STAT305, Assignment No. 4 Fall 2001

1. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed? Define the stochastic process, list the state space.

   Suppose also that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine the the transition matrix.

2. Let the transition probability matrix of a two-state Markov chain be given by

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

   Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{pmatrix} 0.5 + 0.5(2p-1)^n & 0.5 - 0.5(2p-1)^n \\ 0.5 - 0.5(2p-1)^n & 0.5 + 0.5(2p-1)^n \end{pmatrix}.$$

3. Let the one step transition matrix of an MC be

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 < a, \quad b < 1.$$

   Show that the n-step transition matrix

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

   Use matrix multiplication directly to obtain P^3 when $a = b = 0.25$. Verify the result by using the formula you just obtained.

4. Specify the classes of the following Markov chains and determine whether they are transient or recurrent, whether they are periodic or aperiodic. For recurrent states, find their mean recurrence time.

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \mathbf{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P}_3 = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \mathbf{P}_4 = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{P}_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \qquad \mathbf{P}_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix}.$$

$$\mathbf{P}_7 = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \quad \mathbf{P}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 \\ 1/3 & 0 & 1/6 & 1/6 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. A transition matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is

$$\sum_i p_{ij} = 1, \text{ for all } j.$$

If such a chain is irreducible and aperiodic and consists of $M+1$ states, 0, 1, ..., M, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \qquad j = 0, 1, ..., M$$

6. Let $\{X_n\}_{n=0}^{\infty}$ be a Markov Chain with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1) Classify the state space into classes.

2) Which of them are recurrent, or transient?

3) Find the period of state 2. (assume the state space is { 0, 1, 2, 3}).

4) Find the expected inter-recurrent times for all recurrent states. (The answers to some states should be obvious; Limiting probabilities)

7. Consier the transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 \; \frac{2}{3} & 0 & \end{bmatrix}.$$

(a) Show that S consists of 2 closed classes and 2 open classes. What are these classes?

(b) Determine the period of each of the closed classes.

Note it is impossibel to return to either of the transient states 2 and 4 in this chain. In this case, we set the period of the state to be infinity, to indicate that we cannot return.

(c) Find the unique steady state corresponding to each of the closed classes.

(d) Write down the general form of all steady states for P.

(2) If $X_0 = 2$, what is the probability of absorption in to the class $\{0, 1\}$? If $X_0 = 4$, what is the probability of absorption in to the class $\{0, 1\}$?

8. Consier the transition matrix

$$P = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

(a) Show that S consists of two closed classes and one open class.

(b) Find the period of each of the three classes.

(c) Find the unique steady state corresponding to each closed class, and write down the general form of all steady states for P.

(d) Find the probability of absorption into $\{1,3\}$ from state 0 and the probability of absorption into $\{1,3\}$ from state 5. What can you say about the probabilities of absorption in $\{2,4\}$ from states 0 and 5 respectively?

9. Consier the transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Check that P is irreducible and find the period of P.

(b) Solve for the unique steady state of P.

(c) Use the periodic form of the Main Convergence Theorem to find the mean recurrence time of each of the states.

10. Consider a chain with states 0, 1, 2, ..., a with

$$p_{0,1} = 1, \quad P_{a,a-1} = 1$$

and

$$P_{ij} = \{ \begin{array}{ll} i^2/a^2 & j = i-1 \\ (a-i)^2/a^2 & j = i+1 \\ 2i(a-i)/a^2 & j = i, (i \neq 0, a) \end{array}$$

Show that the chain is ergodic and obtain the stationary distribution.

11. One form of a random walk with two reflecting barriers has transition matrix given by

$$P_{00} = 1-p, \quad P_{01} = p;$$

$$P_{j,j-1} = 1, \quad P_{j,j} = 4, P_{j,j+1} = p;$$

$$P_{a,a-1} = q, \quad P_{a,a} = 1-q.$$

where $p+q+r=1$ Whoat that the chain is irreducible and aperiodic. Determine the stationary distribution for this chain. (Can you spot the minor missing assumption we really need?)

12. Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process with the family size distribution given by $P(x=0) = 1/3$, $P(X=2) = 2/3$.

1) State the definition of the Markov chain.

2) Verify that $\{Z_n\}_{n=0}^{\infty}$ is a Markov chain. Calculate the transition probabilities p_{ij}. (Think about situations such as $i=0$; $j=0$; j is odd etc).

3) Classify the state space. Indicate whether they are recurrent or transient. Give a one line explanation.

4) Can you find a stationary distribution?