots.$$

(a) Find the distribution of $S_n$.

(b) Calculate $P(N(t) \geq n)$.

(c) Find $m(t) = E[N(t)]$(not necessarily in a closed form).

105. Mr. Smith works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean 2, then at what rate does Mr. Smith get new jobs?

106. Each time a machine is repaired it remains up for an exponentially distributed time with rate $\lambda$. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machines is exponential with rate $\mu_1$; if it is a type 2 failure, then the repair time is exponential with rate $\mu_2$. Each failure is , independently of the time it took the machines to fail, a type 1 failure with probability $p$ and a

type 2 failure with probability $1-p$. What proportion of time is the machine down due to a type 1 failure? what proportion of time is the machine down due to a type 2 failure? What proportion of time is it up?

107. A machine in use is replaced by a new machine either when it fails or when it reaches the age of $T$ years. If the lifetimes of successive machines are independent with a common distribution $F$ having density $f$ show that

(a) the long-run rate at which machines are replaced equals

$$\left[\int_0^T xf(x)dx + T(1-F(T))\right]^{-1};$$

(b) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\int_0^T xf(x)dx + T[1-F(T)]}.$$

108. Machines in a factory break down at an exponential rate of six per hour. There is a single repairman who fixes machines at an exponential rate of eight per hour. The cost incurred in lost production when machines are out of service is \$10 per hour per machine. What is the average cost rate incurred due to failed machines?

109. The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$C per hour. The manager estimates that, on the average, each customer's time is worth \$1 per hour and should be accounted for the model. If customers arrive at a Poisson rate of 10 per hour, then

(a) what is the average cost per hour if Mary is hired? if Alice is hired?

(b) find C if the average cost per hour is the same for Mary and Alice.

110. Consider a renewal process $\{N(t), t \geq 0\}$ having a gamma $(r, \lambda)$ interarrival distribution. That is, the interarrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0.$$

(a) Show that

$$P\{N(t) \geq n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}.$$

(b) Use (a) to show that

$$m(t) = \sum_{i=r}^{\infty} [\frac{i}{r}] \frac{e^{-\lambda t}(\lambda t)^i}{i!}.$$

where $[\frac{i}{r}]$ is the largest integer less than or equal to $i/r$.

111. Consider a single-server bank for which customers arrive in accordance with a Poisson process with rate $\lambda$. If a customer only will enter the bank if the server is free when he arrives, and if the service time of a customer has the distribution $G$, then what proportion of time is the server busy?

112. Consider the following queueing system. Customers arrive in a Poisson process at rate $\lambda > 0$ and are served, in order of arrival, by a single server. Service times are independent; however, they are not identically distributed since it has been observed that the server works more quickly when there are a number of customers waiting in the queue. To model this phenomenon of stat-dependent serve times assume that when there are $j$ customers in the system the server provides exponential service at rate $j\mu$, $j = 1, 2, \ldots$.

(a) Show that $\{\pi_n\}$, the equilibrium probability distribution for the number of customers in the system (including the one being served), is Poisson with mean $\rho = \lambda/\mu$.